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BRIEF NOTES

Non-polynomial representations of orthotropic tensor functions in the three-dimensional case: an alternative approach

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THE OBJECTIVE of this paper is to extend some of the results obtained in [1] to the three-dimensional case. Functional bases and generators for symmetric second-order orthotropic tensor functions are derived.

1. Introduction

THE THEORY of representation of tensor functions has been developed for more than thirty years [2–5]. The results obtained within the framework of this theory yield general forms of isotropic [6–15] and anisotropic [16–22] tensor functions. Most complete results were obtained for scalar-valued, vector-valued, symmetric and skew-symmetric tensor-valued functions of the second order, dependent on vectors as well as symmetric and skew-symmetric tensors of the second order.

Theoretical foundations of the formulation of anisotropic constitutive relationships were laid, among others, in the books [23–26]. There the group theory and the theory of representation of tensor functions were exploited. Anisotropic materials constitute an important class of structural materials in many fields of engineering. Hence the need for further development of the constitutive theory, where the theory of representation of tensor functions plays an important role, cf. [3, 27–30].

The determination of a representation of a tensor function in the so-called canonical form reduces to finding irreducible sets of basic invariants and generators of this function. One distinguishes polynomial and non-polynomial representations of tensor functions [3, 23]. To find the polynomial representation of a tensor function it is sufficient to determine the relevant integrity basis. Once this basis is established, generators are obtained by a simple process of integration [23]. An integrity basis is said to be irreducible if none of its elements can be expressed as a polynomial in the remaining elements, cf. [23]. A set of invariants is said to constitute a functional basis, for given arguments and a symmetry group of the considered function, if any other invariant of the same arguments can be expressed as a scalar function of these invariants. A non-polynomial representation is irreducible if none of the generators can be expressed as a linear

combination of the remaining generators, with the coefficients being arbitrary functions of the functional basis. WANG [6–8], SMITH [9, 10] and BOEHLER [11] proved that in the general case a non-polynomial representation, if compared with the corresponding polynomial representation, contains less generators and invariants.

The aim of this note is the determination of the representation of a non-polynomial orthotropic scalar function as well as orthotropic, symmetric tensor-valued function of the second order. Our approach is alternative to that used by BOEHLER [18, 19]. Those functions depend on a finite number of symmetric, second order tensors. Thus we extend to the three-dimensional case the results presented in our earlier paper [1].

2. Formulation of the problem

Our aim is to determine the non-polynomial representations of the following functions

$$
\begin{aligned}
& s = f(\mathbf{A}_p; \mathbf{H}), \qquad f: \underbrace{T_s \times \ldots \times T_s}_{(P+1)-\text{times}} \to R, \\
& \mathbf{S} = \mathbf{F}(\mathbf{A}_p; \mathbf{H}), \qquad \mathbf{F}: \underbrace{T_s \times \ldots \times T_s}_{(P+1)-\text{times}} \to T_s,
\end{aligned} \tag{1}
$$

where $\mathbf{A}_p$ are symmetric second order tensors, $\mathbf{A}_p \in T_s$, $T_s = \{\mathbf{A} \in T: \mathbf{A} = \mathbf{A}^T\}$, $p = 1, \ldots, p$ and $T = E \otimes E$; $\mathbf{A}^T$ stands for the transpose of a tensor $\mathbf{A}$. Here E is the three-dimensional Euclidean space and $\mathbf{H}$ is a symmetric, positive-definite tensor of the second order. The tensor $\mathbf{H}$ plays the role of a parametric tensor, i.e. $\mathbf{H} = \text{const}$. The function f is a scalar-valued function while $\mathbf{F}$ is a symmetric, second order tensor function. Suppose that (1) are to be constitutive relationships. Then $\mathbf{A}_p$ are causes, $\mathbf{H}$ models the structure of a material while s and $\mathbf{S}$ are responses or effects. Within the framework of the classical continuum mechanics, such relationships should be invariant with respect to the group of automorphisms of the space E, cf. [25]. In other words, they have to satisfy the so-called principle of isotropy of the physical space. Consequently, the functions appearing in (1) fulfil the following conditions:

$$
\begin{aligned}
\forall \mathbf{Q} \in O: \qquad & f(\mathbf{A}_p; \mathbf{H}) = f\left(\mathbf{Q}\mathbf{A}_p\mathbf{Q}^T; \mathbf{Q}\mathbf{H}\mathbf{Q}^T\right), \\
& \mathbf{Q}\,\mathbf{F}(\mathbf{A}_p; \mathbf{H})\mathbf{Q}^T = \mathbf{F}\left(\mathbf{Q}\mathbf{A}_p\mathbf{Q}^T; \mathbf{Q}\mathbf{H}\mathbf{Q}^T\right),
\end{aligned} \tag{2}
$$

where O denotes the full orthogonal group, that is

$$
O \equiv \left\{\mathbf{Q} \in T: \ \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}\right\}. \tag{3}
$$

Here $\mathbf{I}$ stands for the identity tensor.

According to our assumption, the tensor $\mathbf{H}$ has three distinct eigenvalues, say H_i $(i = 1, 2, 3)$. Thus we may write

$$\mathbf{H} = H_1\mathbf{e}_1 \otimes \mathbf{e}_1 + H_2\mathbf{e}_2 \otimes \mathbf{e}_2 + H_3\mathbf{e}_3 \otimes \mathbf{e}_3\,, \qquad H_1 \neq H_2 \neq H_3 \neq H_1\,, \tag{4}$$

where $\mathbf{e}_i$ are unit eigenvectors of the tensor $\mathbf{H}$. We observe that the group of external symmetries of the tensor $\mathbf{H}$, given by

$$S \equiv \left\{\mathbf{Q} \in O\colon \quad \mathbf{Q}\,\mathbf{H}\,\mathbf{Q}^T = \mathbf{H}\right\}, \tag{5}$$

is the orthotropy group. Moreover, the eigenvectors of $\mathbf{H}$ determine the so-called principal axes of orthotropy of a material. This statement becomes obvious if we compare (4) and (5) with the corresponding definitions given in the papers [3, 18–20, 25].

Let

$$\mathbf{M}_i = \mathbf{e}_i \otimes \mathbf{e}_i \qquad \text{(no summation on } i = 1, 2, 3), \tag{6}$$

then we recover, by taking account of (4) and (6) in (1), provided that (2) is satisfied, the problem considered in the papers [18, 19].

From (2) and (5) it follows that

$$\begin{aligned} \forall \mathbf{Q} \in S\colon \qquad & f(\mathbf{A}_p; \mathbf{H}) = f\left(\mathbf{Q}\,\mathbf{A}_p\mathbf{Q}^T; \mathbf{H}\right), \\ & \mathbf{Q}\,\mathbf{F}(\mathbf{A}_p; \mathbf{H})\mathbf{Q}^T = \mathbf{F}\left(\mathbf{Q}\,\mathbf{A}_p\mathbf{Q}^T; \mathbf{H}\right). \end{aligned} \tag{7}$$

In other words, the functions $f(\ldots; \mathbf{H})$, $\mathbf{F}(\ldots; \mathbf{H})$ are orthotropic functions of the tensors $\mathbf{A}_p$.

3. Determination of the orthotropic functional basis

Since the tensor $\mathbf{H}$ has three distinct eigenvalues, therefore in order to determine the functional basis for the scalar function $(1)_1$ we may exploit the results obtained by SMITH [10]. To this end it is sufficient to consider the case (2ii) studied by SMITH [10, pp. 905–907]. The functional basis derived in this manner is presented in Table 1.

It can easily be proved that the representation of the scalar function $(1)_1$ depicted in Table 1 is equivalent to the results obtained by BOEHLER in [18, 19]. Boehler's orthotropic functional basis is presented in Table 2.

Both functional bases are equivalent because:

$$\begin{aligned} \operatorname{tr}\mathbf{A}_p &= \operatorname{tr}\mathbf{M}_1\mathbf{A}_p + \operatorname{tr}\mathbf{M}_2\mathbf{A}_p + \operatorname{tr}\mathbf{M}_3\mathbf{A}_p\,, \\ \operatorname{tr}\mathbf{A}_p^2 &= \operatorname{tr}\mathbf{M}_1\mathbf{A}_p^2 + \operatorname{tr}\mathbf{M}_2\mathbf{A}_p^2 + \operatorname{tr}\mathbf{M}_3\mathbf{A}_p^2\,, \\ \operatorname{tr}\mathbf{H}^a\mathbf{A}_p^b &= H_1^a\operatorname{tr}\mathbf{M}_1\mathbf{A}_p^b + H_2^a\operatorname{tr}\mathbf{M}_2\mathbf{A}_p^b + H_3^a\operatorname{tr}\mathbf{M}_3\mathbf{A}_p^b\,, \\ \operatorname{tr}\mathbf{A}_p\mathbf{A}_q &= \operatorname{tr}\mathbf{M}_1\mathbf{A}_p\mathbf{A}_q + \operatorname{tr}\mathbf{M}_2\mathbf{A}_p\mathbf{A}_q + \operatorname{tr}\mathbf{M}_3\mathbf{A}_p\mathbf{A}_q\,, \\ \operatorname{tr}\mathbf{H}^a\mathbf{A}_p\mathbf{A}_q &= H_1^a\operatorname{tr}\mathbf{M}_1\mathbf{A}_p\mathbf{A}_q + H_2^a\operatorname{tr}\mathbf{M}_2\mathbf{A}_p\mathbf{A}_q + H_3^a\operatorname{tr}\mathbf{M}_3\mathbf{A}_p\mathbf{A}_q\,, \quad a, b = 1, 2, \end{aligned} \tag{8}$$

where tr stands for the trace of a tensor; for instance $\mathrm{tr}\mathbf{A}\,\mathbf{B} = \mathrm{tr}(\mathbf{A}\,\mathbf{B})$, where $\mathbf{A}\,\mathbf{B} = \mathrm{tr}_{(2,3)}\mathbf{A} \otimes \mathbf{B}$.

Table 1. Functional basis for the scalar function $(1)_1$.

Arguments	Basic invariants
$\mathbf{A}_p$	$\mathrm{tr}\mathbf{A}_p$, $\mathrm{tr}\mathbf{A}_p^2$, $\mathrm{tr}\mathbf{A}_p^3$, $\mathrm{tr}\mathbf{H}\mathbf{A}_p$, $\mathrm{tr}\mathbf{H}^2\mathbf{A}_p$, $\mathrm{tr}\mathbf{H}\mathbf{A}_p^2$, $\mathrm{tr}\mathbf{H}^2\mathbf{A}_p^2$
$\mathbf{A}_p$, $\mathbf{A}_q$	$\mathrm{tr}\mathbf{A}_p\mathbf{A}_q$, $\mathrm{tr}\mathbf{A}_p^2\mathbf{A}_q$, $\mathrm{tr}\mathbf{A}_p\mathbf{A}_q^2$, $\mathrm{tr}\mathbf{H}\mathbf{A}_p\mathbf{A}_q$, $\mathrm{tr}\mathbf{H}^2\mathbf{A}_p\mathbf{A}_q$
$\mathbf{A}_p$, $\mathbf{A}_q$, $\mathbf{A}_r$	$\mathrm{tr}\mathbf{A}_p\mathbf{A}_q\mathbf{A}_r$, $p,q,r = 1,\ldots P$; $p < q < r$

Table 2. Orthotropic functional basis after BOEHLER [19].

Arguments	Basic invariants
$\mathbf{A}_p$	$\mathrm{tr}\mathbf{M}_1\mathbf{A}_p$, $\mathrm{tr}\mathbf{M}_1\mathbf{A}_p^2$, $\mathrm{tr}\mathbf{A}_p^3$, $\mathrm{tr}\mathbf{M}_2\mathbf{A}_p$, $\mathrm{tr}\mathbf{M}_2\mathbf{A}_p^2$, $\mathrm{tr}\mathbf{M}_3\mathbf{A}_p$, $\mathrm{tr}\mathbf{M}_3\mathbf{A}_p^2$
$\mathbf{A}_p$, $\mathbf{A}_q$	$\mathrm{tr}\mathbf{M}_1\mathbf{A}_p\mathbf{A}_q$, $\mathrm{tr}\mathbf{A}_p^2\mathbf{A}_q$, $\mathrm{tr}\mathbf{A}_p\mathbf{A}_q^2$, $\mathrm{tr}\mathbf{M}_2\mathbf{A}_p\mathbf{A}_q$, $\mathrm{tr}\mathbf{M}_3\mathbf{A}_p\mathbf{A}_q$
$\mathbf{A}_p$, $\mathbf{A}_q$, $\mathbf{A}_r$	$\mathrm{tr}\mathbf{A}_p\mathbf{A}_q\mathbf{A}_r$, $p,q,r = 1,\ldots P$; $p < q < r$

4. Determination of generators of an orthotropic tensor-valued function of the second order

In order to derive the representation of the function $(1)_2$ under the condition $(2)_2$, we shall apply the method similar to that used in the papers [1, 13, 14, 31, 32]. This method is based on the idea primarily proposed in the paper by the second author [30]. First, we construct a scalar function, say g, defined by

$$g = \mathrm{tr}\mathbf{F}\,\mathbf{C}, \tag{9}$$

linear with respect to the second argument or $\mathbf{C}$. Here $\mathbf{C}$ is a symmetric second order tensor while $\mathbf{F}$ is the function $(1)_2$. The function g has the following form:

$$g(\mathbf{A}_p, \mathbf{C}; \mathbf{H}) = \tilde{g}(I_t, J_s) = \sum_{s=1}^{S} \phi(I_t) J_s\,, \tag{10}$$

where I_t are invariants listed in Table 1 whereas J_s are invariants linear in $\mathbf{C}$, see Table 3 below.

The canonical form of the tensor-valued function $(1)_2$ is found from

$$\mathbf{F}(\mathbf{A}_p, \mathbf{H}) = \frac{1}{2}\left(\frac{\partial g}{\partial \mathbf{C}} + \frac{\partial g}{\partial \mathbf{C}^T}\right) = \frac{\partial g}{\partial \mathbf{C}} = \sum_{s=1}^{S} \phi_s(I_t)\frac{\partial J_s}{\partial \mathbf{C}} = \sum_{s=1}^{S} \tilde{\phi}_s(I_t)\mathbf{G}_s\,. \tag{11}$$

The results of calculations are summarised in Table 4, where the generators $\mathbf{G}_s$ are listed.

Table 3. Invariants linear in C.

Arguments	Invariants J_s
C	$\mathrm{tr}\mathbf{C}$, $\mathrm{tr}\mathbf{H}\mathbf{C}$, $\mathrm{tr}\mathbf{H}^2\mathbf{C}$
C, $\mathbf{A}_p$	$\mathrm{tr}\mathbf{A}_p\mathbf{C}$, $\mathrm{tr}\mathbf{A}_p^2\mathbf{C}$, $\mathrm{tr}\mathbf{H}\mathbf{A}_p\mathbf{C}$, $\mathrm{tr}\mathbf{H}^2\mathbf{A}_p\mathbf{C}$
C, $\mathbf{A}_p$, $\mathbf{A}_q$	$\mathrm{tr}\mathbf{A}_p\mathbf{A}_q\mathbf{C}$, $p, q, r = 1, \ldots P$; $p < q$

Table 4. Generators of the function $(1)_2$.

Arguments	Generators
	$\mathbf{I}$, $\mathbf{H}$, $\mathbf{H}^2$
$\mathbf{A}_p$	$\mathbf{A}_p$, $\mathbf{A}_p^2$, $\mathbf{H}\mathbf{A}_p + \mathbf{A}_p\mathbf{H}$, $\mathbf{H}^2\mathbf{A}_p + \mathbf{A}_p\mathbf{H}^2$
$\mathbf{A}_p$, $\mathbf{A}_q$	$\mathbf{A}_p\mathbf{A}_q + \mathbf{A}_q\mathbf{A}_p$, $p, q = 1, \ldots P$; $p < q$

The generators obtained in this way are equivalent to those derived by BOEHLER [19] and listed in Table 5. To corroborate this statement, it is sufficient to exploit the following identities:

$$
\begin{aligned}
\mathbf{I} &= \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3\,, \\
\mathbf{H}^a &= H_1^a\mathbf{M}_1 + H_2^a\mathbf{M}_2 + H_3^a\mathbf{M}_3\,, \qquad a = 1, 2, \\
2\mathbf{A}_p &= \mathbf{M}_1\mathbf{A}_p + \mathbf{M}_1\mathbf{A}_p + \mathbf{M}_2\mathbf{A}_p + \mathbf{M}_2\mathbf{A}_p + \mathbf{M}_3\mathbf{A}_p + \mathbf{M}_3\mathbf{A}_p\,, \\
\mathbf{H}^a\mathbf{A}_p + \mathbf{H}^a\mathbf{A}_p &= H_1^a(\mathbf{M}_1\mathbf{A}_p + \mathbf{M}_1\mathbf{A}_p) + H_2^a(\mathbf{M}_2\mathbf{A}_p \\
&\qquad + \mathbf{M}_2\mathbf{A}_p) + H_3^a(\mathbf{M}_3\mathbf{A}_p + \mathbf{M}_3\mathbf{A}_p).
\end{aligned}
\tag{12}
$$

Table 5. Boehler's [5] generators of the orthotropic tensor function.

Arguments	Generators
	$\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$
$\mathbf{A}_p$	$\mathbf{M}_1\mathbf{A}_p + \mathbf{M}_1\mathbf{A}_p$, $\mathbf{M}_2\mathbf{A}_p + \mathbf{M}_2\mathbf{A}_p$, $\mathbf{M}_3\mathbf{A}_p + \mathbf{M}_3\mathbf{A}_p$, $\mathbf{A}_p^2$
$\mathbf{A}_p$, $\mathbf{A}_q$	$\mathbf{A}_p\mathbf{A}_q + \mathbf{A}_q\mathbf{A}_p$, $p, q = 1, \ldots P$; $p < q$

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