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The stochastic vortex method for viscous incompressible flows in a spatially periodic domain

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THE RANDOM VORTEX METHOD for two-dimensional, nonstationary flows of a viscous liquid in a spatially periodic, infinite system of airfoils is considered. The main idea is to approximate the evolution of the vorticity by a large set of small "vortex particles" (vortex blobs), which are transported in the velocity field (convection) and perform random walks according to Wiener process with standard deviation depending on the viscosity (diffusion). The velocity field is due to the induction of vortex blobs and includes also certain potential components. Since the flow domain is not simply connected, additional constraints concerning the vorticity production on the boundaries are introduced. They are necessary to obtain a solution with physically correct, single-valued pressure field. The results of numerical calculations are also presented.

1. Introduction

DURING LAST TWO DECADES, large amount of research work has been devoted to the development of more sophisticated variants of vortex methods, to widening the range of their applications and improving their computational efficiency. Since 1973, when CHORIN published his fundamental paper [1], many authors have applied a stochastic approach to calculate flows with various geometrical configurations. However, a majority of available publications on external flows focus on flows around individual contours only, although, from the engineering point of view, multi-body systems are even more important.

The aim of this paper is to present the random vortex algorithm for flows which are periodic with respect to one spatial variable. The standard engineering example is a flow in a cascade of airfoils, which is used as a model of turbomachinery flows. The numerical method constructed here is a natural extension of the method proposed by STYCZEK [2] and its primary version was also the subject of the thesis of one of the authors (see [3]). The current version includes careful treatment of the pressure problem arising due to multiply connected geometry of the flow domain. More refined numerical results are also obtained.

We remind briefly the general idea of the stochastic approach to viscous liquid motion (more detailed discussion and examples of applications can be found in [2, 4, 5 and 6]). The equation of the vorticity transport (Helmholtz equation) in an incompressible, viscous and two-dimensional flow can be written in the following form:

$$\partial_t \omega + \partial_x (u\omega) + \partial_y (v\omega) = \nu \Delta \omega \,. \tag{1.1}$$

This equation is formally identical to the Fokker–Planck equation corresponding to a diffusive stochastic process with the convective vector equal to the velocity of the flow $\mathbf{V} = [u, v]$ and with the diagonal matrix of diffusion $\mathrm{Diag}[2\nu, 2\nu]$. Thus the evolution of the vorticity field can be described on the "microscopic level" as a movement of a large (theoretically infinite) set of "vortex particles", governed by the following Ito equations

$$
\begin{aligned}
dx(t) &= u\,(t, x(t), y(t))\, dt + \sqrt{2\nu}\, dW_x\,, \\
dy(t) &= v\,(t, x(t), y(t))\, dt + \sqrt{2\nu}\, dW_y\,.
\end{aligned} \tag{1.2}
$$

Here W_x and W_y are independent Wiener processes.

In a numerical simulation "vortex particles" can be constructed in many ways. Here the vortex blobs i.e. small circular vortices with uniform vorticity distribution are used. It should be emphasised that there is no natural, independent boundary condition for the vorticity field – there are only conditions for the velocity. It is known, however, that the vorticity is produced on the boundaries. In the vortex method new vortex blobs are created on the boundaries in each time step in order to satisfy the boundary condition for the velocity. Some of these blobs subsequently enter the flow domain, while the others move randomly across the boundary and are eliminated. This process gives rise to the diffusive flux of the vorticity through the contours of embedded bodies. All vortex blobs are convected in the velocity field which is partly due to the induction, and also has additional potential components necessary to fulfil boundary conditions and providing appropriate asymptotic behaviour of the velocity field (the condition at infinity).

In the case of a cascade flow the domain is not simply connected. Then there exist velocity and vorticity fields which satisfy the continuity and Helmholtz equations, but correspond to meaningless, multivalued pressure distributions. In order to avoid such "solutions", additional constraints should be imposed on the velocity field (see, for instance, [6] or [7]). These constraints have the form of following integral equalities:

$$
\frac{d}{dt}\oint\limits_{C_k} U_g^t(s)\, ds + \int\limits_{C_k} \left(U_g^n \omega - \nu \frac{d}{dn}\omega \right)(s)\, ds = 0, \tag{1.3}
$$

where C_k denotes k-th component of the boundary of a multiply connected flow region and U_g is the boundary velocity distribution. If U_g is fixed in time then we have the condition

$$
\int\limits_{C_k} \left(U_g^n \omega - \nu \frac{d}{dn}\omega \right)(s)\, ds = 0 \tag{1.4}
$$

which means that the total flux (convective and diffusive) of the vorticity through the contour C_k should be zero. In particular, on an impermeable boundary we have simply

$$\oint_{C_k} \frac{d}{dn} \omega \, ds = 0. \tag{1.5}$$

It is interesting that the stochastic vortex method developed by Styczek automatically ensures the equality (1.5) in the case of an external flow around a single contour. If the geometry is more complicated, the conditions (1.4) or (1.5) must be stated explicitly. However, direct implementation of the above equalities requires sufficient regularity of the vorticity field. In the considered method the vorticity is a piecewise constant function of space variables and its normal derivative on the boundary is not properly defined. We show that this difficulty can be overcome by writing explicitly the conditions for the balance between the vorticity production and vorticity flux across the boundaries during one time step.

2. Formulation of the problem

We consider the viscous liquid motion in the exterior of the spatially periodic system of airfoils. The period of the cascade geometry and of the flow field is assumed to be 2π. The inlet line is identified with y-axis. The computational domain is a strip region shown in Fig. 1. Boundary conditions for the velocity field are prescribed on the inlet line segment $\partial \mathbf{D}_W$ and on the contour of the airfoil $\partial \mathbf{D}_P$.

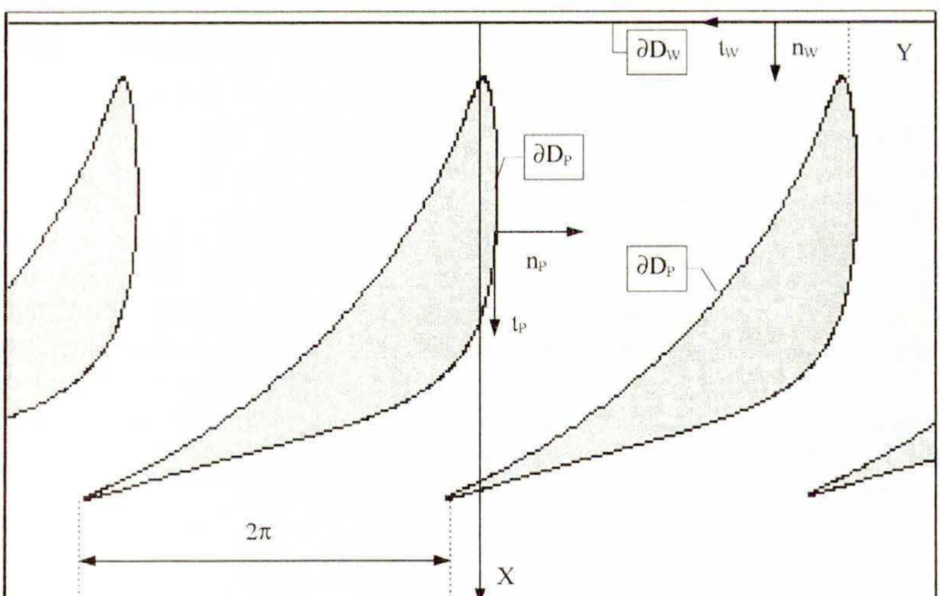


FIG. 1. The computational domain.

The mathematical formulation, which is adequate for the vortex method is following:

Determine the velocity $\mathbf{V} = [u(t,x,y), v(t,x,y)]$ and the vorticity

$$\omega = \omega(t,x,y) = \partial_x v - \partial_y u$$

satisfying

1) Helmholtz and the continuity equation

$$\begin{aligned} \partial_t\omega + u\partial_x\omega + v\partial_y\omega &= \nu\delta\omega, \\ \partial_x u + \partial_y v &= 0; \end{aligned}$$

2) conditions of y-periodicity

$$\begin{aligned} u(t,x,y+2\pi k) &= u(t,x,y), \\ v(t,x,y+2\pi k) &= v(t,x,y), \\ \omega(t,x,y+2\pi k) &= \omega(t,x,y), \qquad k = \ldots,-2,-1,0,1,2,\ldots; \end{aligned}$$

3) boundary conditions

$$\begin{aligned} u\Big|_{\partial D_P} &= 0, & v\Big|_{\partial D_P} &= 0, \\ u\Big|_{\partial D_W} &= u_W(y), & v\Big|_{\partial D_W} &= v_W(y). \end{aligned}$$

This formulation is purely kinematic – the pressure has been eliminated, but it can be recovered *a posteriori* from the velocity and vorticity fields. The results of such calculations are physically sensible provided that the velocity and vorticity were constructed taking pressure correctness conditions (1.4) into account.

3. Elements of the numerical method

3.1. Y-periodic vortex blob

The velocity field induced by the vortex blobs must be y-periodic. To satisfy this demand we use y-periodic vortex blobs (PVB) which are simply infinite, y-periodic systems of ordinary vortex blobs (with identical radii ϱ and charge of vorticity Γ) uniformly spaced with the distance 2π along straight lines parallel to the y-axis. The position of a PVB is a pair (x_0, y_0) of the coordinates of this vortex blob in the system which is located in the computational domain. The velocity induced by a PVB is given by the following expressions

$$(3.1) \qquad V_{\text{ind}} = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \frac{\Gamma}{2\pi i}\,\frac{1}{z-(z_0+2\pi n)} = \frac{\Gamma}{4\pi i}\coth\left(\frac{z-z_0}{2}\right) \\[2ex] \displaystyle\frac{\Gamma}{4\pi i}\coth\left(\frac{z-z_0}{2}\right) - \frac{\Gamma}{2\pi i}\,\frac{1}{z-(z_0+2n\pi i)} + \frac{\Gamma}{2\pi i}\,\frac{\overline{z-(z_0+2\pi i)}}{\varrho^2}. \end{cases}$$

We apply here a convenient complex notation. The upper formula is used when the point $z = x + iy$ is located outside the PVB's vortex cores and is nothing more than the well known formula for spatially periodic system of point vortices (see [9]). The lower formula is applied when the point z happens to drop inside the n-th vortex core of the PVB, the center of which is $z_0 + 2n\pi i$, $z_0 = x_0 + iy_0$.

The velocity field induced by a PVB has an important asymptotic property, namely

$$
(3.2) \qquad \begin{aligned}
&\lim_{x\to\infty} V_{\text{ind}} = \frac{\Gamma}{4\pi i} \quad \Rightarrow u_{\text{ind}} \to 0, \qquad v_{\text{ind}} \to \frac{\Gamma}{4\pi},\\
&\lim_{x\to-\infty} V_{\text{ind}} = -\frac{\Gamma}{4\pi i} \Rightarrow u_{\text{ind}} \to 0, \qquad v_{\text{ind}} \to -\frac{\Gamma}{4\pi}.
\end{aligned}
$$

Thus, if we consider the induction of a system of PVBs, then the behaviour of the velocity at infinity is determined by the total vorticity charge of this system – in particular the velocity vanishes at infinity only when the total charge of vorticity is zero. This is an important difference as compared with any finite system of vortex blobs, where the velocity at infinity tends to zero in any case.

3.2. Y-periodic ideal fluid flow

We are going to construct the total velocity field as a sum of several components. Some of them carry vorticity, the other are potential. It is reasonable to consider separately an ideal liquid flow since it provides a natural way to satisfy a part of boundary conditions on the inlet line and to prescribe the velocity at infinity. Then the following mathematical problem is to be solved:

Determine the potential of the velocity Φ_P such that:

1) Φ_P is a harmonic function in the domain D;

2) the velocity $\mathbf{V}_P = \nabla\Phi_P$ is y-periodic i.e.

$$\mathbf{V}_P(t, x, y + 2k\pi) = \mathbf{V}_P(t, x, y), \qquad k = 0, \pm 1, 2, \ldots;$$

3) the Neumann boundary condition is satisfied:

$$\frac{d\Phi_P}{dn} = \begin{cases} 0 & \text{on} \quad \partial D_P,\\ u_W(y) & \text{on} \quad \partial D_W, \end{cases}$$

where $u_W(y) = u_W(y + 2k\pi)$, $k = 0, \pm 1, 2, \ldots;$

4) the circulation of $\mathbf{V}_P$ along the inlet ∂D_W is given

$$\Gamma\Big|_{\partial D_W} = -\int\limits_{\partial D_W} v_W(y)\,dy.$$

Vector $[u_W, v_W]$ denotes the given velocity distribution on ∂D_W. It is convenient to seek Φ_P in the form of

$$\Phi_P(x,y) = \overline{u}_\infty x + \overline{v}_\infty y + \frac{\Gamma_P}{4\pi} y + \Gamma_P \Phi_C + \Phi_1 + \Gamma_P \Phi_2 \tag{3.3}$$

where meanings of the symbols are the following:

$$\overline{u}_\infty = \frac{1}{2\pi}\int_0^{2\pi} u_W(y)\,dy, \qquad \overline{v}_\infty = \frac{1}{2\pi}\int_0^{2\pi} v_W(y)\,dy, \tag{3.4}$$

Γ_P – the circulation of an airfoil-connected vortex, Φ_C – the potential of the velocity field induced by a unitary airfoil-connected vortex, defined as

$$\Phi_C = \mathrm{Re}\left[\frac{1}{2\pi i}\,\mathrm{Ln}\sinh\frac{z - z_C}{2}\right]. \tag{3.5}$$

Φ_1, Φ_2 – additional y-periodic harmonic functions, their derivatives vanishing at infinity. The potential Φ_P fulfils the imposed boundary conditions if

$$\frac{d}{dn}\left(\overline{u}_\infty x + \overline{v}_\infty y + \Phi_1\right) = \begin{cases} 0 & \text{on } \partial D_P, \\ u_W(y) & \text{on } \partial D_W, \end{cases}$$

(3.6) and

$$\frac{d}{dn}\left(\frac{y}{4\pi} + \Phi_C + \Phi_2\right) = 0 \qquad \text{on } \partial D_P \cup \partial D_W.$$

Thus we obtain the following Neumann conditions for Φ_1 and Φ_2:

$$\frac{d\Phi_1}{dn} = \begin{cases} -\overline{\mathbf{V}}_\infty \cdot \mathbf{n} & \text{on } \partial D_P, \\ u_W(y) - \overline{u}_\infty & \text{on } \partial D_W, \end{cases}$$

$$\frac{d\Phi_2}{dn} = -\frac{d\Phi_C}{dn} - \frac{n_y}{4\pi} \qquad \text{on } \partial D_P \cup \partial D_W, \tag{3.7}$$

where $\mathbf{n} = [n_x, n_y]$ is the internal normal vector on the boundary.

Assume that the functions Φ_1 and Φ_2 have been already determined. Then the differentiation of Φ_P on the boundary yields

$$V_t(s) = \overline{\mathbf{V}}_\infty \cdot \mathbf{t} + \frac{d\Phi_1}{ds} + \Gamma_P\left(\frac{t_y}{4\pi} + \frac{d\Phi_C}{ds} + \frac{d\Phi_2}{ds}\right), \tag{3.8}$$

where $\mathbf{t} = [t_x, t_y]$ denotes the tangent vector on the boundary and s is the arc length coordinate. If we assume that the value $s = 0$ corresponds to the rear stagnation point then the condition $V_t(0) = 0$ yields the circulation of the airfoil-connected vortex Γ_P

$$\Gamma_P = \frac{\overline{\mathbf{V}}_\infty \cdot \mathbf{t} + \dfrac{d}{dt}\Phi_1}{\dfrac{1}{4\pi}t_y + \dfrac{d}{ds}(\Phi_C + \Phi_2)}. \tag{3.9}$$

Now we describe briefly the mathematical technique we apply to find the harmonic functions Φ_1 and Φ_2. If we consider a y-periodic function Φ, harmonic in the domain D and such that

$$\int_{\partial D} \frac{d\Phi(Q)}{dn_Q} ds_Q = 0 \qquad \text{and} \qquad \int_{\partial D} \frac{d\Phi(Q)}{ds_Q} ds_Q = 0$$

i.e. it is not a real or an imaginary part of any multivalued complex function, then the function Φ is the only solution of the boundary integral equation (see [9] and [10])

$$\Phi(P) + \frac{1}{\pi}\int_{\partial D} \operatorname{Re}\left(\frac{1}{2}\coth\frac{z_Q - z_P}{2}\cdot n_Q\right)\Phi(Q)\, ds_Q$$

$$= \frac{1}{\pi}\int_{\partial D} \operatorname{Re}\left(\operatorname{Ln}\sinh\frac{z_Q - z_P}{2}\right)\frac{d\Phi(Q)}{dn_Q}\, ds_Q\,, \tag{3.10}$$

$$n_Q = (n_x + in_y)(s_Q).$$

This is the Fredholm second kind integral equation. If the curvature of the boundary is finite, the kernel is bounded. This equation can be solved numerically using, for instance, the Boundary Element Method.

Having the boundary distribution of the function Φ, we can calculate the vector field $\mathbf{V} = \nabla\Phi$ using the following procedure. First we determine the boundary value of this field

$$V(s_P) = \left(\frac{d\Phi}{ds} - i\frac{d\Phi}{dn}\right)(s_P)\cdot t^*(s_P), \qquad t^*(s_P) = (t_x - it_y)(s_P), \quad P \in \partial D.$$

Next we are able to calculate $V(z)$ for any complex $z = x + iy$ by means of the y-periodic Cauchy integral

$$V(z) = \frac{1}{4\pi i}\int_{\partial D} V(\zeta)\coth\frac{\zeta - z}{2}d\zeta\,. \tag{3.11}$$

It is important that the solution of the boundary integral equation defines the mapping

$$L : \frac{d\Phi}{dn} \to \frac{d\Phi}{ds} \tag{3.12}$$

i.e., in the hydrodynamic context, it transforms the normal velocity to the tangent one. This mapping is a unique, linear and continuous operator. We will use this operator in the next section while deriving an equation for the boundary vortex layer.

3.3. Construction of the complete velocity field

The full velocity of a viscous, y-periodic flow is expressed as

$$\mathbf{V} = \mathbf{V}_P + \mathbf{V}_O + \mathbf{V}_W + \mathbf{V}_A + \mathbf{V}_C + \mathbf{V}_V \,. \tag{3.13}$$

In (3.13) we denote

- $\mathbf{V}_P$ the velocity of the potential flow (previous section),
- $\mathbf{V}_O$ the velocity induced by old i.e. previously created PVBs,
- $\mathbf{V}_W$ the velocity induced by new, boundary PVBs,
- $\mathbf{V}_C$ the velocity induced by an additional, airfoil-connected vortex with the circulation Γ_C,
- $\mathbf{V}_A$ an additional potential velocity field vanishing at infinity,
- $\mathbf{V}_V$ a uniform, vertical stream i.e. $\mathbf{V}_V = [0, v_V]$.

All the velocity components are y-periodic vector fields. In each time step the following unknowns should be calculated:

1) the circulations of new PVBs $\{\gamma_1, \ldots, \gamma_N\}$,
2) the circulation Γ_C,
3) the vertical flow v_V,
4) the potential velocity field V_A.

The role of all unknowns will be explained further on. In general, new PVBs and the velocity V_A are necessary to fulfil the boundary conditions for the velocity. Additional "free parameters" Γ_C and v_V are included in order to satisfy a condition at infinity and to ensure correctness of the pressure.

The velocity decomposition written in natural coordinates for boundary points on ∂D_W yields

$$\begin{aligned} V_P^t + V_A^t + V_C^t + V_O^t + V_W^t - v_V + v_{\mathrm{IN}} &= 0, \\ V_A^n + V_O^n + V_W^n + V_C^t &= 0. \end{aligned} \tag{3.14}$$

We have taken into account that $V_P^n = u_{\mathrm{IN}}$, $V_V^n = 0$ and $V_V^t = -v_{\mathrm{IN}}$.

Analogously, for the points on the airfoil contour ∂D_P we obtain

$$\begin{aligned} V_P^t + V_A^t + V_O^t + V_W^t + V_C^t + V_V^t &= 0, \\ V_A^n + V_O^n + V_W^n + V_C^n + V_V^n &= 0. \end{aligned} \tag{3.14$'$}$$

This time the equality $V_P^n = 0$ has been used.

From (3.14), the normal velocity V_A^n can be expressed as

$$
\begin{aligned}
V_A^n &= -(V_O^n + V_W^n + V_C^n) && \text{on} \quad \partial D_W\,, \\
V_A^n &= -(V_O^n + V_W^n + V_C^n + V_V^n) && \text{on} \quad \partial D_P\,.
\end{aligned}
\tag{3.15}
$$

The boundary operator L applied to V_A^n gives V_A^t expressed in terms of other velocity components. This results in the following equation

$$
\begin{aligned}
V_P^t + V_O^t + V_W^t + V_C^t - v_V + v_{\mathrm{IN}} - L(V_O^n + V_W^n + V_C^n) &= 0 \quad \text{on} \ \partial D_W, \\
V_P^t + V_O^t + V_W^t - L(V_O^n + V_W^n + V_C^n + V_V^n) &= 0 \quad \text{on} \ \partial D_P.
\end{aligned}
\tag{3.16}
$$

We call (3.16) *the equation of the boundary vortex layer* since the unknown here is the distribution of the vorticity (circulation) generated on the boundary. We approximate this vortex layer by a finite set of PVBs located on the boundary and inducing the velocity component $\mathbf{V}_W$. The circulations $\{\gamma_1, ..., \gamma_N\}$ of these PVBs are to be determined. Since new PVBs are born always in the same positions, we can introduce two sets of functions $\{T_i(s),\ \ i = 1, ..., N\}, \{N_i(s),\ \ i = 1, ..., N\}$, which describe tangent and normal velocity distributions induced by the boundary PVBs with unitary circulations. Then the components of $\mathbf{V}_W$ can be written as follows

$$
V_W^t = \sum_{i=1}^{N} \gamma_i T_i(s), \qquad V_W^n(s) = \sum_{i=1}^{N} \gamma_i N_i(s).
\tag{3.17}
$$

Equation (3.16) can be solved in the mean integral sense over a finite set of boundary segments. The division of the boundary lines into segments is quite natural – each boundary PVB overlaps a small part of the inlet line or the airfoil contour. In other words, the boundary is divided into N separate segments, each accompanied by an adjacent PVB. If we now substitute (3.17) to (3.16) and integrate the latter on each segment $\sigma_j = [s_j, s_{j+1}]$ then the following system of linear equations will be obtained:

$$
\begin{aligned}
\sum_{i=1}^{N} \left[\int_{\sigma_j} (T_i - LN_i)(s)\, ds \right] \cdot \gamma_i &= - \int_{\sigma_j} \left(V_P^t + V_O^t + V_C^t + v_{\mathrm{IN}} \right)(s)\, ds \\
&\quad + \int_{\sigma_j} L(V_O^n + V_C^n)(s)\, ds + v_V(s_{j+1} - s_j) \qquad \text{for} \quad \sigma_j \in \partial D_W\,, \\
\sum_{i=1}^{N} \left[\int_{\sigma_j} (T_i - LN_i)(s)\, ds \right] \cdot \gamma_i &= - \int_{\sigma_j} \left(V_P^t + V_O^t + V_C^t + V_V^t \right)(s)\, ds \\
&\quad + \int_{\sigma_j} L(V_O^n + V_C^n + V_V^n)(s)\, ds \qquad \text{for} \quad \sigma_j \in \partial D_P\,.
\end{aligned}
\tag{3.18}
$$

The system (3.18) consists of $N = N_W + N_P$ equations. However, it is not closed since we have two additional unknowns Γ_C (hidden in $\mathbf{V}_C$) and v_V. In order to obtain a solvable problem we have to formulate two equations more.

First we consider the behaviour of the velocity field at infinity. The following condition of asymptotic consistency of nonviscous (potential) flow $\mathbf{V}_P$ and full, viscous flow $\mathbf{V}$ is postulated

$$\lim_{x\to\infty} \mathbf{V} = \lim_{x\to\infty} \mathbf{V}_P \quad \Rightarrow \quad \lim_{x\to\infty} (\mathbf{V}_O + \mathbf{V}_W + \mathbf{W}_C + \mathbf{V}_V) = 0. \tag{3.19}$$

Taking into account the asymptotic formulas (3.2) the condition (3.19) implies that

$$\Gamma_O + \Gamma_W + \Gamma_C + 4\pi v_V = 0. \tag{3.20}$$

The velocity fields $\mathbf{V}_P$ and $\mathbf{V}$ are y-periodic, their circulations along the inlet line ∂D_W are equal and they are asymptotically consistent at infinity. Then, from the Stokes theorem, one concludes that the total charge of the vorticity in the flow is equal to Γ_P. This means, in particular, that the total amount of vorticity in the flow is fixed in time. This conclusion is important for further considerations concerning the pressure condition (1.4). It should be also noticed that total vorticity charge is not identical to total charge of the circulation of PVBs. The reason is that the vortex cores of PVBs have finite dimensions, and some of them protrude partly from the computational domain.

Now we focus on the problem of the pressure correctness. In order to obtain physically meaningful pressure field, the total vorticity production on each boundary line must be equal to zero. Since the total charge of the vorticity within the flow is fixed due to the asymptotic consistency condition (3.19), it suffices to consider the vorticity generation process only on one of the boundary lines – it is more convenient to choose ∂D_P.

The amount of the vorticity created on the airfoil contour in one time step is defined as the difference between the contribution of new PVBs located on this contour and the vorticity charge carried by these PVBs which have left the computational domain in the previous time step by penetrating the interior of the airfoil. More precisely, the flux of the vorticity through the airfoil contour emerges for two reasons:

A) some PVBs protruding from the computational domain into the airfoil interior move to different positions,

B) some PVBs (in particular those located closely to the airfoil contour) can jump randomly out of the computational domain – they are eliminated.

Both types of the events mentioned above give rise to the vorticity flux across ∂D_P. However, the direct calculation of this flux (especially due to events of A type) is a rather strenuous problem. Fortunately, we have a very convenient indicator of the vorticity flux – the circulation of the velocity on the airfoil contour. At the beginning of each time step (i.e. before the PVBs' movement), the boundary conditions are satisfied and the velocity circulation on ∂D_P is exactly zero. As a result of PVBs' motion, the boundary distribution of the velocity is slightly perturbed – its circulation on ∂D_P is, in general, different from zero. This variation is related directly to the amount of vorticity which left the flow domain due to PVBs' motion. This amount should be balanced by the contribution of new PVBs generated on ∂D_P at the beginning of the next time step. The mathematical expression for this balance is the following

$$\Omega_W^{n+1}(\partial D_P) = \Gamma_P + \Gamma_C^n + \overline{\Omega}_O^n(D_P) + \Gamma_{\mathrm{OUT}}^n(\partial D_P). \tag{3.21}$$

In (3.21) we have used the following notation:

$\Omega_W^{n+1}(\partial D_P)$ – the contribution of new PVBs (i.e. created at the beginning of the $(n+1)$-th time step) on ∂D_P,

$\overline{\Omega}_O^n(D_P)$ – the amount of vorticity carried by old PVBs, which sticks out from the computational domain or, equivalently, is inside the airfoil D_P,

$\Gamma_{\mathrm{OUT}}^n(\partial D_P)$ – the sum of circulations of PVBs removed from the computational domain because they have penetrated into D_P.

Now, the following equality holds

$$\Omega_W^{n+1}(\partial D_P) = \Gamma_W^{n+1}(\partial D_P) - \overline{\Omega}_W^{n+1}(D_P), \tag{3.22}$$

where Γ_W^{n+1} denotes the sum of circulations of new PVBs on ∂D_P, while $\overline{\Omega}_W^{n+1}(D_P)$ denotes the amount of vorticity carried by these PVBs, but sticking out from the flow domain into D_P.

From (3.21) and (3.22) we derive the equation

$$\Gamma_W^{n+1}(\partial D_P) = \Gamma_P + \Gamma_C^n + \Gamma_{\mathrm{OUT}}^n(\partial D_P) + \overline{\Omega}_O^n(D_P) + \overline{\Omega}_W^{n+1}(D_P). \tag{3.23}$$

The velocity field $\mathbf{V}$ fulfils the boundary conditions at the beginning of the $(n+1)$-th time step. Thus its circulation along ∂D_P is equal to zero, which implies that

$$\Gamma_P + \Gamma_C^{n+1} + \overline{\Omega}_O^n(D_P) + \overline{\Omega}_W^{n+1}(D_P) = 0. \tag{3.24}$$

The last equation allows for eliminating troublesome quantities $\overline{\Omega}_W^{n+1}(D_P)$ and $\overline{\Omega}_O^n(D_P)$. Finally we obtain the equation involving only the circulations

$$\Gamma_W^{n+1}(\partial D_P) = \Gamma_{\mathrm{OUT}}^n(\partial D_P) - (\Gamma_C^{n+1} - \Gamma_C^n). \tag{3.25}$$

Equations (3.20) and (3.25) supplement the system (3.18) giving together a solvable algebraic problem. However, it is interesting to show that Eq. (3.20) can be replaced by the other one, which is, in a certain sense, symmetric to Eq. (3.25).

If we subtract Eq. (3.20) written for the n-th time step from the same equation but written for the next, $(n+1)$-th step, then the result will be as follows:

$$\Gamma_O^{n+1} - \Gamma_O^n + \Gamma_W^{n+1} - \Gamma_W^n + \Gamma_C^{n+1} - \Gamma_C^n + 4\pi(v_V^{n+1} - v_V^n) = 0. \tag{3.26}$$

Now, from Eq. (3.25) we have

$$\Gamma_C^{n+1} - \Gamma_C^n = \Gamma_{\mathrm{OUT}}^n(\partial D_P) - \Gamma_W^{n+1}(\partial D_P). \tag{3.27}$$

The substitution of (3.27) to (3.26) yields

$$\begin{aligned}\Gamma_O^{n+1} - \Gamma_O^n + \Gamma_W^{n+1} - \Gamma_W^n + \Gamma_{\mathrm{OUT}}^n(\partial D_P) - \Gamma_W^{n+1}(\partial D_P)&\\ +4\pi(v_V^{n+1} - v_V^n) = 0.&\end{aligned} \tag{3.28}$$

Writing the balance of the total charge of circulations of PVBs

$$\Gamma_O^{n+1} = \Gamma_O^n + \Gamma_W^n(\partial D_W) + \Gamma_W^n(\partial D_P) - \Gamma_{\mathrm{OUT}}^n(\partial D_P) - \Gamma_{\mathrm{OUT}}^n(\partial D_W) \tag{3.29}$$

we are able to eliminate Γ_O^{n+1} from (3.28). Moreover, the following equality holds

$$\Gamma_W^{n+1} = \Gamma_W^{n+1}(\partial D_P) + \Gamma_W^{n+1}(\partial D_W). \tag{3.30}$$

After substitution of (3.29) and (3.30) to (3.26) most of the terms cancel and we end up with the following, simple condition

$$\Gamma_W^{n+1}(\partial D_W) = \Gamma_{\mathrm{OUT}}^n(\partial D_W) - 4\pi(v_V^{n+1} - v_V^n). \tag{3.31}$$

Summarising, the linear, algebraic system (3.18) can be completed by the pair of additional equations, which read

$$\begin{aligned}\sum_{i=1}^{N_P} \gamma_i^{n+1} &= \Gamma_{\mathrm{OUT}}^n(\partial D_P) - (\Gamma_C^{n+1} - \Gamma_C^n),\\ \sum_{i=N_P+1}^{N_P+N_W} \gamma_i^{n+1} &= \Gamma_{\mathrm{OUT}}^n(\partial D_W) - 4\pi(v_V^{n+1} - v_V^n).\end{aligned} \tag{3.32}$$

These equations are remarkably symmetrical. The first one describes the variation of the airfoil-connected vortex and involves the information concerning only the airfoil contour. The second equation describes the variation of the additional, vertical stream and involves the information concerning only the inlet line. The vortex and the vertical stream provide the mechanism for controlling the vorticity production on the airfoil and on the inlet line, respectively, which in turn ensures physical correctness of the pressure field.

3.4. Summary of the computational algorithm

We summarise briefly main steps of the numerical method. The calculation of each step of the flow evolution begins with the computation of the right-hand sides of the system (3.18). Then the linear equations (3.18) coupled with the pair of Eqs. (3.32) are solved. As a result, the circulations of new PVBs, the airfoil-connected circulation Γ_C and the vertical stream velocity v_V for the new time step are determined. Next the boundary distribution of V_A^n is evaluated from (3.15). The solution of the boundary integral equation yields the value of the harmonic potential Φ_A and, after differentiation, the tangent velocity V_A^t. This way the complete velocity $\mathbf{V}_A$ on the boundary is known and can be reconstructed in the flow domain (in particular in PVBs centres) via y-periodic Cauchy integral. Other components of the velocity field can be calculated directly from the induction formulas (1.4) ($\mathbf{V}_O$, $\mathbf{V}_W$ and $\mathbf{V}_C$) or are determined in advance ($\mathbf{V}_P$) and interpolated to PVB centres from nodes of an auxiliary grid.

The key problem is the computational efficiency. Actually direct application of the induction formulas for all PVBs leads to enormous computational cost excluding the possibility to perform computations on widely available, small computers. A natural way to overcome this difficulty is to calculate the induced velocity only in the grid points and then interpolate it. However, two problems appear immediately. First, the interpolation of velocity smooths out fine, local variations, which can remove important details of the flow pattern. Secondly, the velocity interpolation should be divergence-free. To avoid these problems we applied a hybrid approach – the interaction between close PVBs is calculated from exact formulas (3.1), while distant induction is determined via an interpolation. The interpolating algorithm is based on the fact that the velocity induced by a PVB is potential outside the vortex core. Thus we can calculate the complex potential function of this velocity in grid points and then interpolate it in grid cells by complex polynomials to obtain, after differentiation, a divergence-free approximation of the velocity. This method has an obvious disadvantage – the approximate velocity field is not continuous on the cell sides. In other words, the approximation of the velocity is divergence-free only in a weak sense. This difficulty can be partly cured by using more complicated, Hermitean interpolation algorithms.

Now the problem of initial condition will be considered shortly. While dealing with external flows we have generally two possibilities:

1) sudden "switching on" of the viscosity, or

2) continuous acceleration from the state of rest.

In the first case the viscosity suddenly appears in an ideal liquid flow, which causes first generation of the vortex particles to be created. In the second one, the flow is viscous from the very beginning and is progressively accelerated by changing the free stream velocity. Both methods have certain good and weak points. The first one is not physical and, which is much worse, the primary generated vortex particles are charged with relatively large circulations – they can induce locally a velocity comparable in magnitude with the free stream velocity. The second method is more natural but during acceleration one has to deal with a more complicated version of the pressure problem. If the acceleration is performed by rescaling the "ideal-flow background" $\mathbf{V}_P$, then the total amount of the vorticity in the flow field changes in time and this fact must be taken into account while formulating the pressure correctness condition. Our choice is the first method supplemented by the concept of vortex particle splitting. The idea is to limit the permissible value of the velocity induced by a single PVB to small fraction of the free-stream velocity, say to several percent. This means that every PVB born on the boundary, which is too "strong", is immediately split into a number of "weaker" PVBs moving separately (their trajectories diverge since they perform separate random walks). Although this procedure brings rapid increase of the number of PVBs, in the computations it has also significant advantages. It provides fast saturation of the computational domain with the vortex particles which is desirable when one is interested mainly in the final, quasi-stationary state, not a transient one.

Another important problem is associated with artificial or numerical viscosity. Although the vortex methods are, at least in principle, grid-free, the built-in vorticity discretization produces inevitably an effect of additional, nonphysical diffusion rate. This phenomenon is connected with two parameters of vortex particles, theoretically infinitesimal, but in practice always finite – a radius and a circulation charge. It is quite obvious that the radii of the PVBs vortex cores should be as small as possible – otherwise the method would be unable to resolve fine-scale details of the vorticity and velocity fields. Large PVBs mean that the flow is too organised spatially – relatively large portions of fluid are in regular ("laminar") movement. In the language of modern dynamical system theory, the number of degrees of freedom of such flow is too small – the corresponding, effective "viscosity" is larger than that assumed in the random walk process. Similar effect is obtained when the vortex particles are too "strong". Regions of weak vorticity cannot be reproduced properly, the vorticity gradients are exaggerated and strong, local variations of the induced velocity make PVBs to spread rapidly in all directions like in a diffusion process. It should be emphasised that the above description is only a simple heuristics – no systematic investigation of the artificial viscosity in vortex methods is known to the authors. The practical experience says that the limit of the induced velocity on the level of several percent is sufficient in a sense, that further splitting of PVBs does not make any visible effects on the velocity and vorticity field. Nevertheless, the "real" Reynolds number obtained in our simulations is surely lower than the "theoretical" one resulting from the assumed value of the viscosity.

4. Results of numerical computations

The general data chosen in sample calculations are the following:

- The inlet line ∂D_W is divided uniformly into 120 segments while the airfoil

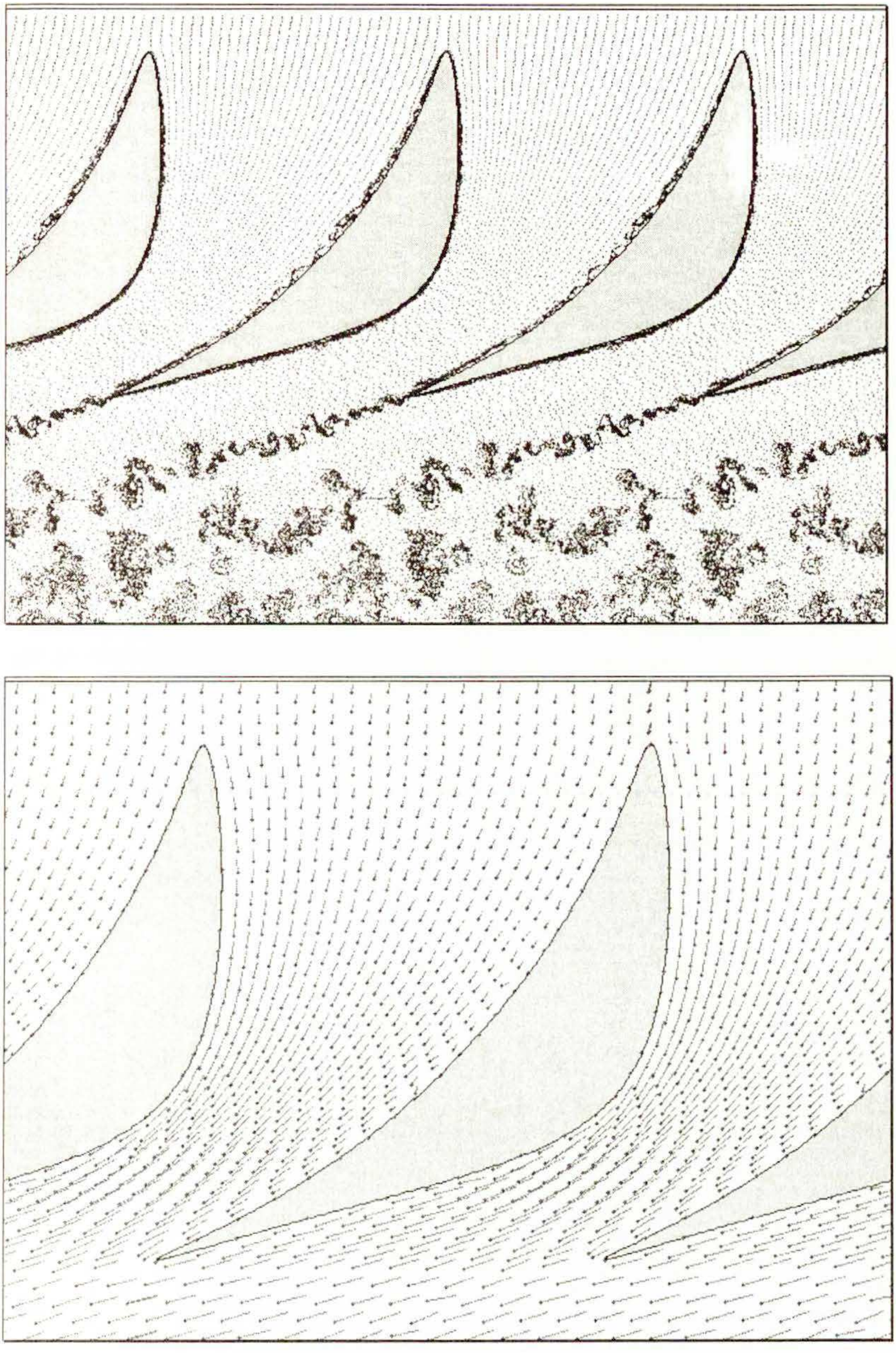


FIG. 2. The PVBs and instantaneous velocity field for t = 12.0 (case I).

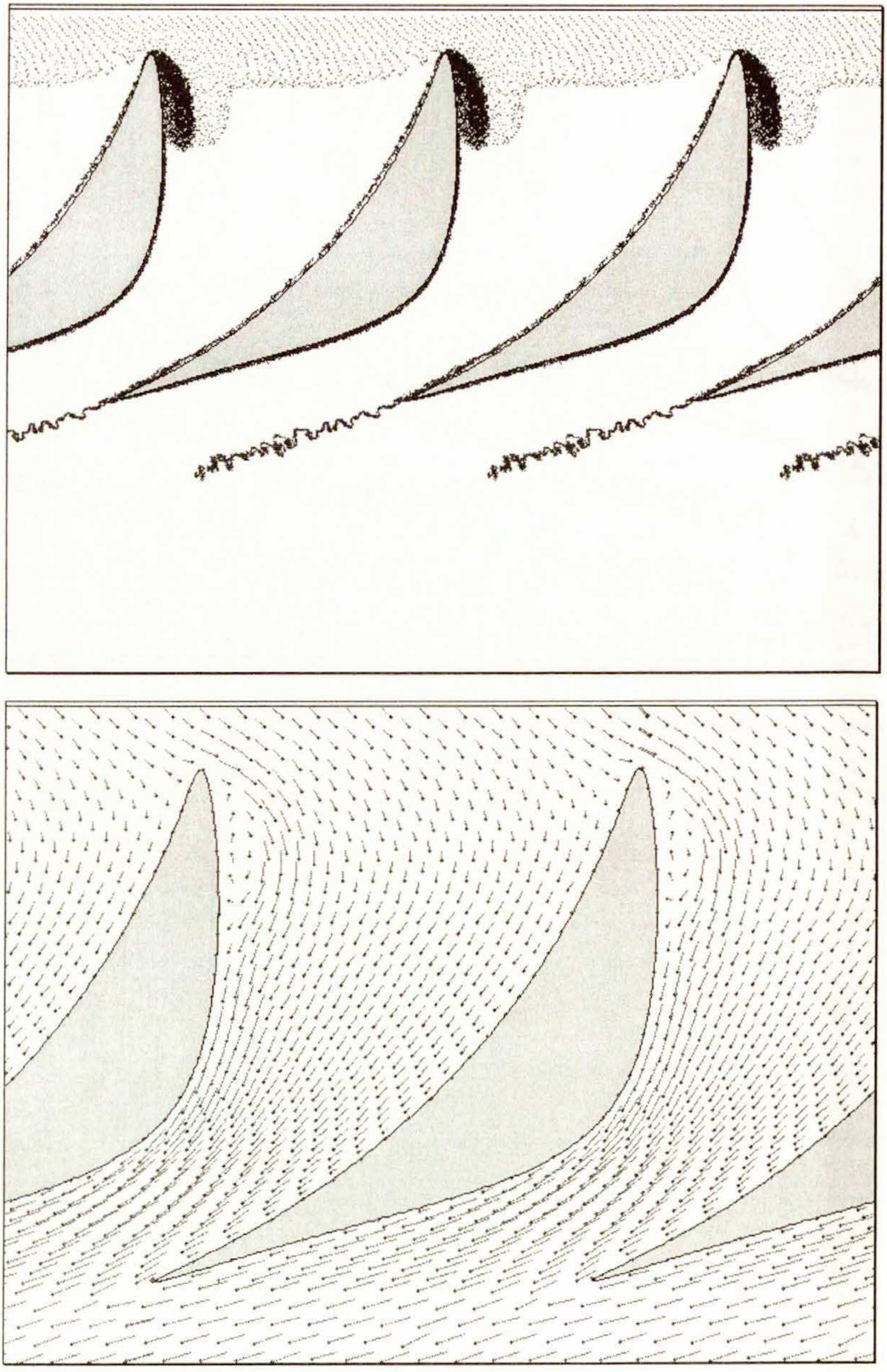
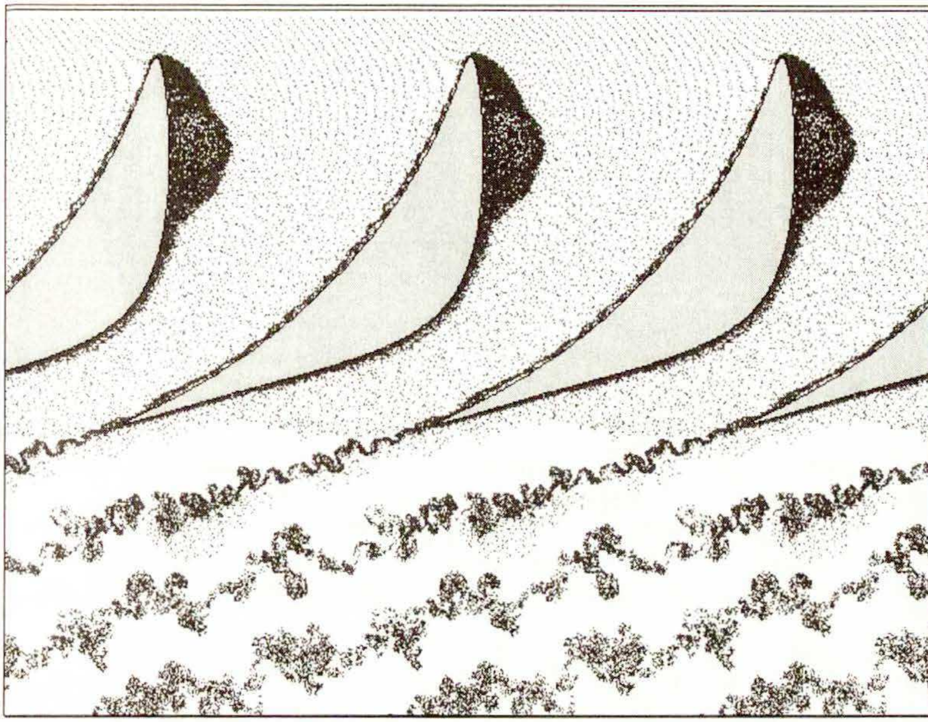


FIG. 3 a. The PVBs and instantaneous velocity field at $t = 2.0$ (case II).



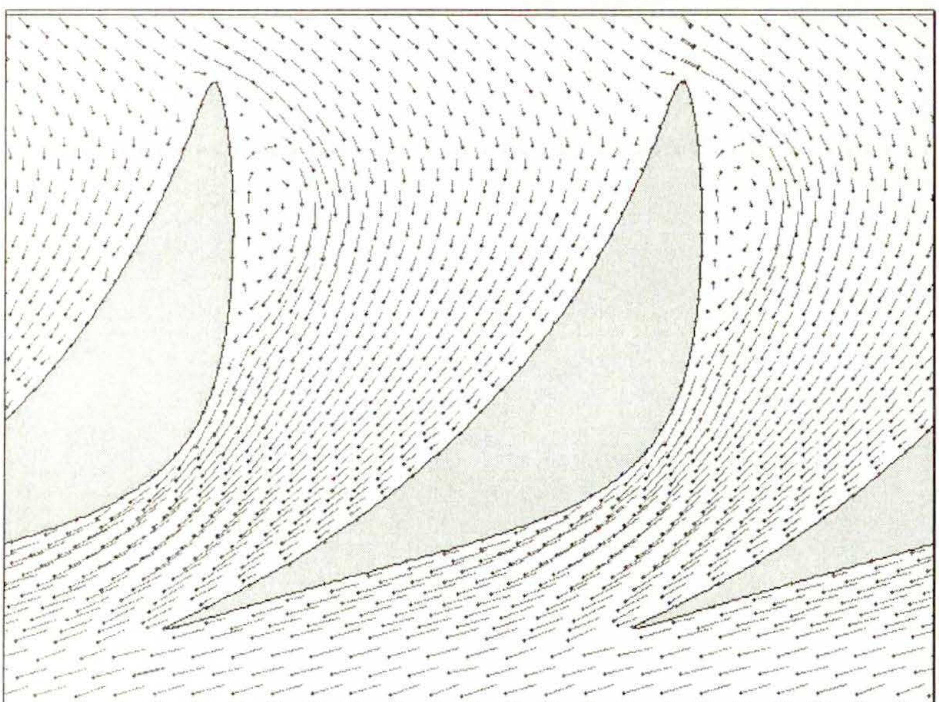


FIG. 3 b. The PVBs and instantaneous velocity field at $t = 6.0$ (case II).

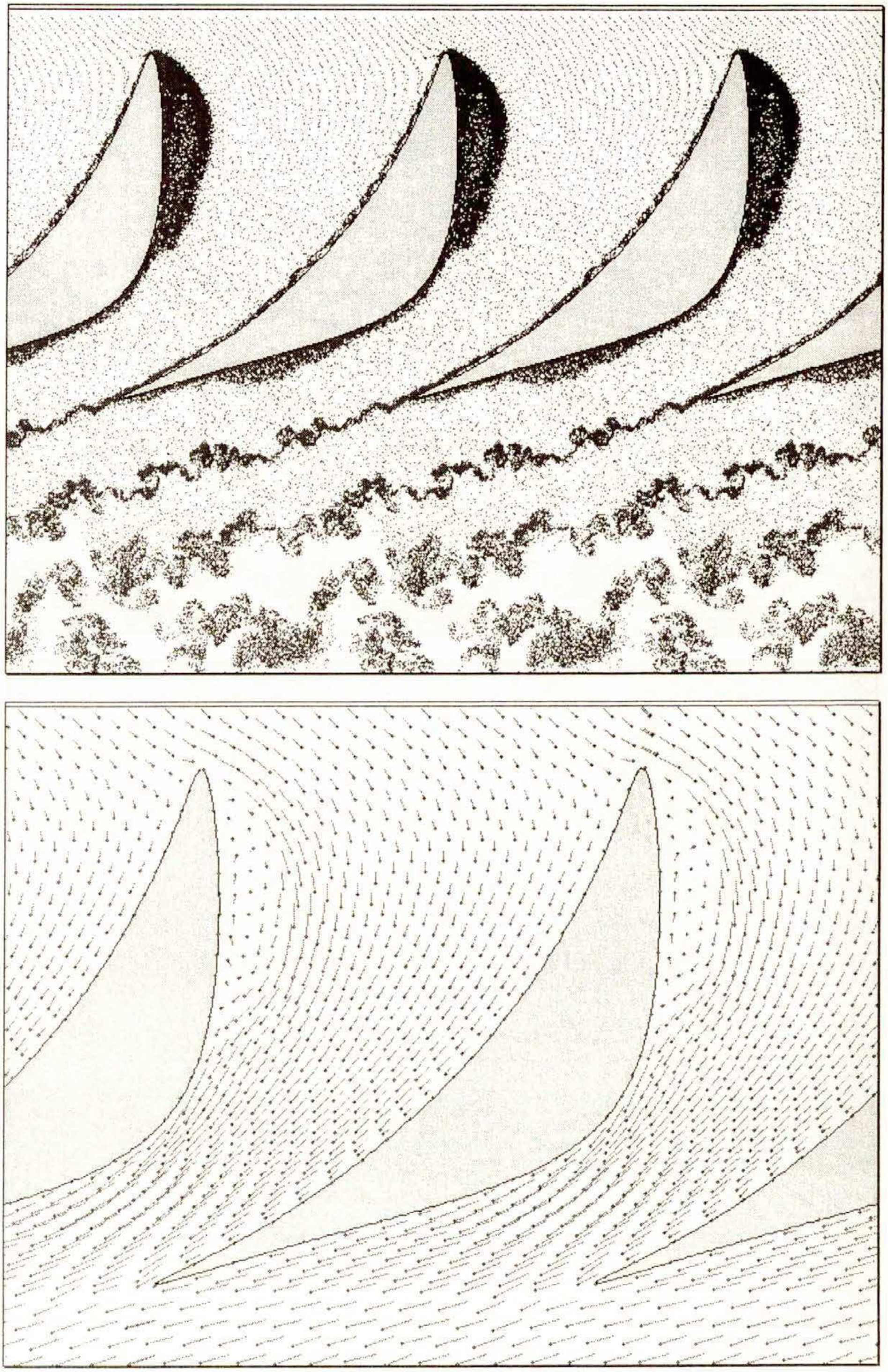
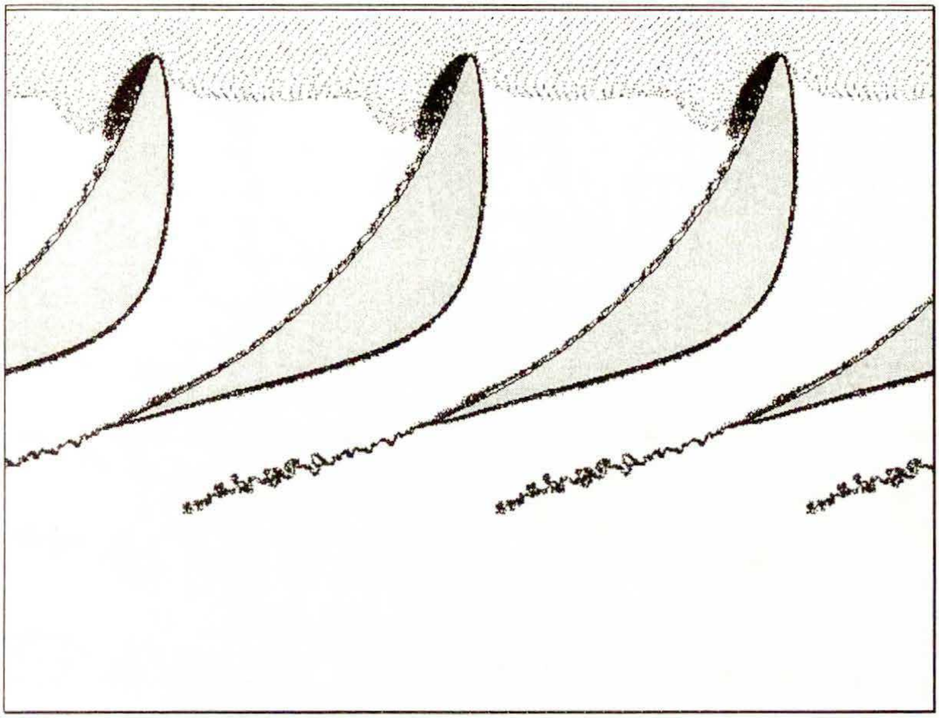


FIG. 3 c. The PVBs and instantaneous velocity field at $t = 12.0$ (case II).



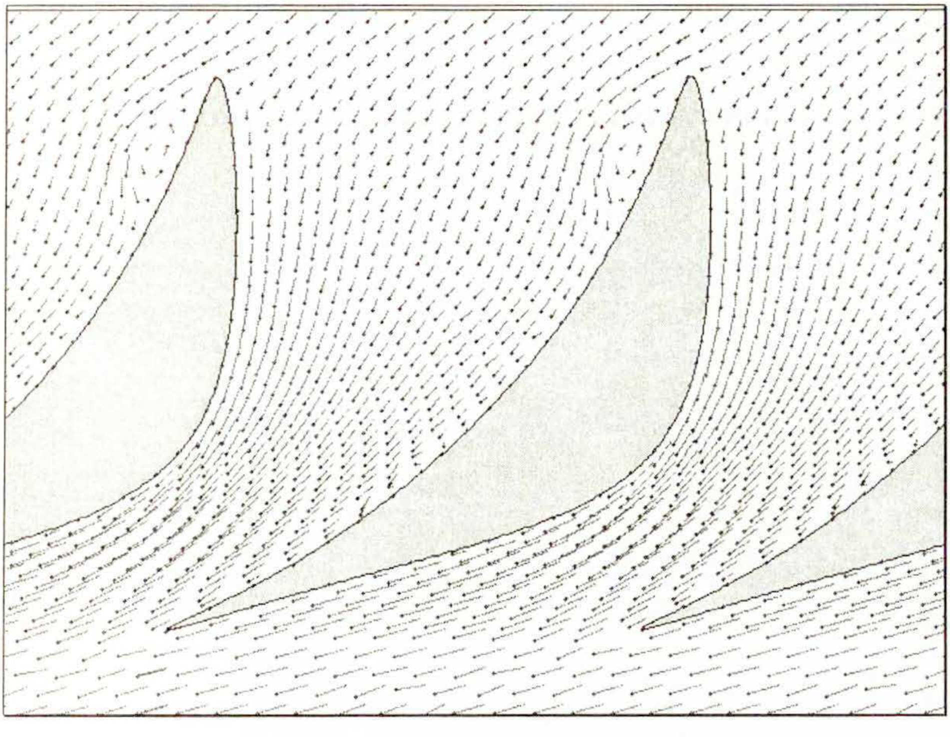


FIG. 4 a. The PVBs and instantaneous velocity field at $t = 2.0$ (case III).

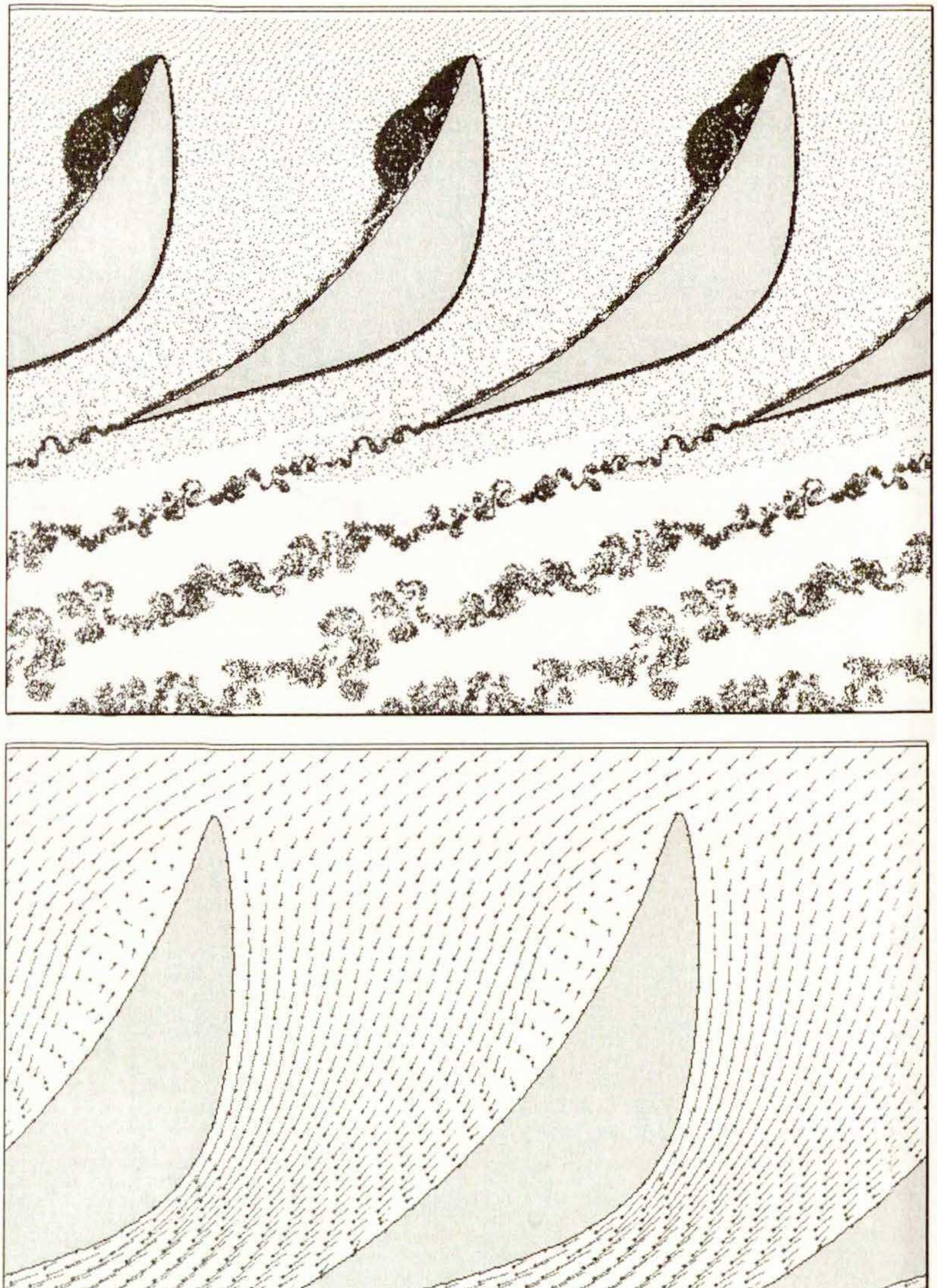


FIG. 4 b. The PVBs and instantaneous velocity field at $t = 6.0$ (case I).

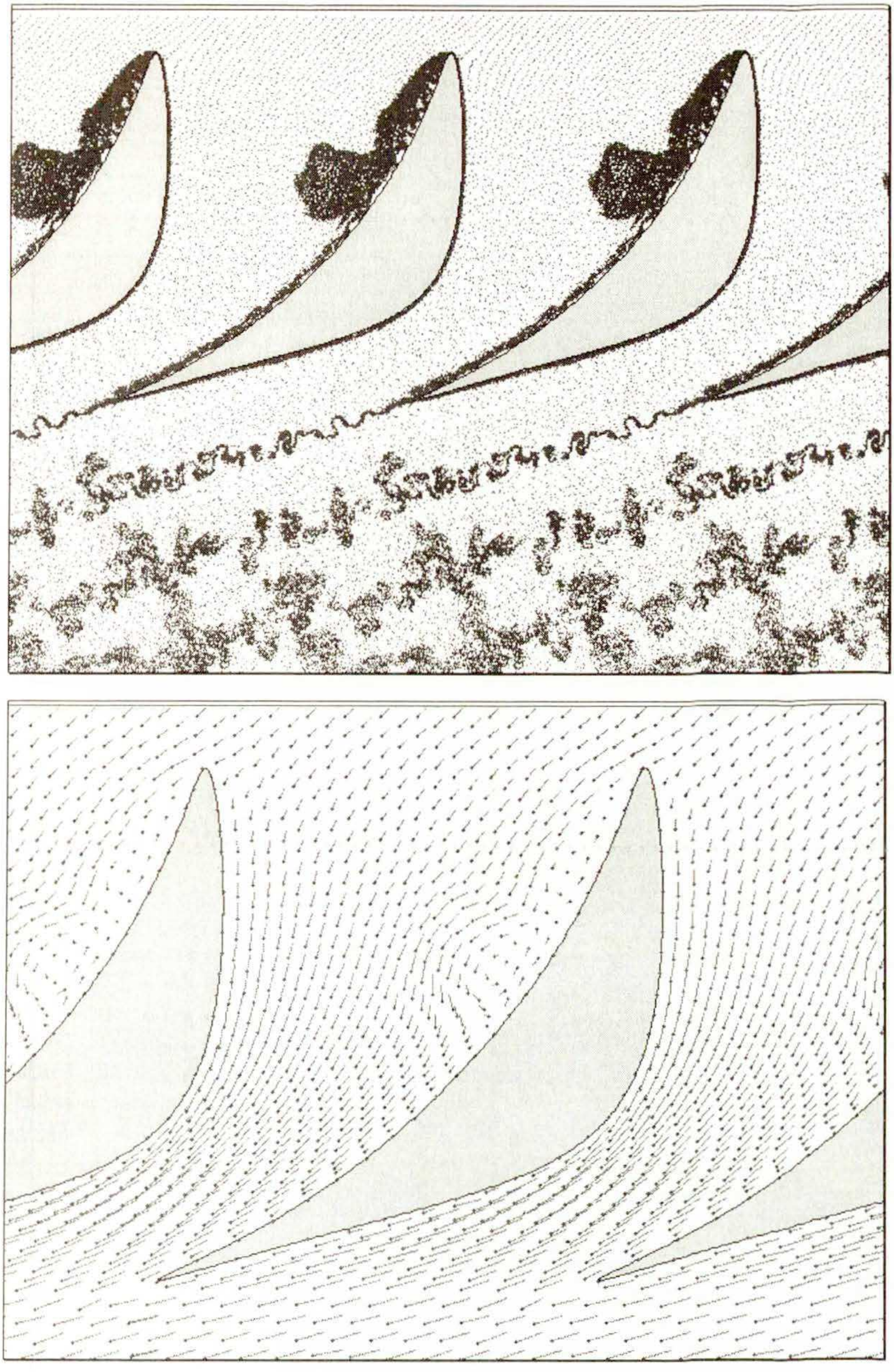


FIG. 4 c. The PVBs and instantaneous velocity field at $t = 12.0$ (case III).

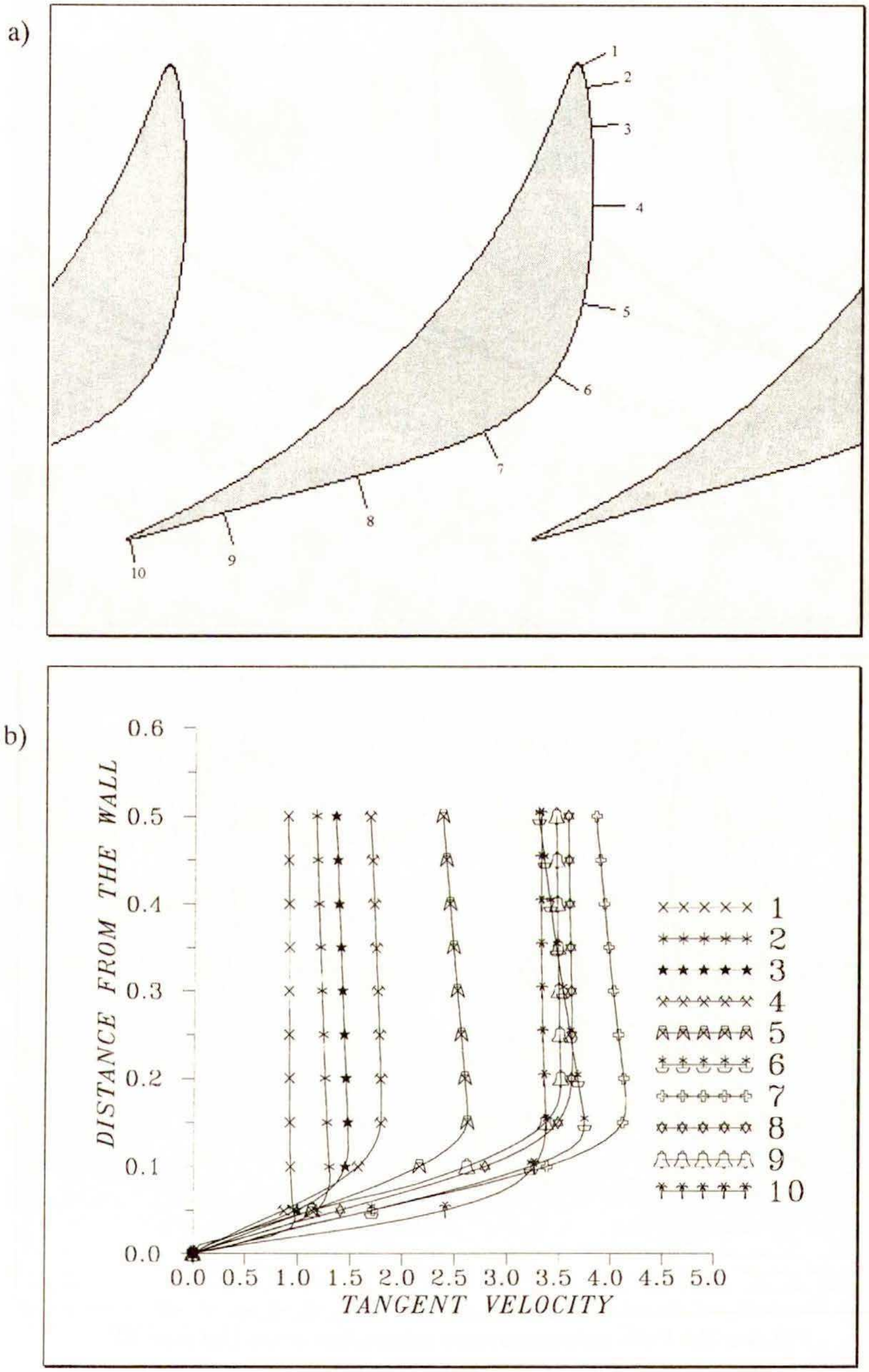


FIG. 5. The averaged boundary layer velocity field distribution calculated at indicated positions.

contour ∂D_P into 520 segments. The dimension of the algebraic system connected with the boundary integral equation is 640;

• Each PVB of next generation is adjacent to four subsequent segments, hence $N_W = 30$ and $N_P = 130$;

• The inlet line velocity distribution is uniform and fixed in time;

• Reynolds number calculated on the basis of a characteristic length (the chord of the airfoil), the assumed viscosity and the value of the inlet velocity is approximately 10^5;

• Time step is fixed ($\Delta t = 0.05$) and the Ito equations are solved by the Euler integration scheme.

Three cases of flow with different inlet conditions are presented:

1) low angle of incidence flow $u_W = 1.0$, $v_W = -0.2$,

2) high positive angle of incidence flow $u_W = 1.0$, $v_W = 1.0$,

3) high negative angle of incidence flow $u_W = 1.0$, $v_W = -1.0$.

The instantaneous positions of PVBs and the velocity field calculated in the first case are presented in Fig. 2. Analogous results for the second case are shown in Fig. 3 and, for the third case, in Fig. 4. In all cases the growth of vortical structures in wakes is apparent. In the cases of a high angle of attack, the closed separation regions appear and evolve in time. Figure 5 a shows locations of the sections perpendicular to the airfoil contour, where the averaged velocity distributions of the boundary layer were calculated in the first case. The computed velocity profiles are shown in Fig. 5 b.

5. Concluding remarks

The stochastic vortex method proposed above seems to be capable of reproducing characteristic features of nonstationary viscous flows in spatially periodic domains. The effect of local separation has been captured and the velocity distribution in the boundary layer exhibits reasonable qualitative features. The boundary layer thickness is, however, much exaggerated. The reason is that the characteristic dimension of PVBs is of the same order (or even grater) as this thickness at the considered Reynolds number. Obviously, flow details of such a spatial scale cannot be properly resolved. It can be expected that significant improvement would be achieved if the number of PVBs were much greater and their vortex cores were much finer. Also some other types of vortex particles (like y-periodic vortex sheets) could be applied in the vicinity of the airfoil contours.

Although only stationary inlet velocity distributions are considered here, it is not difficult to generalise the method to nonstationary or even random inlet conditions. Such generalisation would allow us to perform approximate calculations of multi-stage cascade flows: the velocity behind a row of blades and relative movement of the rows would yield the nonstationary inlet conditions for the next row. Randomness of the inlet conditions can be applied to simulate turbulent fluctuations in an incoming stream.

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