Slow viscous flow about a permeable circular cylinder

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SLOW STEADY two-dimensional motion of a viscous incompressible fluid about a porous circular cylinder is considered, using Darcy law for the flow in the porous region and Jones conditions on the contour of the cylinder. The problem is formulated in terms of Stokes stream function and velocity, and pressure fields of the modified flow in the presence of porous cylindrical boundary are obtained explicitly. It is observed that the Stokes paradox exists even in this case. Several other illustrative examples are presented to justify the usefulness of the method. It is found that the potential (point) singularities in the presence of a cylinder produce uniform flow at large distances, its strength being independent of porosity. However, the Stokes singularities (such as Stokeslet etc.) produce uniform flow at infinity, and its strength depends on the porosity as well as on the location of those singularities in the presence of the cylinder. The known results in two-dimensional Stokes flow are deduced as special cases from our result.

1. Introduction

THERE EXISTS an extensive literature on two-dimensional creeping flow (Stokes flow) problems, in which the inertial effects are negligible in comparison with the viscous effects in a viscous incompressible fluid. The problem, in general, can be reduced to finding solution of biharmonic equation that represents two-dimensional slow viscous flow past a finite body. It is quite well-known that there is no solution of the biharmonic equation for the streaming flow past a finite body, what is widely known as Stokes paradox. However, the slow streaming flow at large distances from a finite body may be obtained from the solution of the biharmonic equation for locally generated two-dimensional flows in an unbounded fluid. JEFFERY [1] has shown that two rigid circular cylinders of equal radius, rotating with equal but opposite angular velocities, produce a uniform stream at large distances. DORREPAAL et al. [2] have also explained such phenomenon by considering a rotlet or a Stokeslet in front of a rigid circular cylinder which lead to a uniform flow at infinity. SMITH [3] considered the simplest situation of a single sink positioned in front of a circular cylinder, and concluded that there was a uniform stream in this case also. The solution due to SMITH [3] was also obtained earlier by AVUDAINAYAGAM and JOTHIRAM [4] by an approach similar to that of DORREPAAL et al. [2].

The purpose of the present paper is to discuss the solution of biharmonic equation representing the two-dimensional Stokes flow in the presence of a porous circular cylinder. The corresponding three-dimensional problem with spherical and plane boundaries have been investigated by several authors in different contexts [5–12]. In this paper, we consider a general Stokes flow past a stationary infinite circular porous cylinder (using Darcy model) in a viscous, incompressible

fluid. The velocity and pressure fields in the Stokes region are obtained explicitly from the stream function which satisfies the biharmonic equation. The Darcy region velocity is derived by using the fact that the Darcy pressure satisfies the Laplace equation. The solutions of the two regions are matched at the contour of the cylinder using the boundary conditions due to JONES [13]. It is shown that the Stokes paradox continues to exist with these conditions at the contour of the cylinder. Several illustrative examples are worked out to justify the usefulness of the present method. It is noted that the point singularities located in front of the cylinder produce a uniform stream at infinity, and its speed

1) depends on their location alone in the case of potential singularities;

2) depends on their location as well as porosity in the case of Stokes singularities.

This fact may be due to the validity of the Darcy equations which are restricted to low porosity of the region. The above observation would have to be checked by using Brinkman model equations which are valid for high porosity.

2. Mathematical formulation

Consider the slow steady flow (creeping flow or Stokes flow) of a viscous incompressible fluid past an infinite circular permeable (porous) cylinder (Darcy region) of radius a. For the flow outside the cylinder, the governing equations are the linearised Navier–Stokes equations or simply the Stokes equations

$$(2.1) \qquad \mu\nabla^2\mathbf{q} = \nabla p,$$

$$(2.2) \qquad \nabla\cdot\mathbf{q} = 0.$$

Here $\mathbf{q}$ is the velocity vector with components $(q_r, q_\theta, 0)$ in the radial and transverse directions (r, θ) respectively, p the pressure and μ the coefficient of viscosity of the fluid.

The flow inside the porous infinite cylinder $(0 \leq r \leq a)$ is governed by Darcy's law

$$\mathbf{Q} = -\frac{k}{\mu}\nabla P,$$

$$(2.3)$$

$$\nabla\cdot\mathbf{Q} = 0,$$

where $\mathbf{Q}$ is the volume rate per unit cross-sectional area, P the Darcy pressure and $k > 0$ is the permeability coefficient.

The appropriate boundary conditions on $r = a$ are as follows:

(i) the pressure is continuous across the boundary of the cylinder

$$(2.4) \qquad p(a, \theta) = P(a, \theta);$$

(ii) the radial velocity is continuous at the boundary of the cylinder

(2.5) $$q_r(a, \theta) = Q_r(a, \theta);$$

(iii) JONES condition [13] for tangential velocity on the cylinder is that the tangential stress is proportional to the difference in the tangential velocities of the two regions, i.e

(2.6) $$T_{r\theta}|_{r=a} = \mu \left[\frac{1}{r} \frac{\partial q_r}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{q_\theta}{r} \right) \right]_{r=a} = \frac{\alpha}{\sqrt{k}} [q_\theta - Q_\theta]_{r=a} \, ,$$

where $T_{r\theta}$ is the tangential stress component and α is a parameter which completely depends on the porous medium.

3. Method of solution

It is well-known that the Stokes equations (2.1) and (2.2) in two dimensions, when expressed in terms of stream function, reduce to

(3.1) $$\nabla^4 \psi = 0,$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

and

(3.2) $$q_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \, ,$$

(3.3) $$q_\theta = \frac{\partial \psi}{\partial r} \, ;$$

q_r, q_θ are the components of velocity along r and θ directions, respectively. The general solution of (3.1) in polar coordinates is given by

(3.4) $$\psi = \sum_{n=0}^{\infty} \left[A_n r^n + B_n r^{n+2} + \frac{C_n}{r^n} + \frac{D_n}{r^{n-2}} \right] (\cos n\theta + \sin n\theta),$$

where we have excluded the terms which give nonzero vorticity at infinity. The constants A_n and B_n are assumed to be known and will be determined from the given flow field. For convenience we proceed further with the terms involving $\sin n\theta$ in the Fourier expansion (3.4) only, since the calculation for the other part

involving $\cos n\theta$ is similar. Now the components of velocity and pressure in the Stokes region obtained from (3.2), (3.3) and (2.1) are

$$q_r = -\sum_{n=1}^{\infty} \left[A_n r^{n-1} + B_n r^{n+1} + \frac{C_n}{r^{n+1}} + \frac{D_n}{r^{n-1}} \right] n \cos n\theta,$$

$$(3.5) \quad q_\theta = \sum_{n=1}^{\infty} \left[nA_n r^{n-1} + (n+2)B_n r^{n+1} - \frac{nC_n}{r^{n+1}} - (n-2)\frac{D_n}{r^{n-1}} \right] \sin n\theta,$$

$$p = p_0 - \mu \sum_{n=1}^{\infty} \left[4(n+1)B_n r^n + 4(n-1)\frac{D_n}{r^n} \right] \cos n\theta.$$

In the porous region (i.e $r < a$) the Darcy pressure satisfies the Laplace equation $\nabla^2 P = 0$. Therefore,

$$(3.6) \qquad\qquad P = P_0 + \sum_{n=1}^{\infty} E_n r^n \cos n\theta.$$

The components of velocity inside the porous cylinder in r and θ directions now become

$$(3.7) \quad\begin{aligned} Q_r &= -\frac{k}{\mu}\frac{\partial P}{\partial r} = -\frac{k}{\mu}\sum_{n=1}^{\infty} nE_n r^{n-1} \cos n\theta, \\ Q_\theta &= -\frac{k}{\mu}\frac{\partial P}{r\partial\theta} = \frac{k}{\mu}\sum_{n=1}^{\infty} nE_n r^{n-1} \sin n\theta. \end{aligned}$$

The stream function for the Darcy region may also be defined and given by

$$(3.8) \qquad\qquad \blacksquare$$

where $\nabla^2 \psi^+ = 0$. It should be noted here that in (3.6) we have omitted the terms which do not produce finite velocities at the origin.

The general expressions for the pressure and velocity fields in both the regions will now be solved for the constants C_n, D_n, E_n expressed in terms of A_n and B_n using the boundary conditions (2.4)–(2.6).

Application of the boundary conditions (2.4)–(2.6) in the general solutions yields

$$(3.9) \quad \frac{C_n}{a^{2n}} = \frac{\left((n-1)\frac{\alpha a}{\sqrt{k}} - \frac{4k}{a^2}n(n-1)^2 \right) A_n}{M_n}$$

$$+ \frac{\left(-2n + \frac{\alpha a}{\sqrt{k}}n + 4\frac{a\sqrt{k}}{a}(n-1)(n+2) \right) a^2 B_n}{M_n},$$

(3.9)
[cont.]

$$\frac{D_n}{a^{2n-2}} = -\frac{\left(2 + \dfrac{\alpha a}{\sqrt{k}}\right) n A_n}{M_n}$$

$$+ \frac{\left(\dfrac{\alpha a}{\sqrt{k}} + \dfrac{4\alpha\sqrt{k}}{a}n + \dfrac{4k}{a^2}n(n+1)\right)(n+1)a^2 B_n}{M_n},$$

$$E_n = -\frac{4\mu}{a^2}\left[\frac{-\left(2 + \dfrac{\alpha a}{\sqrt{k}}\right)n(n-1)A_n}{M_n}\right.$$

$$\left. + \frac{\left(2n - \dfrac{\alpha a}{\sqrt{k}}(n-2)\right)(n+1)a^2 B_n}{M_n}\right],$$

where

$$M_n = 2n + \frac{\alpha a}{\sqrt{k}} + 4\alpha\frac{\sqrt{k}}{a}n(n-1) + \frac{4k}{a^2}n(n+1)(n-1).$$

4. Examples

4.1. Uniform flow along OX

For the uniform flow with a speed U along OX, we have

$$q_r = -U\cos\theta, \qquad q_\theta = U\sin\theta$$

and

(4.1)
$$\psi_0 = Ur\sin\theta.$$

Threfore we have $A_1 = U$, $A_n = 0$ for all $n \geq 2$ and $B_n = 0$ for all n. The coefficients C_n, D_n and E_n as calculated from (3.9)–(3.11) are

(4.2)
$$C_1 = 0, \qquad D_1 = -U, \qquad E_1 = 0.$$

This implies $\psi = 0$. Thus a uniform flow about a porous cylinder is not possible, which is the usual Stokes paradox known in the literature.

4.2. Quadratic potenial flow

In this case

(4.3)
$$\psi_0(r,\theta) = -\frac{U}{3}r^3\sin 3\theta$$

($U/3$ is a shear velocity) and $A_1 = A_2 = 0$, $A_3 = -U/3$, $A_n = 0$ for all $n \geq 4$ and $B_n = 0$ for all n. The coefficients C_3 and D_3 are found from (3.9) and are given by

$$C_3 = \frac{\left(2\frac{\alpha a}{\sqrt{k}} - 48\frac{k}{a^2}\right)\left(-\frac{U}{3}\right) a^6}{\left(6 + \frac{\alpha a}{\sqrt{k}} + 24\frac{\alpha\sqrt{k}}{a} + 96\frac{k}{a^2}\right)},$$

(4.4)
$$D_3 = \frac{\left(2 + \frac{\alpha a}{\sqrt{k}}\right) U a^4}{\left[6 + \frac{\alpha a}{\sqrt{k}} + 24\frac{\alpha\sqrt{k}}{a} + 96\frac{k}{a^2}\right]},$$

$$E_3 = -\frac{4\mu}{a^2} U \left[\frac{2\left(2 + \frac{\alpha a}{\sqrt{k}}\right)}{\left(6 + \frac{\alpha a}{\sqrt{k}} + 24\frac{\alpha\sqrt{k}}{a} + 96\frac{k}{a^2}\right)}\right].$$

Now the complete stream function for the two flow fields are given by

(4.5)
$$\psi = \frac{U}{3}\left[-r^3 - \frac{\left(\frac{2\alpha a}{\sqrt{k}} - \frac{48k}{a^2}\right)}{\left(6 + \frac{\alpha a}{\sqrt{k}} + 24\alpha\frac{\sqrt{k}}{a} + 96\frac{k}{a^2}\right)}\frac{a^6}{r^3}\right.$$

$$\left. + 3\frac{\left(2 + \frac{\alpha a}{\sqrt{k}}\right)}{\left(6 + \frac{\alpha a}{\sqrt{k}} + 24\frac{\alpha\sqrt{k}}{a} + 96\frac{k}{a^2}\right)}\frac{a^4}{r}\right]\sin 3\theta,$$

$$\psi^+ = \frac{8k}{a^2} U \left[\frac{\left(2 + \frac{\alpha a}{\sqrt{k}}\right)}{\left(6 + \frac{\alpha a}{\sqrt{k}} + 24\alpha\frac{\sqrt{k}}{a} + 96\frac{k}{a^2}\right)}\right] r^3 \sin 3\theta.$$

Stream lines in Stokes' region are plotted for different values of porosity in Fig. 1. We observe that in the limit $(\alpha/\sqrt{k}) \rightarrow \infty$, $k = 0$, we recover in (4.5)$_1$ the stream function for the quadratic potential flow past a circular cylinder [14]. When $(\alpha/\sqrt{k}) = 0$, $k = 0$, we obtain the quadratic potential flow past a shear-free cylinder [15].
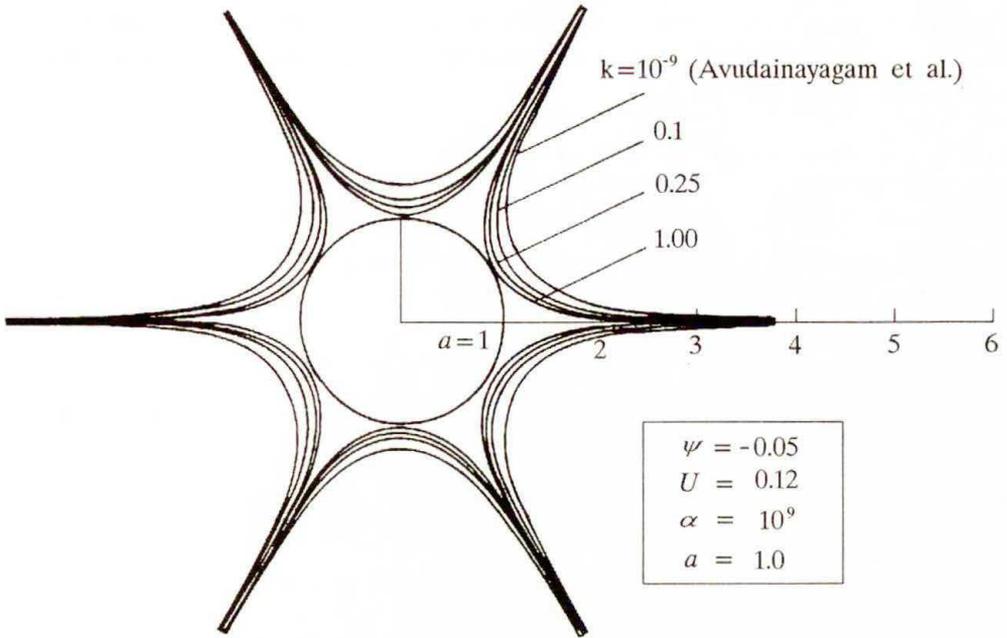
FIG. 1. Stream function $\psi(r, \theta)$ in Stokes region.

Another interesting special case may be deduced from Eq. (4.5)$_1$. If we let $(\alpha/\sqrt{k}) - (1/\lambda\mu)$ and $k = 0$, then (4.5)$_1$ reduces to

$$(4.6) \qquad \psi = \frac{U}{3}\left[-r^3 - \frac{2(1-\beta)}{(\beta+2)}\frac{a^6}{r^3} + \frac{3\beta}{(\beta+2)}\frac{a^4}{r}\right]\sin 3\theta,$$

where $\beta = 1 + (\alpha/2\lambda\mu)$. This solution corresponds to the quadratic flow past a cirular cylinder with mixed slip-stick conditions [16]. In the present case the boundary condition (2.6) becomes $q_\theta = \lambda T_{r\theta}$ on $r = a$ where λ is here the slip parameter. Thus our solution includes all the possible quadratic flows past a cylinder indicating that the boundary conditions (2.4)–(2.6) are assumed in a more general form.

4.3. Source outside a circular cylinder

Consider a source of unit strength located at $(c, 0)$, $c > a$. The stream function corresponding to a source in an unbounded flow is

$$(4.7) \qquad \psi_0(r, \theta) = \tan^{-1}\frac{r\sin\theta}{c - r\cos\theta}.$$

Equation (4.7) may be expanded into a Fourier series as

$$(4.8) \qquad \psi_0 = \sum_{n=1}^{\infty}\frac{r^n}{nc^n}\sin n\theta.$$

Therefore $A_n = 1/nc^n$ and $B_n = 0$ for all n. The coefficients C_n and D_n can be calculated from (3.9) and the modified stream function in the presence of a porous cylinder is:

for $r > a$,

$$(4.9) \qquad \psi(r,\theta) = \sum_{n=1}^{\infty} \left[r^n + \frac{\left((n-1)\dfrac{\alpha a}{\sqrt{k}} - \dfrac{4k}{a^2} n(n-1)^2 \right)}{M_n} \frac{a^{2n}}{r^n} \right.$$

$$\left. - \frac{\left(2 + \dfrac{\alpha a}{\sqrt{k}} \right) n}{M_n} \frac{a^{2n-2}}{r^{n-2}} \right] \frac{1}{nc^n} \sin n\theta;$$

for $r < a$

$$(4.10) \qquad \psi^+ = \frac{4k}{a^2} \sum_{n=1}^{\infty} \frac{\left(2 + \dfrac{\alpha a}{\sqrt{k}} \right)}{M_n} (n-1) \frac{r^n}{c^n} \sin n\theta.$$

It will be of some interest to study the asymptotic behaviour of (4.9) as r approaches infinity. In the limit as $r \to \infty$, Eq. (4.9) becomes

$$(4.11) \qquad \psi = -\frac{1}{c} r \sin \theta.$$

This is a uniform flow along the negative x-direction at large distance from the porous cylinder.

This conclusion has already been drawn by SMITH [3] in the case of a source acting outside a rigid cylinder. We remark that the porosity has no effect on the speed of the uniform stream at large distance. Perhaps, this may be due to the fact that the porosity is small in Darcy flow.

4.4. Stokeslet outside a circular cylinder

Now let us consider a Stokeslet of strength F located at $(0, c)$, $c > a$. The stream function corresponding to the Stokeslet in an unbounded region is

$$(4.12) \qquad \psi_0 = F(r \cos \theta - c) \log R_1,$$

where $R_1^2 = r^2 + c^2 - 2cr \cos \theta$. The constants A_n, B_n, C_n, D_n and E_n can be obtained in the similar way as that explained in the above example. The stream-functions for the two flow fields in the presence of a Stokeslet in front of a

porous circular cylinder may be constructed with these constants. The asymptotic form of the perturbed external flow field as $r \to \infty$ is given by

$$(4.13) \qquad \psi(r, \theta) = \frac{F\left[-\left(2 + \frac{\alpha a}{\sqrt{k}}\right)\frac{1}{2} + \left(\frac{\alpha a}{\sqrt{k}} + 4\frac{\alpha\sqrt{k}}{a} + \frac{8k}{a^2}\right)\frac{a^2}{2c^2}\right]}{\left[2 + \frac{\alpha a}{\sqrt{k}}\right]} r\cos\theta.$$

Hence, at large distances, the Stokeslet produces a uniform flow whose strength depends on the location of the singularity and on the porosity. The variation of the speed for different values of $\alpha/\sqrt{k}$ are shown graphically (see Fig. 2). The
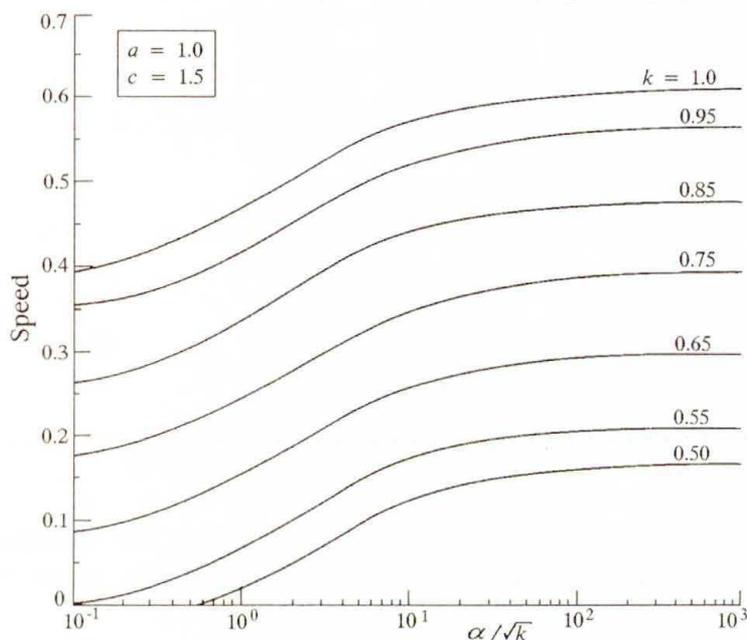

FIG. 2. Stokeslet-cylinder combination-effect of permeability on the speed at large distances.

effect of porosity on the stream function at large distances is shown in Fig. 3. In the limit when $k = 0$ and $(\alpha a/\sqrt{k}) \to \infty$, we recover the result obtained by DORREPAAL et al. [2] for a rigid circular cylinder. In the limit of $(\alpha/\sqrt{k}) = 0$ and $k = 0$ we get

$$(4.14) \qquad \psi(r, \theta) = -\frac{F}{2}r\cos\theta.$$

Therefore a Stokeslet in the presence of a shear-free circular cylinder produces a uniform flow at large distances, its strength being independent of the location and porosity. If we let $\dfrac{\alpha}{\sqrt{k}} = \dfrac{1}{\lambda\mu} = \dfrac{2(\beta - 1)}{a}$, where $\beta = 1 + (a/2\lambda\mu)$ as in example
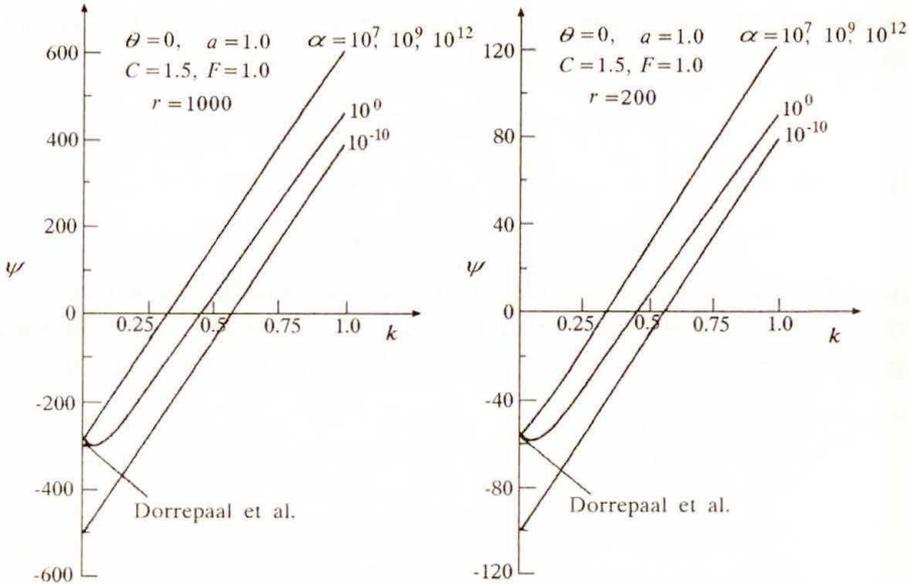
FIG. 3. Stokeslet-cylinder combination-effect of porosity on the stream function at large distance.

(4.2), Eq. (4.13) reduces to

$$(4.15) \qquad \psi = F \left[\frac{-\beta + (\beta - 1)\dfrac{a^2}{c^2}}{2\beta} \right] r \cos \theta.$$

This solution corresponds to the asymptotic behaviour of the Stokeslet in front of the cylinder when mixed slip-stick conditions are applied at the contour of the cylinder.

Acknowledgments

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