Arch. Mech., 49, 1, pp. 35–48, Warszawa 1997

Asymptotic expansion of solution of the torsion problem for an elastic rod with a cavity and a thin bonded multilayer

G. S. MISHURIS (RZESZÓW)

THE FIRST TERM of the asymptotic expansion of the solution of the torsion problem for an elastic rod is derived using the method of a matched asymptotic expansion. The prismatic rod is weakened by an internal cavity with angular points, one of which is situated on the exterior boundary. The exterior boundary of the rod is reinforced by a thin elastic multilayer. Difference between the exact and approximate solution of the problem are estimated by the norm of the Sobolev spaces. Relations for stress intensity factors in the angular points are found and verified.

1. Introduction

STRUCTURAL ELEMENTS reinforced by thin surface layers have found wide application in modern technology. Such elements can seriously change the elastic and strength properties of the structures. The corresponding boundary value problems have been investigated in [2, 3, 4, 20]. In those problems it is assumed that curvature of the thin layers is small. In this way, note paper [9], in which "averaged" boundary conditions are obtained for a thin surface layer with arbitrary curvature by the operator method. All the mentioned problems are related to the so-called boundary value problems with regular perturbations of the boundaries [7, 8].

However, in the cases when stress concentrators are situated near the thin layer, singular perturbations of the boundaries appear. The methods of solution of such problems have been proposed in [6, 12, 19]. One of them is the method of matched asymptotic expansion. It consists in the solution of the limiting (internal and external) problems, and later – in their coordination in some intermediate region [6, 12].

In paper [15] the method of solving the boundary value problems in infinite domains represented by wedges and layers is proposed. For some values of the parameters, homogeneous problems discussed in [15] have nontrivial solutions, which are of some class of solutions of the internal limiting boundary value problems. These solutions can be calculated by functions belonging to the kernel of special singular integral operators [14, 15]. In [13] the numerical method of deriving the functions from the kernel of the operators has been introduced.

In the paper, a singular perturbed boundary value problem is considered, which corresponds to the torsion problem of a prismatic rod with a cavity and a thin multilayer. A similar problem for a homogeneous rod with a linear crack was investigated in [1].

2. Formulation of the problem

Let us consider a domain Ω_h with compact closure $\overline{\Omega}_h \subset \mathbb{R}^2$, smooth exterior boundary Γ_e (for example, $\Gamma_e \in C^1$), and piecewise smooth interior boundary Γ_0 ($\partial\Omega_h = \Gamma_e \cup \Gamma_0$). By Γ_1 we denote the closed curve: $\Gamma_1 = \{P \in \Omega_h : \mathrm{dist}(P, \partial\Omega_h) = h\}$, (see Fig. 1).

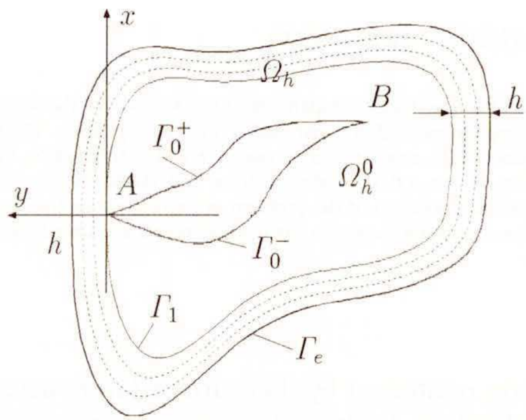

FIG. 1.

Assume that $A, B \in \Gamma_0$ are corner points which divide the closed curve $\Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-$, and

$$
(2.1)\qquad
\begin{array}{ll}
\text{(i)} & \mathrm{dist}(A, \Gamma_e) = h \ll 1, \qquad r_{\Omega_e} \geq 1, \qquad \mathrm{dist}(B, \Gamma_e) \sim 1, \\
\text{(ii)} & \angle(\Gamma_1, \Gamma_0^{\pm})_{|A} = \pi/2 \mp \phi_A, \qquad \angle(\Gamma_0^+, \Gamma_0^-)_{|B} = 2\phi_B, \\
\text{(iii)} & k_{\Gamma_1}(A) = k_{\Gamma_0^{\pm}}(A) = k_{\Gamma_0^{\pm}}(B) = 0,
\end{array}
$$

where $\phi_A, \phi_B \in (0, \pi/2)$, $k_{\Gamma_1}(A)$, $k_{\Gamma_0^{\pm}}(A)$ are curvatures of the curves Γ_1, and $\Gamma_0^{\pm}$ in point A, but $r_{\Omega_e} = \sup\{r : B_r \subset \Omega_e\}$ is the Chebyshev radius of the domain Ω_e (here $\partial\Omega_e = \Gamma_e$, and B_r is open disk of a radius r).

Let (s, n) be a local coordinate system connected with the curve Γ_1. Its origin is at the point $A \in \Gamma_1$, and $n > 0$ along the outer normal. A Cartesian coordinate system (x, y) coincides with the local system (s, n) at point A $(A = (0, 0))$.

If $m \in \mathbb{N}$, $\mu_0, \mu_j \in \mathbb{R}_+$ $(j = 1, 2, ..., m)$ are some positive constants, and $0 = h_0 < h_1 < ... < h_j < ... < h_{m-1} < h_m = h$, then we consider the step function:

$$
(2.2)\qquad \mu(s, n) = \begin{cases} \mu_{j+1}, & (s, n) \in \Omega_h \wedge h_j < n < h_{j+1}, \\ \mu_0, & (s, n) \in \Omega_h \wedge -\infty < n < 0, \end{cases}
$$

and from the assumption it follows

$$
(2.3)\qquad 0 < \min_{0 \leq j \leq m} \{\mu_j\} = \underline{\mu} \leq \mu(x, y) \leq \overline{\mu} = \max_{0 \leq j \leq m} \{\mu_j\} < \infty.
$$

We shall use also the symbols ($j = 0, 1, \dots, m$):

$$(2.4)\qquad \begin{aligned} \Omega_h^j &= \Omega_h \cap \{(x, y) : \mu(x, y) = \mu_j\}, \\ \Gamma_{j+1} &= \{(s, n) : (s, n) \in \Omega_h \wedge n = h_j\}. \end{aligned}$$

We shall seek a harmonic function $u(x, y)$ in each domain Ω_h^j (the torsion function [18]), satisfying along the interior boundaries Γ_j ($j = 1, 2, \dots, m$) between different materials the conditions:

$$(2.5)\qquad (u_j - u_{j-1})_{|\Gamma_j} = 0, \qquad \frac{\partial}{\partial n}(\mu_j u_j - \mu_{j-1} u_{j-1})_{|\Gamma_j} = f_j(x, y).$$

But along $\partial\Omega_h$ we have

$$(2.6)\qquad \mu_m \frac{\partial}{\partial n} u_{m|\Gamma_e} = f_m(x, y), \qquad \mu_0 \frac{\partial}{\partial n} u_{0|_{\Gamma_0^\pm}} = -f_0^\pm(x, y),$$

with some functions f_j, $f_0^\pm \in C^\infty(\Gamma_j)$ (see [18]), so that the following conditions are satisfied:

$$(2.7)\qquad f_j(0, h_j), f_0^\pm(0, 0) = 0, \qquad \frac{\partial}{\partial s} f_j(0, h_j), \frac{\partial}{\partial s} f_0(0, 0) \sim 1.$$

For solvability of the problem we should assume, in addition [18], that

$$(2.8)\qquad \sum_{j=0}^{m+1} \int_{\Gamma_j} f_j(s)\, ds = 0,$$

where $\Gamma_{m+1} = \Gamma_e$, but to secure the uniqueness of the solution we normalize it by the condition:

$$(2.9)\qquad u(B) = 0.$$

Using the results from [10], one can show that the linear problem (2.4) – (2.8) has the unique solution u_h in the space $W_2^1(\Omega_h, B) \equiv \{u \in W_2^1(\Omega_h) \wedge u(B) = 0\}$. It can be easily seen on the basis of the results of [5], that the solution belongs to $C^\infty(\Omega_h^j)$. Besides, we can prove that $u_h \in C(\overline{\Omega}_h)$, however, $u_h \notin W_2^2(\Omega_h)$. To verify the first fact, it is sufficient to investigate the asymptotic behaviour of the solution near any point situated on the interior boundary Γ_j ($j = 1, \dots, m$); but to check the second conclusion, we should know the behaviour of the solution in the neighbourhood of points A or B. We shall consider in detail only the second proposition. Namely, let us represent the solution near these points in the form: $u_h = \chi(r/\varepsilon)u_h + (1 - \chi(r/\varepsilon))u_h$ with some small $\varepsilon > 0$ ($\varepsilon < h_1$). Here and further on, by $\chi \in C^\infty(\mathbb{R}_+)$, we shall understand a cut-off function defined by

$$(2.10)\qquad \chi(t) = \begin{cases} 1, & 0 \le t \le 1/3, \\ 0, & 2/3 \le t < \infty. \end{cases}$$

Let us note that the function $u_\varepsilon|_l = \chi(r/\varepsilon)u_h|_l \in L_1(\mathbb{R}_+)$, where l is an arbitrary radius with origin at point A (B) so that $l \cap \Omega_h \neq \emptyset$. Then applying the Mellin transform technique to the corresponding problem for the function $u_\varepsilon = \chi(r/\varepsilon)u_h$, and taking into account the assumptions on curvatures (2.1), we obtain

$$\begin{aligned} u_h(h, r, \phi) &= d_A + c_A \nu_A^{-1} r^{\nu_A} F(\phi) + O(r^{\delta_A}), \quad r \to 0, \\ u_h(h, r, \phi) &= d_B + c_B \nu_B^{-1} r^{\nu_B} F(\phi) + O(r^{\delta_B}), \quad r \to 0, \end{aligned} \tag{2.11}$$

where (r, ϕ) are local coordinates connected with point A (or B), and the angle ϕ calculated with respect to the bisector of the corresponding corner angles, are situated in the domains Ω_h^1 (Ω_h^0, respectively), but

$$F(\phi) = \begin{cases} \dfrac{\sin \phi\, \nu}{\sin(\pi\nu/2)}, & |\phi| \le \pi/2, \\ \operatorname{sign}\phi \, \dfrac{\cos(\pi - \phi_0 - |\phi|)\, \nu}{\cos(\pi/2 - \phi_0)\, \nu}, & \pi/2 < |\phi| < \pi - \phi_0, \end{cases} \tag{2.12}$$

where $\phi_0 = \phi_A(\phi_B)$, $d_B = 0$ $(u_h \in W_2^1(\Omega_h, B))$, but constants $\nu_A, \nu_B \in (0, 1)$ are the first zeros of the function:

$$\Delta_c(s) = \kappa \cos \phi_0 s - \cos(\pi - \phi_0)s, \qquad \kappa_A = \frac{\mu_0 - \mu_1}{\mu_0 + \mu_1}, \qquad \kappa_B = 0,$$

which are the nearest to the imaginary axis. Since $\kappa_B = 0$, the relation for the function $F(\phi)$ at point B has a similar form for $|\phi| \le \pi/2$ as well as for $|\phi| > \pi/2$. Here the values of the parameters $\delta_A, \delta_B \in (1, 2)$ in (2.11) are calculated as follows:

$$\delta_A = \min\{\nu_A^{(2)}, \tau_A\}, \qquad \delta_B = \min\{\nu_B^{(2)}, \tau_B\},$$

where $\nu_A^{(2)}, \nu_B^{(2)}$ are the second zeros of the function $\Delta_c(s)$, but τ_A, τ_B are the first zeros $(\tau_A, \tau_B > 0)$ of the function: $\Delta_s(s) = s^{-1}[\kappa \sin \phi_0 s + \sin(\pi - \phi_0)s]$, with the respective value of the parameter κ (κ_A, κ_B).

The constants c_A, c_B in (2.11) play an important role in fracture mechanics [17] (stress intensity factors). The next mechanical parameter which can be calculated from the solution u_h of the problem (2.5) – (2.9) is the stiffness [18]:

$$C = \iint_{\Omega_h} \mu(x, y) \left(x^2 + y^2 + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) u_h(h, x, y) \right) d\Omega. \tag{2.13}$$

However, the numerical process used for solving the problem (2.5) – (2.9) is difficult in view of the existence of the small parameter h, and of the singularity of the solution in the neighbourhood of point A situated near the exterior boundary of the domain. Further on, we find the first term of the asymptotic expansion of the solution $\tilde{u}_h$, which is close to u_h in the norm $W_2^1(\Omega_h)$, and makes it possible to obtain the values of coefficients c_A, c_B, C from (2.11), (2.13).

3. Limiting boundary value problems

3.1. External problem

Now we consider similar problem but the domain will be somewhat different. Namely, by Ω_0 we denote the simply connected domain with boundary $\partial\Omega_0 = \partial\Omega_h \cup M_0^+ \cup M_0^-$, where $M_0^\pm = \{(x,y) : 0 < y < h \wedge x = 0\pm\}$. Along the curves $M_0^\pm$ we define functions $f_M^\pm(s) \equiv 0$, hence, the condition (2.9) holds true and the function along the boundary $\partial\Omega_0$ is continuous. Problem (2.5)–(2.9) in the domain Ω_0 also has a unique solution u_0, belonging to $W_2^1(\Omega_0, B)$. Besides, $u_0 \in C^\infty(\Omega_0^j)$, $u_0 \in C(\Omega_0)$, but $u_0 \notin C(\overline{\Omega}_0)$. This is because the domain Ω_0 has not the "segment" property (see [10]), and $u_0 \in W_2^1(\Omega_0, B)$ is a multifunction near the parts $M_0^\pm$ of the boundary $\partial\Omega_0$ (as (x,y) tends to a point $(0, y_*)$ on the boundaries $M_0^\pm$ from different sides of the domain Ω_0, the function u_0 has different limiting values).

The solution u_0 exhibits the asymptotic behaviour $(2.11)_2$ near point B with a constant c_B^0, but in the neighbourhood of the point A

$$u_0(h, x, y) = \pm d_0^\pm + O(r^{\tau_A}), \qquad r \to 0, \qquad 0 < \pm\phi < \pi - \phi_A \,. \tag{3.1}$$

Hence, u_0 cannot be considered as an approximation of u_h near the zero point.

3.2. Green's function

We shall also need the Green function $\mathcal{G}_A(x,y)$ for this problem in the domain Ω_0, with delta-functions concentrated near point A. It will be normalized by the relation (2.9). Asymptotic behaviour of the Green function near point B is of the form (2.11) (similar to u_h and u_0) with $d_B = 0$ and the constant $c = g_B$, but near the zero point

$$\mathcal{G}_A(h, x, y) = \pm \ln r \pm g_0^\pm + O(r^{\tau_A}), \qquad r \to 0, \qquad 0 < \pm\phi < \pi - \phi_A \,, \tag{3.2}$$

where $g_0^\pm$ are some constants.

Let us note that the Green function $\mathcal{G}_A$ is uniquely determined, and can be calculated using the representation

$$\mathcal{G}_A = \chi(r/h) \cdot \operatorname{sign}\phi \cdot \ln r + v_0 \,,$$

where the function $v_0 \in W_2^1(\Omega_0, B)$ satisfies Poisson equation with the right-hand side: $\operatorname{sign}\phi \cdot (\ln r \Delta\chi(r/h) + 2(rh)^{-1}\chi'(r/h))$ and the boundary conditions (2.5), (2.6) with functions $\hat{f}_j(s) = \dfrac{\partial}{\partial n}[\chi(r/h)\ln r]$ along the curves Γ_j. All these functions are smooth, and $\hat{f}_M^\pm(y) \equiv 0$, $\hat{f}_0^\pm(A)$, $\hat{f}_1(A) = 0$, in view of the assumption (2.1) for curvatures of the curves near point A. Hence, the problem for the function $v_0 \in W_2^1(\Omega_0, B)$ and the problem of the Subs. 3.1 for the function $u_0 \in W_2^1(\Omega_0, B)$ are similar from the point of view of their numerical realization.

3.3. Internal problem

Now let us consider the infinite domain $G = G_0^{\pm} \cup G_j$ represented in Fig. 2, and try to find nontrivial harmonic function $w(x', y')$ satisfying the homogeneous internal boundary conditions (2.6) along the boundaries $\zeta_{j+1} = \{(x', y') : y' = y_j = h_j/h, x' \in \mathbb{R}\}$ between the domains G_{j-1}, G_j $(j = 1..., m)$, and homogeneous conditions (2.8) along the boundaries $\zeta_{m+1}, \zeta_0^{\pm}$.

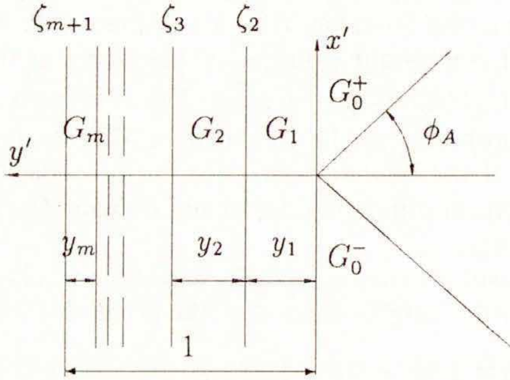

FIG. 2.

At infinity we assume, in addition, that $w = O(\ln r)$, $r \to \infty$. There are two linearly independent harmonic functions satisfying such conditions: $w_1(x', y') =$ const – even function with respect to argument x', and odd function $w_2(x', y')$. The function $w_2(x', y')$ can be calculated, using the inverse Fourier transform, by the nontrivial solution $z(\xi)$ of the singular integral equation obtained in [15] (the corresponding equation (3.16)). From theorem B.4 [15], it follows that $z \in W_{(l)}^{p,\alpha,\beta}(\mathbb{R}_+)$ for any $l \in \mathbb{N}$, $p \in [1, \infty)$, $\alpha > 0, \beta < \nu_A$, and

$$
\begin{aligned}
z(\xi) &= \ln \xi + z_0 + O(\xi^2), \qquad \xi \to 0, \\
z(\xi) &= z_\infty \xi^{-\nu_A} + O(\xi^{-\nu_A^{(2)}}), \qquad \xi \to \infty.
\end{aligned}
\tag{3.3}
$$

Here, $W_{(l)}^{p,\alpha,\beta}(\mathbb{R}_+)$ is the space of functions, which are summable (together with their l-derivatives) with a special weight (see [14]). The space $W_{(l)}^{p,\alpha,\beta}(\mathbb{R}_+)$ does not coincide with usual Sobolev spaces $W_p^l(\mathbb{R}_+)$. In turn, the method of numerical calculation of this nontrivial solution has been proposed in [13]. Finally, $w_2(x', y')$ can be determined (with accuracy to a multiplier) from the relation:

$$
w_2(x', y') = 2\int_0^\infty \left[\operatorname{ch} y'\xi + [\xi\mu_1 M_p(\xi)]^{-1} \operatorname{sh} y'\xi\right] z(\xi)\sin(x'\xi)d\xi, \tag{3.4}
$$

$$(x', y') \in G_1,$$

(3.4) [cont.]
$$w_2(x', y') = \frac{1}{\pi i} \int_{-i\infty-\delta}^{i\infty-\delta} \Gamma(s) \sin(\pi s/2) \frac{\cos(\pi - \phi_A - \phi)s}{\cos(\pi/2 - \phi_0)s} \cdot \int_0^\infty z(\xi)(r\xi)^{-s}\, d\xi\, ds, \qquad (x', y') \in G_0^+,$$

where $0 < \delta < \nu_A$, the function $M_p(\xi)$ can be calculated by recurrence formulae from [15], and besides, $M_p(\xi) = O(\xi^{-2})$, $\xi \to 0$, $M_p(\xi) = -(\mu_1 \xi)^{-1} + O(e^{-2\xi x_1})$, $\xi \to \infty$.

Using this information, we can show that the asymptotic behaviour of the function $w_2(x', y')$ near the zero point is of the form (2.11), with the constant $c_w = 2\pi^{-1} z_\infty \Gamma(1 - \nu_A) \sin(\pi \nu_A/2)$, $d_w = 0$, and $\nu_A^{(2)}$ instead of the parameter δ; but at infinity we obtain

$$(3.5) \qquad w_2(x', y') = \pm \begin{cases} \ln r + \gamma + z_0 , & (x', y') \in G_0 , \\ \ln |x'| + \gamma + z_0 , & (x', y') \in G_j , \end{cases} + O(r^{-2}), \qquad r \to \infty, \qquad \pm x' > 0,$$

where $\gamma = \Gamma'(1)$ is the Euler constant.

4. Main result

Using the method of matched asymptotic expansion (see [6, 19]), we shall consider function $w_2(s/h, n/h) + \text{const}$ as an approximation of the solution u_h in the neighbourhood of point A, but a linear combination of the functions $u_0(h, x, y)$, $\mathcal{G}_A(h, x, y)$ in the remaining part of domain Ω_h. Let $\alpha \in (0, 1)$ be some constant, and

$$(4.1) \qquad \tilde{u}_h^{(1)}(h, x, y) = (1 - \chi(r/h^\alpha)) [u_0(h, x, y) + D\mathcal{G}_A(h, x, y)] + \chi(r/h^\alpha)[Dw_2(s/h, n/h) + E].$$

Unknown constants D, E should be calculated in such a way that both parts (internal and external) of the solution (4.1) will coincide on the distance $r = h^\alpha/2$:

$$(4.2) \qquad \begin{aligned} u_0(h, x, y) + D\mathcal{G}_A(h, x, y) - Dw_2(s/h, n/h) - E &= O\left(h^{\min\{\tau_A \alpha, 2-2\alpha\}}\right), \\ \nabla[u_0(h, x, y) + D\mathcal{G}_A(h, x, y) - Dw_2(s/h, n/h) - E] &= O\left(h^{\min\{\tau_A \alpha, 2-2\alpha\}-\alpha}\right), \end{aligned}$$

for $h^\alpha/3 < r < 2h^\alpha/3$ uniformly with respect to the angular coordinate θ; then

$$(4.3) \qquad D = \frac{d_0^+ + d_0^-}{2(z_0 + \gamma - \ln h) - g_0^+ - g_0^-}, \qquad E = \frac{1}{2}[d_0^+ - d_0^- + D(g_0^+ - g_0^-)].$$

Let us note, that the function $\widetilde{u}_h^{(1)}$ from (4.1) belongs to the space $W_2^1(\Omega_h, B)$, and the constants in the main terms of asymptotics (2.11) near points A, B are:

$$\text{(4.4)} \qquad \tilde{c}_A = \frac{2}{\pi} z_\infty D h^{-\nu_A} \Gamma(1-\nu_A)\sin(\pi\,\nu_A/2), \qquad \tilde{c}_B = c_B^0 + D g_B\,.$$

THEOREM 1. *Let $\alpha \in (0,1)$ and $h \ll 1$, then for the function $\widetilde{u}_h^{(1)} \in W_2^1(\Omega_h, B)$, the following estimates hold true*:

$$\begin{aligned}
\|u_h - \widetilde{u}_h^{(1)}\|_{W_2^1} &= O\left(h^{\min\{\alpha(\tau_A-1),2-3\alpha\}}\right),\\
C - \widetilde{C} &= O\left(h^{\min\{\alpha(\tau_A-1),2-3\alpha\}}\right),\\
c_A - \tilde{c}_A &= O\left(h^{\min\{\alpha(\tau_A-\nu_A),2-\alpha(2+\nu_A)\}}\right),\\
c_B - \tilde{c}_B &= O\left(h^{\min\{\alpha(\tau_A+\nu_A),2-\alpha(2-\nu_A)\}}\right).
\end{aligned}$$

P r o o f. First of all note, that the difference between u_h and $\widetilde{u}_h^{(1)}$ in each domain Ω_h^j satisfies the Poisson equation with the right-hand side $\mathcal{R}^{(1)}(h,x,y)$:

$$\begin{aligned}
\mathcal{R}^{(1)}(h,x,y) &= \mathcal{R}_1^{(1)}(h,x,y) - \mathcal{R}_2^{(1)}(h,x,y),\\
\mathcal{R}_1^{(1)}(h,x,y) &= [u_0(h,x,y) + D\mathcal{G}_A(h,x,y) - Dw_2(s/h,n/h) - E]\Delta\chi(r/h^\alpha)\\
&\quad + 2\nabla[u_0(h,x,y) + D\mathcal{G}_A(h,x,y) - Dw_2(s/h,n/h) - E]\nabla\chi(r/h^\alpha),\\
\mathcal{R}_2^{(1)}(h,x,y) &= D\chi(r/h^\alpha)\Delta_{x,y} w_2(s/h,n/h),
\end{aligned}$$

and fulfills the boundary conditions (2.5), (2.6) with the functions

$$\begin{aligned}
\tilde{f}_j^{(1)} &= \chi(r/h^\alpha) f_j + (\mu_{j-1} - \mu_j)\Big[u_0 + D\mathcal{G}_A - Dw_2(s/h,n/h)\\
&\qquad - E\Big]\frac{\partial}{\partial n}\chi(r/h^\alpha),\\
\tilde{f}_0^{(1)} &= \chi(r/h^\alpha) f_0 - \mu_0\Big[u_0(x,y) + D\mathcal{G}_A(x,y) - Dw_2(s/h,n/h)\\
&\qquad - E\Big]\frac{\partial}{\partial n}\chi(r/h^\alpha),\\
\tilde{f}_m^{(1)} &= \chi(r/h^\alpha) f_m + \mu_m\Big[u_0(x,y) + D\mathcal{G}_A(x,y) - Dw_2(s/h,n/h)\\
&\qquad - E\Big]\frac{\partial}{\partial n}\chi(r/h^\alpha),
\end{aligned}$$

instead of f_j. Such a problem (for the function $u_h - \widetilde{u}_h^{(1)}$) has also a unique solution in the space $W_2^1(\Omega_h, B)$.

Taking into account (4.2), we can obtain for $h \to 0$

$$\mathcal{R}_1^{(1)}(h,x,y) = O(h^{\min\{\alpha(\tau_A-2),2-4\alpha\}}),$$
$$\operatorname{supp}\mathcal{R}_1^{(1)} = \{(x,y) \in \Omega_h : h^\alpha/3 < r < 2h^\alpha/3\},$$

but to estimate the function $\mathcal{R}_2^{(1)}(h,x,y)$ ($\operatorname{supp}\mathcal{R}_2^{(1)} = \{(x,y) \in \Omega_h : 0 < r < h^\alpha/3\}$), the Laplace operator should be considered in the curvilinear coordinate system (s,n):

$$\Delta_{x,y} w_2(s/h, n/h) = \frac{1}{1-nk(s)}\left[\frac{\partial}{\partial n}\left((1-nk(s))\frac{\partial w_2}{\partial n}\right) + \frac{\partial}{\partial s}\left(\frac{1}{1-nk(s)}\frac{\partial w_2}{\partial s}\right)\right].$$

Denoting $\xi = s/h, \eta = n/h$, we can conclude, in view of assumption (2.1) on the curves Γ_1, and taking into account the asymptotic formula (3.5) for the function w_2, that $\mathcal{R}_2^{(1)}(h,x,y) = \mathcal{R}_3^{(1)}(\xi,\eta) + O(h)$, where

$$\mathcal{R}_3^{(1)}(\xi,\eta) = O(\rho^{\nu_A}), \qquad \rho \to 0, \qquad \mathcal{R}_3^{(1)}(\xi,\eta) = O(1), \qquad \rho \to \infty.$$

The functions $\tilde{f}_j^{(1)}$ in the boundary conditions (2.5), (2.6) can be represented as a sum $\tilde{f}_j^{(1)} = \tilde{f}_{j1} + \tilde{f}_{j2}$, which at $h \to 0$ have the properties:

$$\tilde{f}_{j1} = O(h^\alpha), \qquad \operatorname{supp}\tilde{f}_{j1} = \{(x,y) \in \Omega_h : 0 < r < 2h^\alpha/3\},$$
$$\tilde{f}_{j2} = O(h^{\min\{2-3\alpha,\alpha(\tau_A-1)\}}), \qquad \operatorname{supp}\tilde{f}_{j2} = \{(x,y) \in \Omega_h : h^\alpha/3 < r < 2h^\alpha/3\}.$$

We can then conclude that

$$\|\mathcal{R}_1^{(1)}\|_{L_2(\Omega_h)} = O(h^{\min\{\alpha(\tau_A-1),2-3\alpha\}}), \qquad \|\mathcal{R}_2^{(1)}\|_{L_2(\Omega_h)} = O(h^\alpha),$$
$$\|\tilde{f}_{j1}^{(1)}\|_{L_2(\Gamma_j)} = O(h^{\min\{\alpha(\tau_A-1/2),2-5\alpha/2\}}), \qquad \|\tilde{f}_{j2}^{(1)}\|_{L_2(\Gamma_j)} = O(h^{3\alpha/2}).$$

Now, the first conclusion of Theorem 1 follows from the results [10]. However, the constant in the estimate ($\|u_h - \tilde{u}_h^{(1)}\| \leq \mathrm{Const}\, h^{\min\{\alpha(\tau_A-1),2-3\alpha\}}$) cannot be effectively obtained. It depends on the norm of the inverse operator of problem (2.5)–(2.9). The second relation follows immediately from the Hölder inequality.

For estimation of the constants c_A, c_B in the main terms of the asymptotics (2.11), we shall use the MAZ'YA, PLAMENEVSKY method [11]. Following [11] (see also [17]), we can define "non-energetic" harmonic function $\Psi_A^- \in L_2(\Omega_h)$ satisfying the homogeneous problem (2.5)–(2.9) with asymptotic behaviour (2.11) near point B, but in the neighbourhood of point A satisfying the condition

$$\Psi_A^-(x,y) = r^{-\nu_A}F(\phi) + O(r^{\nu_A}), \qquad r \to 0, \tag{4.5}$$

where function $F(\phi)$ is defined in (2.11). The function $\Psi_A^-(x, y)$ can be calculated from the representation ($\varepsilon < h_1$):

$$\Psi_A^-(x, y) = \chi(r/\varepsilon) r^{-\nu_A} F(\phi) + \Psi_\varepsilon^-(x, y), \qquad \Psi_\varepsilon^- \in W_2^1(\Omega_h, B),$$

because the corresponding problem for function Ψ_ε^- has a unique solution in the space $W_2^1(\Omega_h, B)$. Further on we define $\omega_\varepsilon = \{(x, y) : r < \varepsilon\}$ and write the Green formulae for the functions as $\hat{u}_h = u_h - \tilde{u}_h^{(1)}$ and Ψ_A^- in the domains of $\Omega_h^0 \setminus \omega_\varepsilon$, $\Omega_h^1 \setminus \omega_\varepsilon$, Ω_h^j ($j = 2, \ldots, m$). The sum of the corresponding relations is in the form of:

$$\iint\limits_{\Omega_h \setminus \omega_\varepsilon} \mu(x, y) \left[\Psi_A^- \Delta \hat{u}_h - \hat{u}_h \Delta \Psi_A^-\right] d\Omega = \int\limits_{\Gamma_{m+1}} \mu_m \left[\Psi_A^- \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi_A^-}{\partial n}\right] d\sigma$$
$$+ \sum_{j=1}^{m} \int\limits_{\Gamma_j \cap (\Omega_h \setminus \omega_\varepsilon)} \left\{\mu_j \left[\Psi_A^- \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi_A^-}{\partial n}\right] - \mu_{j-1} \left[\Psi_A^- \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi_A^-}{\partial n}\right]\right\} d\sigma$$
$$- \int\limits_{\Gamma_0 \cap (\Omega_h \setminus \omega_\varepsilon)} \mu_0 \left[\Psi_A^- \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi_A^-}{\partial n}\right] d\sigma - \int\limits_{\partial \omega_\varepsilon} \mu(x, y) \left[\Psi_A^- \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi_A^-}{\partial n}\right] d\sigma,$$

or taking into account the equations and the boundary conditions for functions $\hat{u}_h$ and Ψ_A^-, this relation can be rewritten as follows ($\varepsilon < h_1$):

$$\text{(4.6)} \qquad \int\limits_{\partial \omega_\varepsilon} \mu(x, y) \left[\Psi_A^- \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi_A^-}{\partial n}\right] d\sigma$$
$$= \sum_{j=2}^{m+1} \int\limits_{\Gamma_j} \Psi_A^- \bar{f}_j^{(1)} d\sigma + \int\limits_{\Gamma_1 \cap (\Omega_h \setminus \omega_\varepsilon)} \Psi_A^- \bar{f}_1^{(1)} d\sigma + \int\limits_{\Gamma_0 \cap (\Omega_h \setminus \omega_\varepsilon)} \Psi_A^- \bar{f}_0^{(1)} d\sigma$$
$$- \iint\limits_{\Omega_h \setminus \omega_\varepsilon} \mu(x, y) \Psi_A^- \left[\mathcal{R}_1^{(1)}(h, x, y) - \mathcal{R}_2^{(1)}(h, x, y)\right] d\Omega.$$

The net result will be obtained by passing to the limit $\varepsilon \to 0$:

$$\text{(4.7)} \qquad \Phi(\mu_0, \mu_1, \phi_A)(c_A - \tilde{c}_A)$$
$$= \sum_{j=0}^{m+1} \int\limits_{\Gamma_j} \Psi_A^- \left[\bar{f}_{j1}^{(1)} + \bar{f}_{j2}^{(1)}\right] d\sigma - \iint\limits_{\Omega_h} \mu \Psi_A^- \left[\mathcal{R}_1^{(1)} - \mathcal{R}_2^{(1)}\right] d\Omega.$$

Here we use information (2.11) and (4.5) about the asymptotic behaviour of the functions $\hat{u}_h$, Ψ_A^- near point A for calculating the integral on the left-hand side of (4.6):

$$(4.8)\qquad \Phi(\mu_0,\mu_1,\phi_A) = 2\left\{\mu_1\frac{\pi\nu_A - \sin\pi\nu_A}{1-\cos\pi\nu_A} + \mu_0\frac{(\pi-2\phi_A)\nu_A + \sin(\pi-2\phi_A)\nu_A}{1+\cos(\pi-2\phi_A)\nu_A}\right\}.$$

The first and the fourth terms on the right-hand side of (4.7) are estimated as $O(h^{\alpha(2-\nu_A)})$, but the remaining two terms are $O(h^{\min\{\alpha(\tau_A-\nu_A),2-\alpha(2+\nu_A)\}})$. Consequently, the third conclusion of Theorem 1 is proved. The remaining estimation of Theorem 1 is performed in a similar manner. For this purpose, we should take the "non-energetic" function Ψ_B^- (instead of Ψ_A^-), which exhibits the asymptotic behaviour (2.11) near the point A, but in the neighbourhood of point B in the form of (4.5) with ν_B. Then, repeating the same reasoning, we obtain the fourth conclusion of Theorem 1. Let us note that the constants in the last two estimates have been obtained effectively.

COROLLARY 1. The optimal value of the parameter α is $\alpha_* = 2/(2+\tau_A)$, then the estimates are:

$$\|u_h - \widetilde{u}_h^{(1)}\|_{W_2^1}, |C-\widetilde{C}| = O(h^{2-3\alpha_*}),$$

$$c_A - \tilde{c}_A = O(h^{2-\alpha_*(\tau_A+\nu_A)}), \qquad c_B - \tilde{c}_B = O(h^{2-\alpha_*(\tau_A-\nu_A)}).$$

REMARK 1. As it follows from the proof of Theorem 1, the results would be improved, if we could more precisely estimate the terms of solution u_0 and the Green function $\mathcal{G}_A$ of the asymptotic behaviour: $O(r^{\tau_A})$, $r \to 0$. For this purpose, note that the corresponding problem for function u_0 is the perturbation boundary value problem with the regular boundary layer near $\Gamma_{m+1} = \Gamma_e$. The main terms of such problems have been constructed in [4]. Basing on the results from [4], one can show that the term $O(r^{\tau_A})$ in (3.1) can be estimated as: $\text{const}(h)F_*(\phi)r^{\tau_A}$, where $\text{const}(h) = O(h^\beta)$ with some $0 < \beta \leq \tau_A^0 - \tau_A$. Here, τ_A^0 is the corresponding parameter in (3.1) for the solution $u_0^0(x,y)$ of the nonperturbed problem ($\mu(x,y) = \mu_0$, $h = 0$). In a similar manner, the estimation of the corresponding term of the Green function (3.2) can be obtained. Then we can formulate

THEOREM 2. *Let* $\alpha \in (0,1)$ *and* $h \ll 1$, *then for function* $\widetilde{u}_h^{(1)} \in W_2^1(\Omega_h, B)$ *estimates hold true*:

$$\begin{aligned}
\|u_h - \widetilde{u}_h^{(1)}\|_{W_2^1} &= O(h^{\min\{\alpha,\beta+\alpha(\tau_A-1),2-3\alpha\}}),\\
C - \widetilde{C} &= O(h^{\min\{\alpha,\beta+\alpha(\tau_A-1),2-3\alpha\}}),\\
c_A - \tilde{c}_A &= O(h^{\min\{\alpha(2-\nu_A),\beta+\alpha(\tau_A-\nu_A),2-\alpha(2+\nu_A)\}}),\\
c_B - \tilde{c}_B &= O(h^{\min\{\alpha(2+\nu_A),\beta+\alpha(\tau_A+\nu_A),2-\alpha(2-\nu_A)\}}).
\end{aligned}$$

COROLLARY 2. Then the optimal value of the parameter α in Corollary 1 is

$$\alpha_* = \max\{1/2, (2-\beta)/(2+\tau_A)\}.$$

5. Remarks and conclusions

In this section we propose some generalizations under which the mentioned results of the theorems will hold true.

First of all note that from [18] it follows that $f_j = (\mu_{j-1} - \mu_j)[y\cos(n,x) - x\cos(n,y)]$, $f_{m+1} = \mu_m[y\cos(n,x) - x\cos(n,y)]$, $f_0 = \mu_0[y\cos(n,x) - x\cos(n,y)]$. Consequently, these functions satisfy the conditions (2.7). Nevertheless, the results still remain valid, if the functions are "little affected" in the neighbourhood of point A. For this purpose, it is sufficient to find the solution in the form: $u_h = \bar{u}_h + \chi(r/h)v_1(s,n)$, where the function is $v_1 = a_j + b_j s + c_j n$ in each region Ω_h^j. The constants a_j, b_j, c_j should be calculated so that v_1 is continuous along Γ_j, but for function $\bar{u}_h$ the conditions (2.7) have been satisfied.

Further on, note that the conditions (iii) in (2.1) can be weakened like this: $k_{\Gamma_1}(A)$, $k_{\Gamma_0^\pm}(A)$, $k_{\Gamma_0^\pm}(B) \sim 1$. The angle of corner A can be nonsymmetric with respect to the normal to the boundary Γ_0 at this point, in contrast to (ii). Then the functions $F(\phi)$ in (2.11) and the transcendental functions $\Delta_{s(c)}(s)$ (necessary to determine the parameters τ, ν) should be corrected; but the corresponding internal boundary value problems can be calculated by solving of the systems of singular integral equations [15], instead of the singular integral equations as it is in the symmetric cases.

The step function $\mu(x,y)$ allows for the following generalization:

1. The boundaries of discontinuity Γ_j of function $\mu(x,y)$ can be defined as in (2.2) with functions $h_j(s)$ instead of parameters h_j. We should assume only that: $h_j(s) > h_{j-1}(s)$, $h_m(s) = O(h)$, $h_j'(0) = 0$, $h_j''(0) \sim 1$.

2. In each domain Ω_h^j the conditions are true: $\mu \in C^2(\Omega_h^j)$, and $\dfrac{\partial}{\partial x}\mu(0,y) = 0$, $\dfrac{\partial^2}{\partial x^2}\mu(0,y) \sim 1$, $(0,y) \in \Omega_h^j$ $(j > 0)$, $\dfrac{\partial}{\partial r}\mu(0,\theta) = 0$, $\dfrac{\partial^2}{\partial r^2}\mu(0,\theta) \sim 1$, $(0,\theta) \in \Omega_h^0)$. The function $\mu(x,y)$ depends weakly on the argument x in the multilayer near the angle vertex. Then we shall find solution u_h of equation $\nabla(\mu(x,y)\nabla u_h) = 0$ instead of the Laplace equation $\Delta u_h = 0$ used in the paper. Such a problem corresponds to the general case of a nonhomogeneous elastic rod. Note in this connection that the internal boundary value problems (Sec. (3.2)) can be also solved in this case by the method [15] (see Appendix in [16]).

The boundary conditions can be also generalized. Namely, the first of the conditions (2.5) can be represented in the form: $[u_h] - a(s)\dfrac{\partial}{\partial n}u_h = \bar{f}_j$, $a'(0) = 0$, $a''(0) \sim 1$, instead of $[u_h] = 0$. The corresponding internal boundary value problems can be solved by the same method [15, 16].

Let us note in conclusion, that the first two conditions (i) cannot be modified, of course (these conditions make it possible to use the asymptotic methods). If the third condition is not true and $\mathrm{dist}(B, \Gamma_e) = O(h)$, then the asymptotic expansion of the solution can also be constructed. However, the corresponding external boundary value problems are different from those shown in the paper (Sec. (3.1)), and the representation of the solution (4.1) should be changed. In [19], such a problem in a homogeneous domain with the linear crack has been considered.

Acknowledgment

The author is grateful to Professor Z.S. OLESIAK for a helpful discussion.

References

1. O.B. AGALARYAN and S.A. NAZAROV, *On change of stress intensity factors in soldering of crack in prismatic rod*, Dokl. Akad. Nauk Arm. SSR, **72**, 1, 18–21, 1981.
2. V.M. ALEKSANDROV and S.M. MHITARYAN, *Contact problems for bodies with thin covers and layers* [in Russian], Nauka, Moscow 1983.
3. V.M. ALEKSANDROV, B.I. SMETANIN and B.V. SOBOL, *Thin stress concentrators in elastic bodies* [in Russian], Nauka, Moscow 1993.
4. H.N. ARUTYUNYAN and B.L. ABRAMYAN, *Torsion of elastic bodies* [in Russian], Nauka, Moscow 1968.
5. D. GILBARG and N.S. TRUDINGER, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin – New-York – Tokyo 1983.
6. A.M. ILIN, *On the method of matched asymptotic expansions of boundary value problems* [in Russian], Nauka, Moscow 1989.
7. L.A. IVANOV, L.A. KOTKO and S.G. KREIN, *Boundary value problems in varying domains* [in Russian], [in:] Differential Equations and Their Applications, **19**, 6–161, 1977.
8. T. KATO, *Theory of perturbed linear operators* [in Russian], Mir, Moscow 1972.
9. Yu.M. KOLYANO and M.E. CHOMIAKIEWICZ, *Generalized heat conduction in solids with thin surface layer of an arbitrary curvature* [in Russian], J. Engng. Physics and Thermophysics, Minsk, **65**, 6, 745–749, 1993.
10. O.A. LADYZHENSKAYA, *The boundary-value problems of mathematical physics*, Applied Mathematical Science, Vol. **49**, Springer-Verlag, 1985.
11. V.G. MAZ'YA and B.A. PLAMENEVSKY, *On coefficients in asymptotics of elliptic boundary value problems in domains with conical points* [in Russian], Mathematische Nachrichten, **76**, 29–60, 1977.
12. V.G. MAZ'YA, S.A. NAZAROV and B.A. PLAMENEVSKY, *Asymptotics of solutions of singular perturbed elliptic boundary value problems* [in Russian], Tbilisi 1981.
13. G. MISHURIS, *On nontrivial solution of one singular integral equation*, Folia Sci. Univ. Tech. Resoviensis, Math., **16**, 113–130, 1994.
14. G.S. MISHURIS and Z.S. OLESIAK, *On boundary value problems in fracture of elastic composites*, Euro. J. Appl. Math., **6**, 591–610, 1995.
15. G. MISHURIS, *Boundary value problems for Poisson's equations in multi-wedge – multi-layered region*, Arch. Mech., **47**, 2, 295–335, 1995.
16. G. MISHURIS, *Boundary value problems for Poisson's equations in multi-wedge – multi-layered region. Part II. General type of interfacial conditions*, Arch. Mech., **48**, 4, 711–745, 1996.
17. N.F. MOROZOV, *Mathematical problems of the crack theory* [in Russian], Nauka, Moscow 1984.

18. N.I. MUSCHELISHVILI, *Mathematical problems of theory of elasticity* [in Russian], Nauka, Moscow 1966.

19. S.A. NAZAROV, *Introduction to asymptotic methods of the theory of elasticity* [in Russian], Edition of Leningrad State University, 1983.

20. G.Ya. POPOV, *Stress concentration near stamps, cracks, thin inclusions and covers* [in Russian], Nauka, Moscow 1982.

DEPARTMENT OF MATHEMATICS,
RZESZÓW UNIVERSITY OF TECHNOLOGY
e-mail: miszuris@prz.rzeszow.pl

Received May 20, 1996.