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Subsonic flutter calculation of an aircraft with nonlinear control system based on center-manifold reduction

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THE PAPER PRESENTS a method of calculation of limit cycle subsonic flutter oscillations caused by structural nonlinearities. Numerical examples assume the nonlinearities to be concentrated in the hinges of the aircraft control surfaces. Since nonlinear flutter is essentially the Hopf bifurcation, these oscillations tend asymptotically to a certain two-dimensional attracting subspace called the center manifold. Consequently, an asymptotic motion of the entire aircraft in the neighbourhood of bifurcation point is fully described by only two equations. The method of center-manifold reduction consists in a nonlinear change of coordinates, and transforms the initial multi-dimensional nonlinear integro-differential flutter equation into a system of two nonlinear ordinary differential equations of the first order, having phase-shift symmetry. Under the assumption that the nonlinear term has a formal power series expansion with respect to generalized coordinates (multi-variable Taylor series), the transformation can be also expressed in the form of a power series, and the limit cycle amplitude and frequency can be easily calculated.

1. Introduction

DEFORMATIONS of an aircraft structure under aerodynamic loads during flight are responsible for occurrence of self-excited oscillations, called flutter. These often destructive oscillations are driven by the transfer of energy from the airstream to the aircraft structure. The most widely used linear flutter analysis is focused on the particular critical value of flight velocity, above which the steady motion of an aircraft becomes unstable. All velocities below this point are considered to be safe in the sense that any imposed disturbances decay asymptotically in time, regardless of their initial magnitude. This is no more true if either the flow or the structure characteristics are nonlinear. It is known that in a nonlinear case, sufficiently high initial disturbance (e.g. a gust) can trigger self-excited oscillations even below the critical flutter velocity. Since the flutter phenomenon must be completely prevented from occurring within the flight envelope, nonlinear flutter analysis is also of great practical importance.

In the unsteady subsonic motion, the aerodynamic forces depend on the history of motion as a result of shedding of the vortex wake behind an aircraft. Consequently, the aerodynamic operator, relating the unsteady aerodynamic forces to the deflection of an aircraft structure (generalized coordinates), is always of the form of the convolution integral. Thus, in a time domain, the flutter equation is an integro-differential equation (sometimes with infinite delay). This property is the main source of difficulties in nonlinear approach, contrary to other aeroelastic systems described by ordinary differential equations (e.g. supersonic or panel flutter).

It is well known from the theory of dynamical systems [1] that their qualitative behaviour is essentially the same, no matter what physical background they originate from. Therefore, if the steady solution, such as a horizontal flight of an aircraft, bifurcates into the finite amplitude oscillations then the limit cycle attractor appears in the phase space of the system and the Hopf bifurcation takes place. Since the point of interest is an asymptotic motion of an aircraft, it is sufficient to determine only the limit cycle amplitude and frequency for a given velocity in the neighbourhood of the bifurcation point. In the paper, methods of the local bifurcation theory are applied thus restricting the validity of analysis to some finite interval of velocity.

Hopf bifurcation is two-dimensional what means that limit cycle oscillations are described by only two generalized coordinates, no matter how many degrees of freedom are used in order to describe the original aeroelastic system. A two-dimensional subspace containing these asymptotic oscillations is called the center manifold. Thus, as far as an asymptotic analysis is concerned, it is possible to obtain the limit cycle for an entire aircraft from only two differential equations. Calculation procedure for an aeroelastic system of N degrees of freedom goes through the following steps [2]:

- Replacement of the initial N flutter equations of the second order by a system of $2N$ nonlinear integro-differential equations of the first order (all the methods of the bifurcation theory apply to the first order equations).

- Determination of the bifurcation point (critical flutter velocity) by solving the completely linearized flutter equation.

- Unfolding of the aeroelastic system by expanding all functions into power series with respect to velocity U, and also considering the velocity being temporarily an additional variable – this increases the total number of equations by one, and is done in order to work on an interval in velocity space in the vicinity of a bifurcation point.

- Projection of the aeroelastic system onto the appropriate center manifold by means of nonlinear transformation of variables, which transforms the initial $(2N+1)$-dimensional system of integro-differential equations into a two-dimensional system of ordinary differential equations of the first order.

- Normalization of the reduced system by applying the so-called near-identity change of coordinates, resulting in a much simpler system of equations with rotational symmetry.

- Calculation of the limit cycle amplitude and frequency for a given flight velocity – if all nonlinear terms are expanded into multi-variable Taylor series, then the limit cycle parameters are determined by roots of certain polynomials with real coefficients.

It is worth noting here that projection onto the center manifold preserves all information about asymptotic behaviour of the complete initial system and does not introduce any simplifying assumptions. Numerical algorithm for the above scheme worked out for systems with many degrees of freedom is given in Ref. [3], and Ref. [4] presents the full nonlinear analysis for a single two-dimensional airfoil.

2. Flutter equation

Displacements of an aircraft during unsteady motion are described by the M-dimensional vector of physical coordinates $\mathbf{u}(t)$ being functions of time t. In the steady motion with undeflected structure all coordinates are equal to zero, $\mathbf{u}(t) = 0$. Usually, for a conventional aircraft structure, the number M cannot be less than a few hundreds. This is too many even for the classical (linear) flutter analysis. The routine procedure saves much of the computing time by using modal coordinates in order to reduce the total number of equations. Such an approach assumes the vector of physical coordinates $\mathbf{u}(t)$ as a linear combination of natural vibration modes with coefficients forming new generalized coordinates. It is sufficient for the flutter analysis to set the number of modal coordinates to nearly twenty. Modal coordinates can also be used in nonlinear approach without any changes [5]. It means that no attempt is made to generalize the natural modes for nonlinear structures but the same linear modes are applied.

In the absence of external aerodynamic forces and under the assumption that the problem has been fully linearized, the natural frequencies ω_j and modes $\mathbf{\Phi}_j$ $(j = 1, 2, ..., N; j \leq M)$ can be calculated from the eigenvalue problem:

$$\omega_j^2 \mathbf{M}\,\mathbf{\Phi}_j = \mathbf{K}\,\mathbf{\Phi}_j\,, \tag{2.1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are mass and stiffness matrices, respectively. The set of eigenfunctions of Eq. (2.1) is assumed to describe nonlinear limit cycle oscillations with sufficient accuracy. The vector $\mathbf{q}(t)$ of modal coordinates is defined by the relation

$$\mathbf{u}(t) = \mathbf{\Phi}\,\mathbf{q}(t), \tag{2.2}$$

and in the absence of the structural damping forces, satisfies the equation of motion [6]:

$$\ddot{\mathbf{q}}\,(t) + \mathbf{K}_\omega \mathbf{q}(t) + \mathbf{k}(\mathbf{q}) = \mathbf{F}_A(\mathbf{q}), \tag{2.3}$$

where $\mathbf{F}_A(\mathbf{q})$ is the vector of generalized unsteady aerodynamic forces. The matrix $\mathbf{\Phi}$ is built out of eigenvectors of Eq. (2.1). The diagonal generalized stiffness matrix $\mathbf{K}_\omega$ is composed of squares of the natural frequencies ω_j^2 $(j = 1, 2, ..., N)$.

Although the source of the nonlinear term $\mathbf{k}(\mathbf{q})$ can be either aerodynamics or the aircraft structure, it is assumed here that only the structure is nonlinear.

At present, the only general method of describing the center manifold is based on multi-variable Taylor series [7]. In what follows, it is also assumed that the nonlinear term $\mathbf{k}(\mathbf{q})$ is of the form of a power series of nonlinear coordinates $\mathbf{q}$. For structural nonlinearities such an expansion can be easily obtained. Let $\mathbf{f}_\delta$ be a m-dimensional vector of nonlinear forces corresponding to the vector of displacements δ in a finite number of structure points:

$$\mathbf{f}_\delta = \sum_{j\geq 2} \mathbf{K}_j \delta^j, \tag{2.4}$$

where $\mathbf{K}_j$ are diagonal matrices of known numbers, and the symbol δ^j means that each vector component is raised to the power of j separately. In practical calculations, the number of terms of Eq. (2.4) remains finite. In particular, the vector $\mathbf{f}_\delta$ can include nonlinear springs present in the structure and modeling an aircraft control system. On the other hand, Eq. (2.4) can also describe the properly discretized distributed nonlinearities.

For a given structure it is always possible to find a rectangular matrix $\mathbf{R}$ of order $m \times M$ relating the m-dimensional displacement vector δ to the M physical coordinates $\mathbf{u}$:

$$\delta = \mathbf{R}\,\mathbf{u}\,. \tag{2.5}$$

After using Eqs. (2.4) and (2.5), the vector $\mathbf{k}(\mathbf{q})$ of nonlinear generalized forces can be written as:

$$\mathbf{k}(\mathbf{q}) = (\mathbf{R}\,\mathbf{\Phi})^T \mathbf{f}_\delta = (\mathbf{R}\,\mathbf{\Phi})^T \sum_{j\geq 2} \mathbf{K}_j (\mathbf{R}\,\mathbf{\Phi}\,\mathbf{q})^j. \tag{2.6}$$

The aim is to find the critical flutter speed for the Eq. (2.3) and also the limit cycle amplitude and frequency in the neighbourhood of the critical point.

Since the aeroelastic system is nonlinear, it is not possible to assume any given form of the motion during the limit cycle oscillations. Therefore, unsteady aerodynamic forces must be written in a general form valid for an arbitrary motion:

$$\mathbf{F}_A(\mathbf{q}) = \frac{\varrho U^2}{2} \int_{-\infty}^{0} \mathbf{g}(-\tau)\mathbf{q}\left(t + \frac{b}{U}\tau\right) d\tau, \tag{2.7}$$

where U and ϱ denote the flow velocity and density, respectively, and b stands for the characteristic length. Elements of the matrix $\mathbf{g}$ are response functions corresponding to the impulsive changes of generalized coordinates $\mathbf{q}$. Finally, the equation of motion (2.3) takes the form of an integro-differential equation containing an integral of convolution type.

The classical linearized flutter analysis assumes oscillatory motion of an aircraft:

$$\mathbf{q}(t) = \widehat{\mathbf{q}} e^{st}, \tag{2.8}$$

where the complex coefficient

$$s = \gamma + i\omega \tag{2.9}$$

includes circular frequency ω and damping factor γ. For such a motion the vector of unsteady aerodynamic forces is given by the simple linear relation

$$\mathbf{F}_A(\mathbf{q}) = \mathbf{A}(s;U)\widehat{\mathbf{q}}e^{st}, \tag{2.10}$$

where

$$\mathbf{A}(s;U) = \frac{\varrho U^2}{2}\int_0^\infty \mathbf{g}(\tau)e^{-\frac{sb}{U}\tau}\,d\tau \tag{2.11}$$

is called the aerodynamic matrix. The only case for which it is possible to calculate the aerodynamic matrix analytically (in terms of Bessel functions) is a thin airfoil in an incompressible flow [8]. More complex aerodynamic models rely entirely on numerical methods. There are many of them in the literature (a list of the most important ones can be found in [9]), all suited for direct calculation of the aerodynamic matrix, mostly for pure harmonic motion ($\gamma = 0$), without evaluating the response matrix $\mathbf{g}$. Although the present method does not assume a harmonic motion, it does not require the knowledge of the response matrix either.

Local bifurcation theory of dynamical systems [7] has been developed for the first-order equations. By introducing a $2N$-dimensional vector of new coordinates $\mathbf{y}(t)$:

$$\mathbf{y}(t) = \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{Bmatrix}, \tag{2.12}$$

the first-order flutter equation is obtained:

$$\dot{\mathbf{y}}(t) = \mathbf{D}_U\,\mathbf{y}(t) + \int_{-\infty}^{0} \mathbf{G}_U(-\Theta;U)\mathbf{y}(t+\Theta)d\Theta + \mathbf{f}_U(\mathbf{y}), \tag{2.13}$$

where square matrices of order $2N$, $\mathbf{D}_U$, $\mathbf{G}_U$, and the nonlinear term $\mathbf{f}_U(\mathbf{y})$ are given by:

$$\mathbf{D}_U = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_\omega & \mathbf{0} \end{bmatrix}, \qquad \mathbf{G}_U(-\Theta;U) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \dfrac{\varrho U^3}{2b}\mathbf{g}\Big(-\dfrac{U}{b}\Theta\Big) & \mathbf{0} \end{bmatrix}, \qquad \mathbf{f}_U(\mathbf{y}) = \begin{Bmatrix} \mathbf{0} \\ -\mathbf{k}(\mathbf{q}) \end{Bmatrix},$$

with $\mathbf{k}(\mathbf{q})$ given by Eq. (2.6). For oscillatory motion (2.8), the linearized flutter equation reduces to the eigenvalue problem

$$\Big(\mathbf{A}(s;U) - \mathbf{K}_\omega\Big)\widehat{\mathbf{q}} = s^2\widehat{\mathbf{q}}. \tag{2.14}$$

Loss of stability occurs when damping drops to zero ($\gamma = 0$ in Eq. (2.9)) and the flutter boundary is determined by the real negative eigenvalue of Eq. (2.14)

$$s^2 = -\omega_0^2, \tag{2.15}$$

corresponding to the critical flutter velocity $U = U_0$.

The critical bifurcation point of the first order equation (2.13) is defined by the eigenvalues of its linear part corresponding to $\mathbf{f}_U(\mathbf{y}) = 0$. It can be shown [7, 10] that also in the presence of convolution integral within the linear part, the eigenfunctions have the form

$$\mathbf{y}(t) = \widehat{\mathbf{y}} e^{st},$$

where s is given by (2.9). The resulting eigenvalue problem is:

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{A}(s;U) - \mathbf{K}_\omega & \mathbf{0} \end{bmatrix} \widehat{\mathbf{y}} = s\widehat{\mathbf{y}}. \tag{2.16}$$

It follows from comparison with (2.14) that at the flutter boundary, the characteristic matrix of linearized first-order flutter equation has a pair of complex-conjugate, pure imaginary eigenvalues $s = \pm i\omega_0$.

The eigenvalue problem (2.16) of the linearized flutter analysis can be derived in a more formal way by applying the Laplace transform, which replaces the convolution integral in Eq. (2.13) by the product of two functions. In nonlinear approach there are two possible ways: either the application of Laplace transform in frequency domain or solution of the problem in time domain. The first method is suitable for handling convolution integrals but faces more difficulties due to nonlinear terms. On the other hand, working in time domain shifts the whole problem to proper treatment of the convolution integral. The present paper uses the time-domain method.

The qualitative changes in a behaviour of the nonlinear dynamical system are always indicated by the purely imaginary (or zero) eigenvalues of the linearized operator of the governing equation. For the nonlinear flutter equation (2.13) this operator is of the form:

$$\mathcal{L}\mathbf{y}(t) = \mathbf{D}_U\,\mathbf{y}(t) + \int_{-\infty}^{0} \mathbf{G}_U(-\Theta;U)\mathbf{y}(t+\Theta)\,d\Theta.$$

Since the operator $\mathcal{L}$ maps a space of continuous functions onto the Euclidean space, then the eigenvalue problem $\mathcal{L}\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi}$ cannot be posed directly. Instead, an extension of $\mathcal{L}$ is made in order to map a space of continuous functions onto itself. An extended operator is the following [10, 11]:

$$(2.17)\qquad \mathcal{L}_U \boldsymbol{\varphi}(\Theta) = \begin{cases} \dfrac{d\boldsymbol{\varphi}(\Theta)}{d\Theta}, & \text{for } \; -\infty < \Theta < 0, \\ \mathbf{D}_U\, \boldsymbol{\varphi}(0) + \displaystyle\int_{-\infty}^{0} \mathbf{G}_U(-\tau; U)\boldsymbol{\varphi}(\tau)d\tau, & \text{for } \; \Theta = 0, \end{cases}$$

and the flutter equation takes the form:

$$(2.18)\qquad \frac{d\mathbf{y}_t(\Theta)}{d\Theta} = \mathcal{L}_U \mathbf{y}_t(\Theta) + \begin{cases} 0, & \text{for } \; -\infty < \Theta < 0, \\ \mathbf{f}(\mathbf{y}_t(0)), & \text{for } \; \Theta = 0, \end{cases}$$

where the following notation has been introduced:

$$\mathbf{y}_t(\Theta) = \mathbf{y}(t + \Theta).$$

Now, the eigenvalue problem $\mathcal{L}_U \boldsymbol{\varphi} = \lambda \boldsymbol{\varphi}$ can be formulated. First, the form of the eigenfunction is determined $(-\infty < \Theta < 0)$:

$$\frac{d\boldsymbol{\varphi}(\Theta)}{d\Theta} = \lambda \boldsymbol{\varphi}(\Theta) \quad \Rightarrow \quad \boldsymbol{\varphi}(\Theta) = \boldsymbol{\varphi}(0)e^{\lambda t},$$

and next the eigenproblem for the Euclidean vector $\boldsymbol{\varphi}(0)$ is posed

$$(2.19)\qquad \mathbf{D}_U \boldsymbol{\varphi}(0) + \left(\int_{-\infty}^{0} \mathbf{G}_U(-\tau; U)e^{\lambda\tau} d\tau \right) \boldsymbol{\varphi}(0) = \lambda \boldsymbol{\varphi}(0).$$

As can be seen, both eigenvalue problems (2.16) and (2.19) are identical. Therefore, since at criticality there is a pair of pure imaginary eigenvalues, flutter instability is the Hopf bifurcation [10].

3. Center-manifold reduction

If any bifurcation occurs in a dynamical system, then the phase space splits in general into three manifolds: stable – generated by eigenvalues with $\mathrm{Re}(\lambda) < 0$, unstable – generated by eigenvalues with $\mathrm{Re}(\lambda) > 0$, and center manifold, corresponding to $\mathrm{Re}(\lambda) = 0$ [12]. Center manifold is invariant, locally attracting and asymptotically stable. Moreover, it is of finite dimensions – for the Hopf bifurcation it is two-dimensional. It means that in the space of all solutions to Eq. (2.18), bifurcating solution tends asymptotically to a two-dimensional attracting subspace. The asymptotic solution (limit cycle oscillations) satisfies a certain system of two nonlinear ordinary differential equations of the first order, which can be derived from the integro-differential equation (2.18), written for many degrees of freedom. This procedure of obtaining a low-dimensional system of equations from the initial multi-dimensional system is called center-manifold reduction.

There are two problems associated with the center-manifold reduction. Since the aim is to calculate asymptotic limit cycle oscillations for a general form of the nonlinear term $\mathbf{f}_U(\mathbf{y})$, this term is assumed to have a formal power series expansion with respect to generalized coordinates $\mathbf{y}$. Consequently, the method of center-manifold reduction is also based on such expansions. The second problem concerns the way the velocity U should be treated in. The critical flutter conditions correspond to a certain critical value of the velocity $U = U_0$, which in turn determines the existence of purely imaginary eigenvalues of Eq. (2.19) and the center manifold, as well. At this critical branch point the amplitude of oscillations tends to zero and, in order to obtain the finite amplitude limit cycle oscillations, the value of velocity must be different from the critical one. Unfortunately, if $U \neq U_0$, the characteristic matrix of Eq. (2.19) no longer possesses pure imaginary eigenvalues and the center manifold simply does not exist. On the other hand, the existence of the center manifold has been proven in a certain neighbourhood of equilibrium solution $\mathbf{y}_0(t)$, corresponding to $U = U_0$, in the space of solutions $\mathbf{y}(t)$ [10]. For that reason, the center-manifold reduction usually applies to the so-called suspended systems. Suspended aeroelastic system is derived from Eq. (2.13) by introducing the difference

$$u = U - U_0 \tag{3.1}$$

as an additional variable satisfying the equation $\dot{u} = 0$. The $2N+1$-dimensional vector of new generalized coordinates is the following:

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \\ u \end{Bmatrix}, \tag{3.2}$$

and satisfies the equation

$$\dot{\mathbf{x}}(t) = \mathbf{D}\,\mathbf{x}(t) + \int\limits_{-\infty}^{0} \mathbf{G}(-\Theta; u)\mathbf{x}(t+\Theta)\,d\Theta + \mathbf{f}(\mathbf{x}), \tag{3.3}$$

where square matrices of order $2N+1$ $\mathbf{D}$, $\mathbf{G}$, and the nonlinear term $\mathbf{f}(\mathbf{x})$ are given by

$$\mathbf{D} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -\mathbf{K}_\omega & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix}, \qquad \mathbf{G}(-\Theta; u) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \dfrac{\varrho(U_0+u)^3}{2b}\mathbf{g}\left(-\dfrac{U_0+u}{b}\Theta\right) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix},$$

$$\mathbf{f}(\mathbf{x}) = \begin{Bmatrix} \mathbf{0} \\ -\mathbf{k}(\mathbf{q}) \\ 0 \end{Bmatrix}.$$

Since the matrix $\mathbf{G}(-\Theta; u)$ now includes the independent variable u instead of the bifurcation parameter U, the integral in Eq. (3.3) is no longer linear with respect to $\mathbf{x}$. In what follows, the matrix $\mathbf{G}$ is replaced by the Taylor series

$$\mathbf{G}(-\Theta; u) = \mathbf{G}(-\Theta; 0) + \sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^j \mathbf{G}(-\Theta; 0)}{du^j} u^j. \tag{3.4}$$

It is also assumed that the multi-variable power series expansion for the non-linear function $\mathbf{f}(\mathbf{x})$ at the right-hand side of Eq. (3.3) is known:

$$\mathbf{f}(\mathbf{x}) = \sum_{\nu \geq 2} \frac{1}{\nu!} \mathbf{f}_\nu \mathbf{x}^\nu, \tag{3.5}$$

where

$$\mathbf{x}^\nu = \left\{ x_1^{\nu_1} \cdot x_2^{\nu_2} \cdots x_{2N+1}^{\nu_{2N+1}} \right\}, \qquad \sum_{j=1}^{2N+1} \nu_j = \nu, \qquad \nu_j \geq 0.$$

The number of components of the vector $\mathbf{x}^\nu$ and also the number of columns of each matrix $\mathbf{f}_\nu$ changes from one term to another and equals the number $c_{\nu,2N+1}$ of compositions of ν into $2N+1$ parts

$$c_{\nu,2N+1} = \binom{\nu + 2N}{\nu - 1}. \tag{3.6}$$

The elements of matrices $\mathbf{f}_\nu$ can be easily calculated from Eq. (2.6). Substitution of series (3.4) into Eq. (3.3) yields the integro-differential equation valid in a certain neighbourhood of the critical bifurcation point:

$$\dot{\mathbf{x}}(t) = \mathbf{D}\,\mathbf{x}(t) + \int_{-\infty}^{0} \mathbf{G}(-\Theta; 0)\mathbf{x}(t + \Theta)\, d\Theta + \mathbf{h}(\mathbf{x}), \tag{3.7}$$

where $\mathbf{h}(\mathbf{x})$ equals

$$\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \sum_{\eta \geq 2} \frac{1}{(\eta - 1)!} \int_{-\infty}^{0} \frac{d^{\eta-1}\mathbf{G}(-\Theta; 0)}{du^{\eta-1}} \mathbf{x}^\eta(t + \Theta)\, d\Theta, \tag{3.8}$$

with $\mathbf{x}^{\eta} = \left\{x_1^{\eta_1} \cdot x_2^{\eta_2} \cdots x_{2N+1}^{\eta_{2N+1}}\right\}$, and always $\eta = \eta_{2N+1} + 1$ ($x_{2N+1} \equiv u$) which implies that $\sum_{j=1}^{2N} \eta_j = 1$. Equation (3.7) will be reduced on the center manifold.

The linear spectrum of Eq. (3.7) includes one eigenvalue with zero real part more than the previous spectrum of the non-suspended system (2.13). Hence the center manifold corresponding to Eq. (3.7) is larger than that of Eq. (2.13) and has the dimension of three.

Since the center manifold is tangent to the linear subspace spanned by eigenvectors $\boldsymbol{\varphi}$, corresponding to the bifurcating eigenvalues of the extended linear operator $\mathcal{L}_0$ derived from Eq. (3.7)

$$\mathcal{L}_0 \boldsymbol{\varphi}(\Theta) = \begin{cases} \dfrac{d\boldsymbol{\varphi}(\Theta)}{d\Theta}, & \text{for} \quad -\infty < \Theta < 0, \\ \mathbf{D}\,\boldsymbol{\varphi}(0) + \displaystyle\int_{-\infty}^{0} \mathbf{G}(-\tau; 0)\boldsymbol{\varphi}(\tau)\, d\tau, & \text{for} \quad \Theta = 0, \end{cases} \tag{3.9}$$

then it is convenient to introduce the three-dimensional vector $\mathbf{z}(t)$ of center-manifold coordinates as follows:

$$\mathbf{x}_t(\Theta) = \sum_{j=1}^{3} z_j(t)\boldsymbol{\varphi}_j(\Theta) + \mathbf{w}(\Theta, t), \tag{3.10}$$

with the yet unknown function $\mathbf{w}(\Theta, t)$ satisfying the conditions:

$$\mathbf{w}(\Theta, t) = \mathbf{w}(\Theta, \mathbf{z}(t)), \qquad \mathbf{w}(\Theta, 0) = 0, \qquad \frac{d\mathbf{w}(\Theta, 0)}{d\mathbf{z}} = 0. \tag{3.11}$$

The above conditions, besides tangency, reflect invariant properties of the center manifold.

In order to restrict the aeroelastic system to the center manifold, the projection operator P must be determined, satisfying relations

$$\begin{aligned} P\mathbf{x}_t(\Theta) &= \mathbf{z}(t), \\ P\mathbf{w}(\Theta, \mathbf{z}(t)) &= 0. \end{aligned} \tag{3.12}$$

The projection procedure is based on the so-called outer product [10, 11], associated with the extended linear operator $\mathcal{L}_0$:

$$\langle \mathbf{x}^*, \mathbf{x} \rangle = \overline{\mathbf{x}}^{*T}(0)\mathbf{x}(0) - \int_{-\infty}^{0} \int_{0}^{\eta} \overline{\mathbf{x}}^{*T}(\xi - \eta)\mathbf{G}(-\eta; u)\mathbf{x}(\xi)\, d\xi\, d\eta, \tag{3.13}$$

with two continuous functions $\mathbf{x}(\xi)$ and $\mathbf{x}^*(\eta)$ defined over intervals $-\infty < \xi < 0$ and $0 > \eta > \infty$, respectively. The adjoint operator is defined in a standard way by the relation:

$$\langle \mathbf{x}^*, \mathcal{L}_0 \mathbf{x} \rangle = \langle \mathcal{L}_0^* \mathbf{x}^*, \mathbf{x} \rangle .$$

The eigenvalues and eigenvectors of two eigenproblems $\mathcal{L}_0 \boldsymbol{\varphi} = \lambda \boldsymbol{\varphi}$ and $\mathcal{L}_0^* \boldsymbol{\psi} = \lambda^* \boldsymbol{\psi}$ satisfy the equalities $\lambda^* = \overline{\lambda}$, $\langle \boldsymbol{\psi}_k, \boldsymbol{\varphi}_l \rangle = \delta_{kl}$. By using Eqs. (3.9)–(3.13), the simple set of three nonlinear first-order ordinary differential equations describing asymptotic motion on the center manifold is obtained [10]:

$$\dot{\mathbf{z}} = \boldsymbol{\Lambda}\, \mathbf{z} + \overline{\boldsymbol{\Psi}}^T(0) \mathbf{h}_0 , \tag{3.14}$$

where $\boldsymbol{\Lambda}$ denotes the diagonal matrix of eigenvalues $i\omega_0$, $-i\omega_0$, 0, and the matrix $\boldsymbol{\Psi}$ is composed of the corresponding eigenfunctions $\boldsymbol{\psi}_j$ $(j = 1, 2, 3)$. The $(2N+1)$-dimensional vector function $\mathbf{w}(\Theta, \mathbf{z}(t))$ defines essentially the center manifold and the projection operator as well. It satisfies the integro-differential equation:

$$\dot{\mathbf{w}} - \mathcal{L}_0 \mathbf{w} = \begin{cases} -\sum\limits_{j=1}^{3} \overline{\boldsymbol{\psi}}_j^T(0) \mathbf{h}_0 \boldsymbol{\varphi}_j(\Theta), & \text{for} \quad -\infty < \Theta < 0, \\ -\sum\limits_{j=1}^{3} \overline{\boldsymbol{\psi}}_j^T(0) \mathbf{h}_0 \boldsymbol{\varphi}_j(0) + \mathbf{h}_0, & \text{for} \quad \Theta = 0 \end{cases} \tag{3.15}$$

and also the orthogonality conditions, which have not yet been implicitly imposed:

$$\langle \boldsymbol{\psi}_j, \mathbf{w} \rangle = 0, \qquad j = 1,\ 2,\ 3.$$

Both equations (3.14) and (3.15) are coupled by the right-hand side nonlinear term:

$$\mathbf{h}_0 = \mathbf{h}(\mathbf{x}_t(0)) = \mathbf{h}\left(\sum_{j=1}^{3} z_j(t) \boldsymbol{\varphi}_j(0) + \mathbf{w}(\mathbf{z}, 0) \right) .$$

Although the assumption (3.5) describing the nonlinear term by multi-variable power series has not been used so far, it seems to be rather necessary in order to solve the system of Eqs. (3.14) and (3.15). In what follows, also the function $\mathbf{w}$ is expanded into such a series

$$\mathbf{w}(\mathbf{z}, \Theta) = \sum_{\mu \geq 2} \frac{1}{\mu!} \mathbf{w}_\mu(\Theta) \mathbf{z}^\mu(t). \tag{3.16}$$

In terms of power series, the Eq. (3.15) takes the form

$$\begin{aligned} &\sum_{\mu \geq 2} \frac{1}{\mu!} \left(\mathbf{w}_\mu(\Theta) \boldsymbol{\Lambda}_\mu - \mathcal{L}_0 \mathbf{w}_\mu(\Theta) \right) \mathbf{z}^\mu \\ &\qquad = \sum_{\nu \geq 2} \frac{1}{\nu!} \mathbf{r}_\nu(\Theta) \mathbf{z}^\nu + \begin{cases} 0, & \text{for} \quad -\infty < \Theta < 0, \\ \sum\limits_{\nu \geq 2} \frac{1}{\nu!} \mathbf{h}_{0\nu} \mathbf{z}^\nu, & \text{for} \quad \Theta = 0, \end{cases} \end{aligned} \tag{3.17}$$

where

$$(\mathbf{\Lambda}_\mu)_{kk} = \sum_{j=1}^{3} \lambda_j \mu_j\,, \qquad k = 1, 2, \ldots, c_{\mu,2N+1},$$

and the first right-hand series of (3.17) is given by

$$\sum_{\nu \geq 2} \frac{1}{\nu!} \mathbf{r}_\nu(\Theta)\mathbf{z}^\nu = -\sum_{j=1}^{3} \overline{\boldsymbol{\psi}}_j^T(0)\mathbf{h}_0 \frac{\partial}{\partial z_j} \sum_{\mu \geq 2} \frac{1}{\mu!} \mathbf{w}_\mu(\Theta)\mathbf{z}^\mu.$$

The method of recursive calculations of coefficients of equations (3.14) and (3.17) is described in details in Ref. [3]. It is worth noting here that calculations can be carried out up to the desired order of approximation.

From a quite formal point of view, the center-manifold reduction is equivalent to the appropriate nonlinear change of coordinates given in the form of a series, linking (3.10) and (3.16):

$$\mathbf{x}(t + \Theta) = \sum_{\mu \geq 1} \frac{1}{\mu!} \mathbf{w}_\mu(\Theta)\mathbf{z}^\mu(t), \tag{3.18}$$

where the vector $\mathbf{z}(t)$ of new coordinates has only three components. The matrices $\mathbf{w}_\mu(\Theta)$ of order $(2N+1) \times c_{\mu,3}$, where $c_{\mu,3}$ denotes the number of compositions of μ into 3 parts (3.6), are composed of continuous functions defined in the interval $\Theta \in (-\infty, 0]$. The algorithm of center-manifold reduction provides the way of calculating these functions and also the method of simultaneous derivation of the first-order ordinary differential equation describing the limit cycle oscillations in terms of new variables $\mathbf{z}$:

$$\dot{\mathbf{z}}(t) = \mathbf{\Lambda}\, \mathbf{z}(t) + \sum_{\mu \geq 2} \frac{1}{\mu!} \mathbf{d}_\mu\, \mathbf{z}^\mu, \tag{3.19}$$

where $\mathbf{\Lambda}$ denotes, as before, the diagonal matrix of eigenvalues $i\omega_0$, $-i\omega_0$, 0, and $\mathbf{d}_\mu$ are rectangular matrices built out of the already known complex numbers. The way in which the suspended system has been introduced implies that $z_3 \equiv u$ and also $\dot{z}_3(t) = 0$, which means that an asymptotic motion is essentially two-dimensional. The third variable u acts once again as a parameter, while the suspended system serves as a convenient tool for deriving the series expansion with respect to it.

The next important conclusion drawn from the algorithm of center-manifold reduction says that there is no need to know the response functions forming elements of the matrix $\mathbf{G}(-\Theta; 0)$. This is because the columns $\mathbf{w}_{\mu k}(\Theta)$, $k = 1, 2, \ldots, c_{\mu,2N+1}$, $\mu \geq 1$, of each matrix $\mathbf{w}_\mu(\Theta)$ of the series (3.16), can be only of the elementary form [3]:

$$\mathbf{w}_{\mu k}(\Theta) = \widehat{\mathbf{w}}_{\mu k} \Theta^j e^{s\Theta},$$

with integer $j \geq 0$, and s being an imaginary number. Consequently, all integrals involving the response functions within the algorithm can be carried out as follows:

$$\int_{-\infty}^{0} \frac{d^r \mathbf{G}(-\Theta;0)}{du^r} \Theta^j e^{s\Theta}\, d\Theta = \frac{\partial^{r+j} \mathbf{A}(s;U_0)}{\partial U^r \partial s^j}, \tag{3.20}$$

where $r \geq 0$, and the only non-zero block of the matrix

$$\mathbf{A}(s;U_0) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathsf{A}(s;U_0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix}$$

is the aerodynamic matrix $\mathsf{A}(s;U)$ given by Eq. (2.11) and calculated for a pure harmonic motion and the critical velocity U_0.

Since Eq. (3.19) is an ordinary differential equation, it can be easily transformed to the so-called Poincaré normal form either by the Lie transforms [7] or by recursive change of coordinates [13]. Both methods introduce new variables $\zeta(t)$ related to $\mathbf{z}(t)$ by the near-identity transformation

$$\mathbf{z}(t) = \zeta(t) + \sum_{\nu \geq 2} \frac{1}{\nu!} \mathbf{b}_\nu \zeta^\nu(t). \tag{3.21}$$

This transformation retains the form of Eq. (3.19) also with respect to new coordinates $\zeta(t)$. The calculation of elements of matrices $\mathbf{b}_\nu$ requires to make as many coefficients $\mathbf{d}_\mu$ equal to zero as possible. The simplification achieved lies in the phase-shift symmetry introduced by the transformation (3.21). The normal form of Hopf bifurcation in polar coordinates r, θ:

$$\zeta_1 = r\, e^{i\theta}, \qquad \zeta_2 = \overline{\zeta}_1 \tag{3.22}$$

may be written as [2]:

$$\begin{aligned} \dot{r} &= r \left(\gamma(u) + \sum_{j=1}^{\infty} a_j(u) r^{2j} \right), \\ \dot{\theta} &= \omega(u) + \sum_{j=1}^{\infty} b_j(u) r^{2j}, \end{aligned} \tag{3.23}$$

where $\gamma(u) \pm i\omega(u)$ is the pair of complex-conjugate eigenvalues ($\gamma(0) = 0$, $\omega(0) = \omega_0$). All functions $\gamma(u)$, $\omega(u)$, $a_j(u)$, $b_j(u)$ are real and have the form of power series expansions with respect to u. In practical calculations, Eqs. (3.23) are implemented up to some finite order n ($j \leq n$). Therefore, the amplitude r_H of the limit cycle oscillations satisfies an algebraic equation obtained from Eq. (3.23)$_1$ by setting $\dot{r} = 0$:

$$\gamma(u) + \sum_{j=1}^{n} a_j(u) r_H^{2j} = 0. \tag{3.24}$$

For any given u, the left-hand side of Eq. (3.24) is of the form of a polynomial in r_H. Hence all possible limit cycle amplitudes are determined by the real positive roots of this polynomial. Since limit cycle oscillations $\zeta_1 = \zeta_H(t)$ on the center manifold are purely harmonic [10]:

$$\zeta_H = r_H e^{i\omega_H t}, \tag{3.25}$$

then for each amplitude r_H the corresponding frequency ω_H is calculated from

$$\omega_H = \omega(u) + \sum_{j=1}^{n} b_j(u) r_H^{2j}. \tag{3.26}$$

The sequence of transformations of variables given by Eqs. (3.22), (3.21), (3.18), (3.2), (2.12) and (2.2) yields the final limit cycle oscillations of physical variables $\mathbf{u}(t)$. Since two of these transformations are nonlinear, the physical variables do not oscillate harmonically in time, contrary to the center-manifold variables $\zeta(t)$.

Flutter analysis of an aircraft imposes a number of requirements not satisfied by solutions of the Hopf bifurcation for functional differential equations, available in the literature. First of all, it is not sufficient to take into account only the highest order term of (3.24), which gives the characteristic square-root growth of the limit cycle amplitude

$$r_H = \sqrt{\beta u},$$

where

$$\beta = -\frac{1}{a_1(0)} \frac{d\gamma(0)}{du}, \tag{3.27}$$

because the region of validity of this approximation is too close to the bifurcation point to be of practical importance. An example of such a limited analysis is included in [10] and has given a good starting point for the present method. A two-term approximation, however not using the center-manifold reduction, is given in [14], but because of the very special method of solution of the problem, it cannot be directly extended to the arbitrary number of terms and to systems with many degrees of freedom.

4. Numerical examples

All numerical examples presented in this section assume that the nonlinearities are concentrated in the points of connection between the lifting and control surfaces of an aircraft, producing nonlinear restoring moments when the control surfaces perform rotation about the hinge lines. It is also assumed that each hinge moment M_δ is a cubic function of the local angle of rotation δ

$$M_\delta = K_\delta(\delta + c\delta^3), \tag{4.1}$$

where K_δ is a standard linear spring constant, and the coefficient c describes the strength of nonlinearity. The last assumption means that there is only one non-zero matrix $\mathbf{K}_2$ in Eq. (2.4).

Since each nonlinear analysis is essentially an extension of the corresponding linearized problem, it is impossible to calculate the limit cycle parameters for an aircraft without having a suitable computer program for the linear flutter analysis. The standard output of such program includes critical flutter velocity U_0, the corresponding frequency ω_0 and the flutter mode in the form of a right eigenvector $\widehat{\mathbf{q}}$ of Eq. (2.14). For a nonlinear flutter analysis the following additional data should be supplied:

- elements of the aerodynamic matrix (2.11) corresponding to the flutter point,
- a set of derivatives (3.20) of the aerodynamic matrix corresponding to the flutter velocity and calculated for $s = \pm i\omega_0$, $\pm 2i\omega_0$, $\pm 3i\omega_0$ ... up to the desired order of approximation,
- elements of the matrix $\mathbf{R}$ (2.5) defining locations of nonlinear springs within the aircraft structure.

Since the aerodynamic matrix is essentially a function of nondimensional variable $p = \omega b/U$, the derivatives of the aerodynamic matrix with respect to variables s and U can be easily evaluated if the corresponding derivatives with respect to p are known. For the n-th order of approximation of the Eqs. (3.23), the highest derivatives are of order $2n - 1$. Although some simpler unsteady aerodynamic models allow for an analytical calculation of derivatives (e.g. strip theory), it seems that in general, the only efficient way is numerical differentiation. This is because in most cases the aerodynamic matrix is known only numerically (i.e. as a set of numbers). It has been found that satisfactory results, especially for higher-order derivatives, gives a simple integration scheme based on the Cauchy integral in the complex p-plane:

$$\frac{d^j a_{kl}(p)}{dp^j} = \frac{j!}{2\pi i} \oint_C \frac{a_{kl}(z)}{(z-p)^{j+1}} dz \approx \frac{j!}{2\pi i} \sum_{r=1}^{m} \frac{a_{kl}(z_r)}{(z_r - p)^{j+1}} \Delta z_r \, ,$$

where $a_{kl}(p)$ denotes an element of the aerodynamic matrix. Integration nodes z_r are placed on a small circle C with an origin in the point p. All values of argument p of the derivatives appearing in center-manifold reduction are purely imaginary numbers, hence the standard numerical methods for calculation of the aerodynamic matrix can be applied.

The number of degrees of freedom of an aircraft may cause some computational problems since the amount of numerical work required grows very fast. For an aircraft with only six degrees of freedom (modal coordinates) and four-term center-manifold approximation ($n = 4$ in Eqs. (3.24) and (3.26)), the number of components of the last, 9-th vector $\mathbf{x}^\eta$ in Eq. (3.8) equals 293930. Therefore, it is very important to select only the most significant natural modes out of all the modes included in the flutter mode, in order to save both the computer time and memory. Since the center manifold is tangent to the linear subspace spanned by two complex-conjugate eigenvectors of the linear operator (2.17), such a selection is done in the same way as in the conventional linear flutter analysis.

Sample calculations of the limit cycle amplitude and frequency were made for the aileron and flap flutter of two gliders. All hinge springs of the control surfaces were assumed to produce hardening cubic nonlinearities. The number of physical degrees of freedom used to calculate the natural modes was equal to nearly 200. Six modal coordinates were taken into account, including two or three rigid modes. The first glider revealed symmetric and also antisymmetric flap-aileron flutter at velocities 187 km/h and 178 km/h, respectively. Similar antisymmetric flutter at 225 km/h occurred for the second glider.

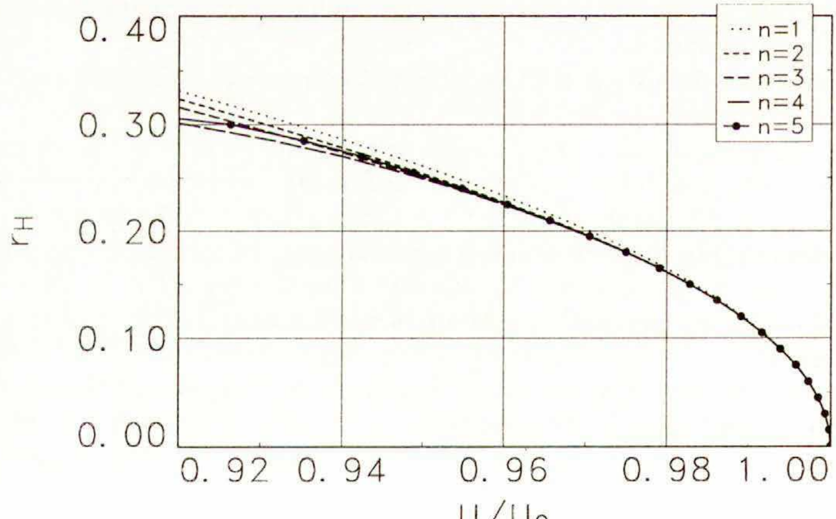

FIG. 1. Amplitude of center-manifold Hopf limit cycle (symmetric flutter).

Both gliders had one nonlinear aileron hinge spring with $c = 50$ (4.1). Results of calculations for the first glider are presented in Figs. 1–8. Figures 9–12 concern the second glider. Symbol n in all figures denotes the number of terms of the

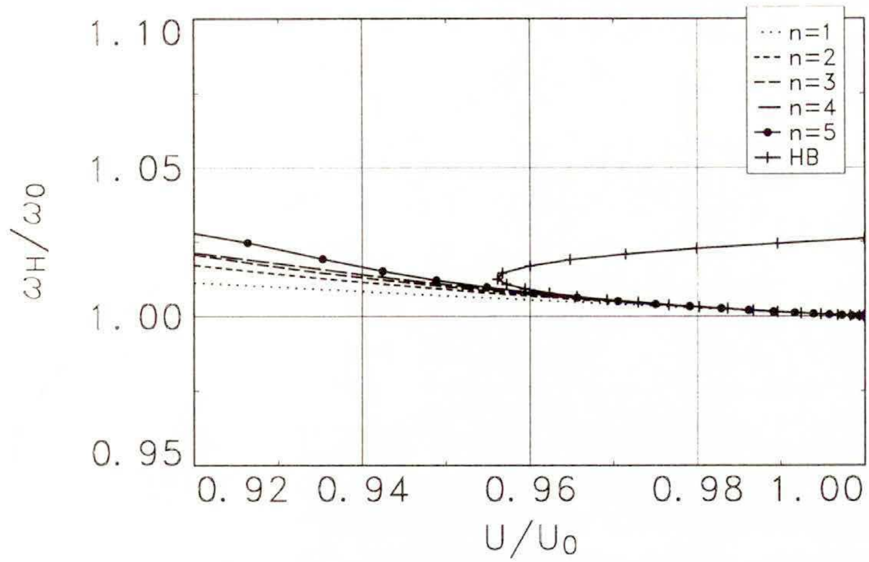

FIG. 2. Frequency of center-manifold Hopf limit cycle (symmetric flutter).

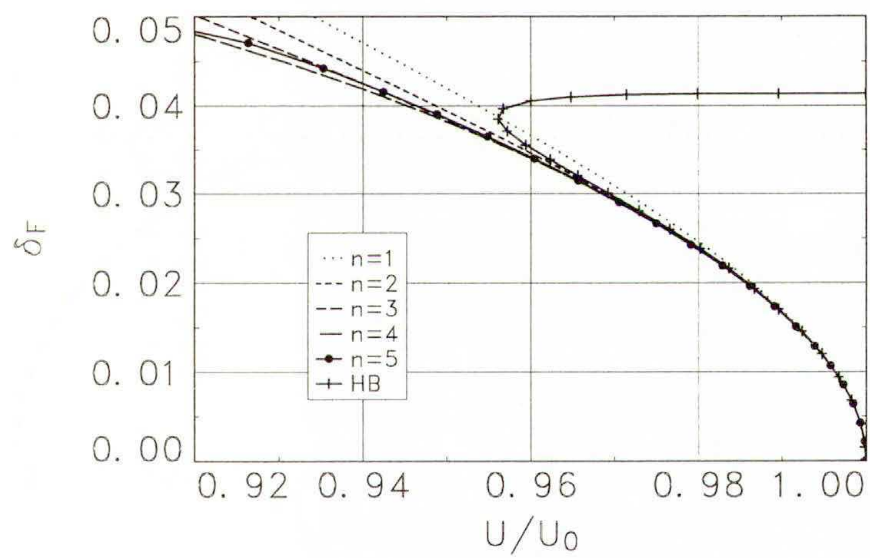

FIG. 3. Flap limit cycle amplitude (symmetric flutter).

series (3.24) and (3.26). As the final results of calculations, the Hopf limit cycle amplitude r_H (3.24), normalized with respect to $\sqrt{\beta}$ (3.27), and frequency ω_H/ω_0 (3.26) are plotted against the nondimensional velocity U/U_0. There is a sequence of five approximations in each chart, corresponding to $n = 1, 2, 3, 4, 5$. Note that

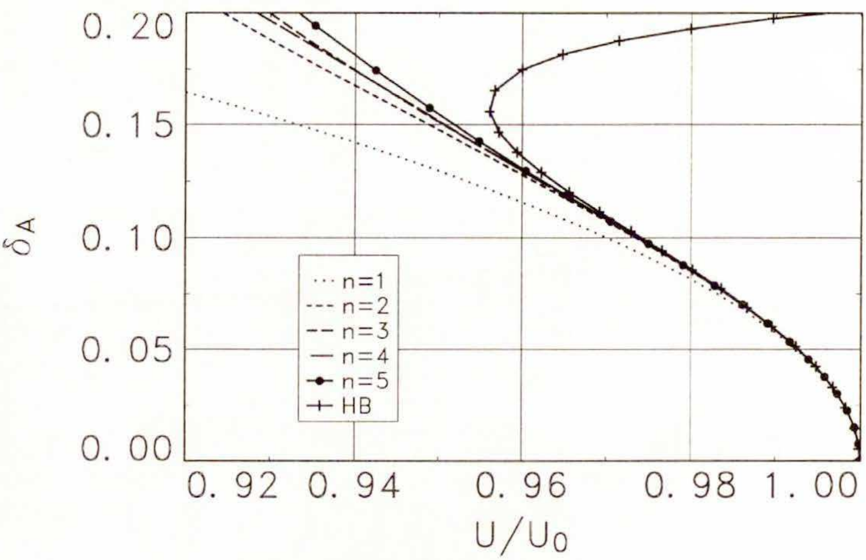

FIG. 4. Aileron limit cycle amplitude (symmetric flutter).

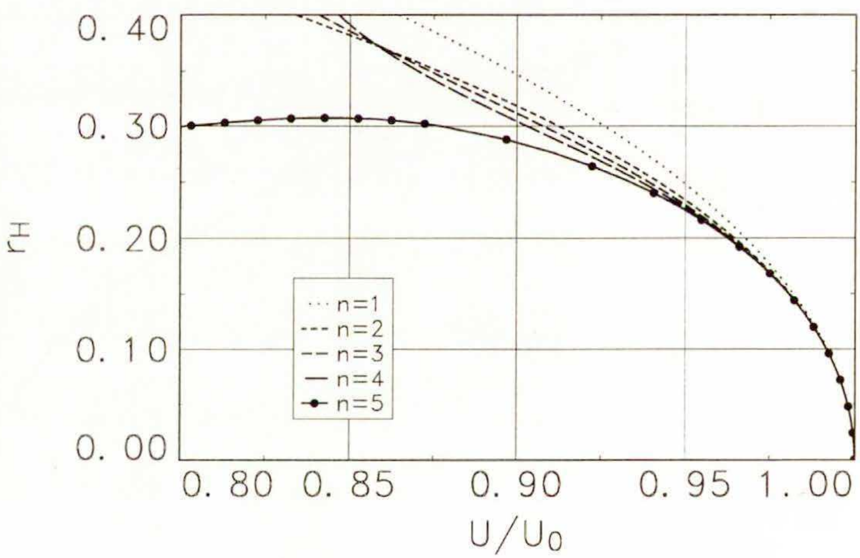

FIG. 5. Amplitude of center-manifold Hopf limit cycle (antisymmetric flutter).

n-th order approximation of a center-manifold limit cycle requires $2n + 1$ terms in the power series expansion (3.18).

Once the center manifold limit cycle parameters are known, it is possible to calculate the physical deflections of a glider during oscillations. Only two of them

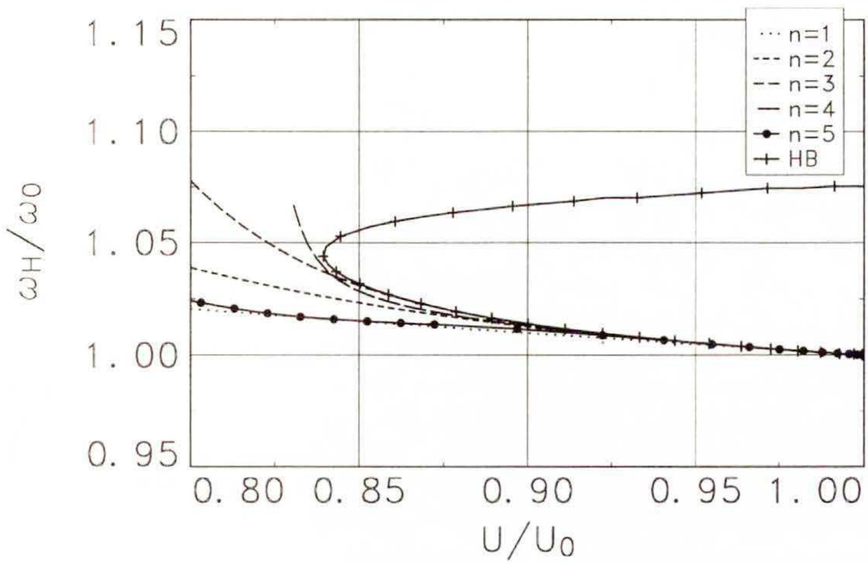

FIG. 6. Frequency of center-manifold Hopf limit cycle (antisymmetric flutter).

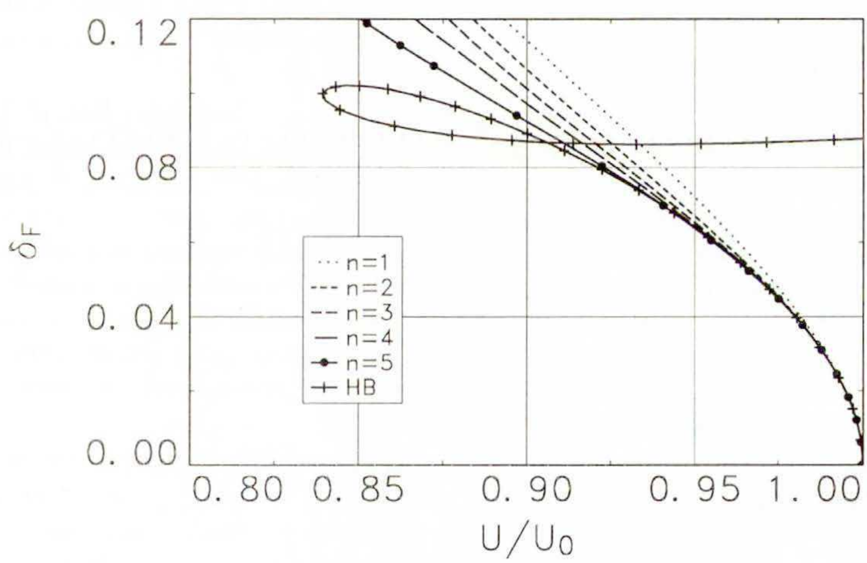

FIG. 7. Flap limit cycle amplitude (antisymmetric flutter).

are plotted: local hinge-line rotation of flap δ_F and aileron δ_A. Both correspond to the location of nonlinear springs and are measured in radians. Because physical coordinates do not oscillate harmonically in time (though in a very similar manner), the amplitude of oscillations is not well-defined. Therefore, δ_F and δ_A

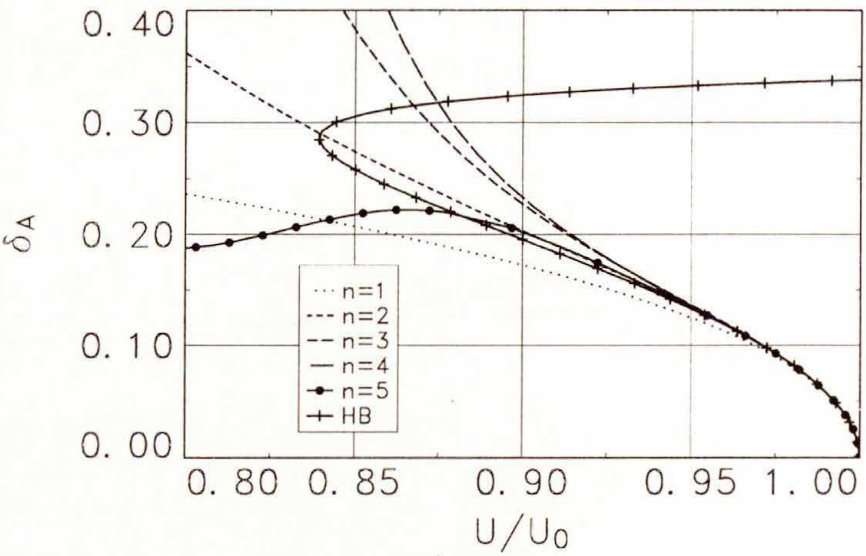

FIG. 8. Aileron limit cycle amplitude (antisymmetric flutter).

denote maximum values of the rotation angle reached during a single period. In all figures the unstable limit cycles appear in the vicinity of the corresponding bifurcation points.

In almost every chart there is an additional line taken from Ref. [17], and denoted HB, describing the amplitude of limit cycle oscillations calculated by the harmonic balance method [15], by using the continuation subroutines package [16]. Harmonic balance method replaces each nonlinear restoring force by the first term of its Fourier transform. If there is only one nonlinear force present in a system, then for any given limit cycle amplitude the linearized flutter equation can be solved for the corresponding flight velocity. Multiple nonlinearities result in greater complexity of calculations, because the amplitudes of aircraft deflections at concentration points are not known prior to the calculations, but their ratios are determined by the resulting flutter mode.

There is a very good agreement between the results of the present method and the harmonic balance method, in a range of a few percent below the linear flutter velocity U_0. However, beyond this interval a qualitative discrepancy of the results of both methods are observed, and also the power series derived by the present method are not convergent anymore.

It was impossible to establish the real behaviour of limit cycle oscillations of the gliders because neither the flight tests nor direct numerical integration of the nonlinear flutter equation were performed. Nevertheless, it is important that the limit cycle oscillations are detected below the linear flutter velocity despite the fact that their amplitude is uncertain. These oscillations can be initiated by a

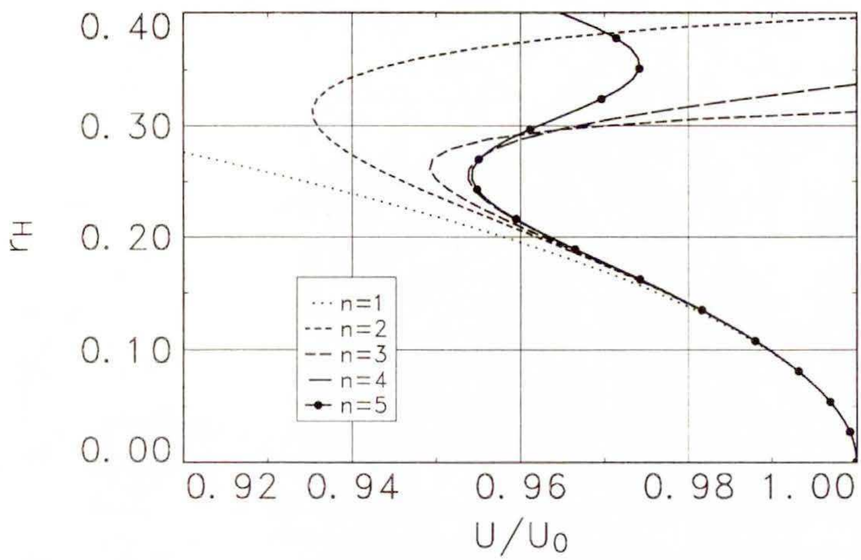

FIG. 9. Amplitude of center-manifold Hopf limit cycle (antisymmetric flutter).

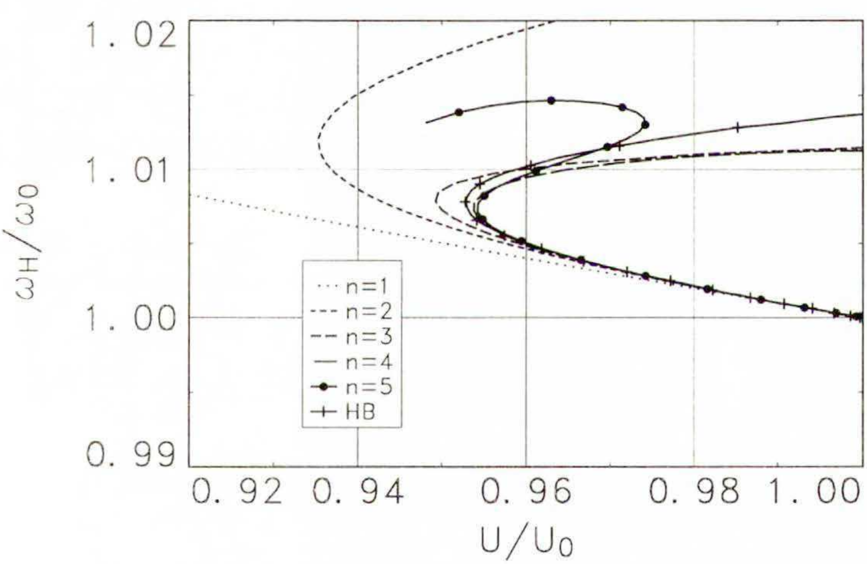

FIG. 10. Frequency of center-manifold Hopf limit cycle (antisymmetric flutter).

sufficiently high disturbance, the magnitude of which is known from the presented results of calculations and which is given by the unstable branch of amplitude curves (the part of plots between the bifurcation point and the turning point in Figs. 11 and 12).

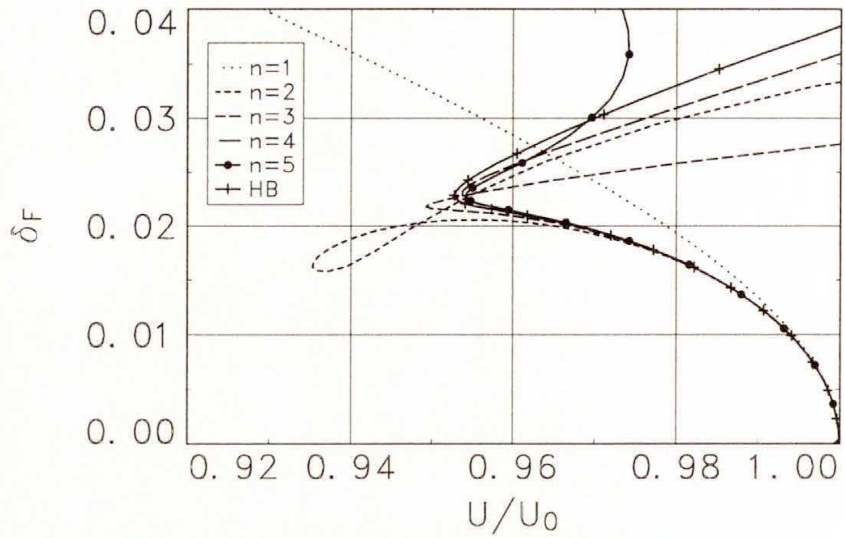

FIG. 11. Flap limit cycle amplitude (antisymmetric flutter).

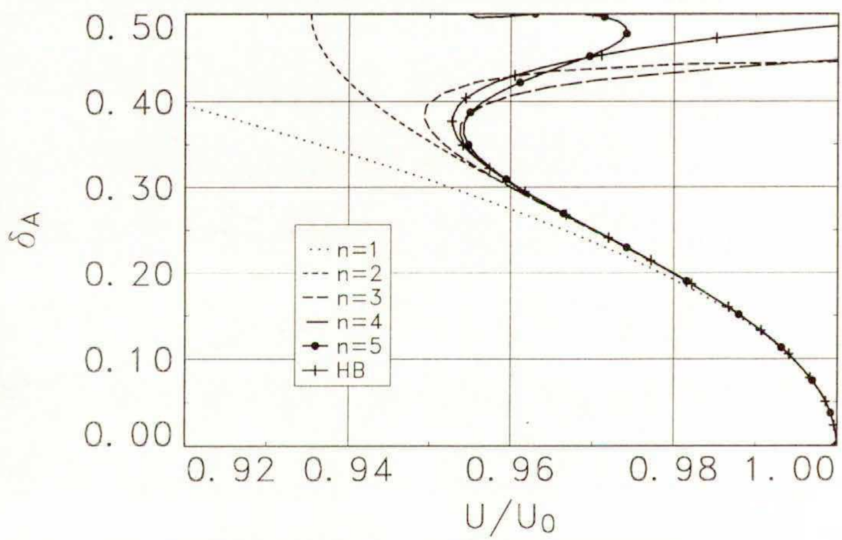

FIG. 12. Aileron limit cycle amplitude (antisymmetric flutter).

5. Concluding remarks

The discrepancy between the present method and the harmonic balance method in a region located not very close to the bifurcation point is not an unexpected result. The harmonic balance method assumes pure harmonic oscillations of a structure, that may not be satisfied, and also treats nonlinear springs in a simplified manner. The method of center-manifold reduction is a method of local validity and, afterwards, is based on asymptotic series expansions, the usefulness of which cannot be expected in a wide range of velocity. Nevertheless, there is a good agreement between these two methods locally. Hence, the main advantage of the center-manifold reduction lies in a possibility of extension of this method to such aeroelastic systems for which harmonic balance method cannot be handled easily (e.g. multiple concentrated nonlinearities), and to systems for which the direct numerical integration method cannot be used in a sufficiently effective way.

The method of center-manifold reduction does not limit the number of degrees of freedom. The problem of treatment of higher degree of freedom systems affects only the efficiency of calculations. The method itself (and the corresponding computer code as well) can be applied to any number of degrees of freedom "as it is". However, the hardware used may bound this number significantly if there is not enough RAM available. It has been found that the computer direct access memory is the bottle-neck of the calculations. The reason is that the main series (3.18) is not a series of numbers but rather a series of functions. These functions are described by a rapidly growing number of parameters, when the number of terms increases, and moreover, all of them must be stored in memory during the entire computation process. On the other hand, not very high number of terms is sufficient to determine the behaviour of the aeroelastic system under considerations in the neighbourhood of a bifurcation point.

The method of center manifold reduction is an asymptotic and local method (i.e. looking near a single point) and, therefore, is not suited for treatment of more complex global bifurcations or transition to chaotic oscillations. Such oscillations appear also in aeroelastic systems.

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