# Can the system of discrete vortices imitate a boundary layer?

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THE PROBLEM of dissipative flow of superfluid due to the vortex interaction with the boundary is considered within the hydrodynamics approximation. The numerical simulations were applied to show that, when the boundary starts moving, the vortices pinned to microscopic surface irregularities can stretch. The array of the growing vortices give rise to the specific boundary layer, which in some aspects is similar to the boundary layer in viscous fluids.

SUPERFLUID $^{4}$He behaves as an ideal fluid with rotation restricted to quantized vortex filaments. The experiments of AWSCHALOM and SCHWARZ [1] suggest that some remnant vortices are always expected to occur. The essentially hydrodynamics description of its dynamics is valid down to a scale comparable to the core radius of the vortex $a_0$ which is of order $1\overset{\bullet}{A}$. In the zero temperature limit, when the interaction between the vortex and the thermal excitation gas (the normal fluid) may be neglected, the motion of an individual quantized vortex $S(\xi, t)$ (in local induction approximation – LIA) is accurately described by [2]

$$\dot{S} = \beta S' \times S'' + V_s,$$ (1)

where $V_s$ is the local average superfluid velocity, and $\beta = (\kappa/4\pi) * \ln(c_1/S''a_0)$, with $c_1$ constant of order 1 and $\kappa = h/m_{He}$ quantum of circulation. The primes denote differentiation with respect to arc length. The equation must be supplemented by a nonlocal interaction term when the vortex approaches another vortex or a boundary.

The aim of present paper is to consider the dynamics of vortices terminating on the flat infinite boundary. The problem of vortex dissipative line dynamics in relatively narrow channels has been already studied by SCHWARZ [3, 4] who pointed earlier [4] that the moving vortex may be pinned to the microscopic surface irregularities. We recall [5] that a vortex filament terminating on a perfectly smooth surface will move without hindrance. When the end of the vortex encounters a bump, however, it will remain pinned there until it bows over up to some critical angle with the surface. Next it jumps off and resumes its motion. Quantized vortices may pin on bumps of only a few Angströms, so that in practice this process is always expected to occur. Moreover Schwarz, while considering the static case, concluded that the depinnig angle (angle between the vortex end and the normal to the surface) depends logarithmically on the size of the pinning site. It means that the leading role may be played by small protrusions which are more abundant. SCHWARZ [3, 4] found that the pinning and release process

makes the vortex line elongate across the channel. In such a process the energy is dissipated by being fed into the growing vortex lines which then annihilate at the opposite wall. The vortices also transfer the momentum between the boundary and the superfluid; the vortex exerts the stream-wise force on the boundary via its interaction with the pinning site. Respectively, the boundary must be exerting a retarding force on the superfluid via its interaction with the vortex.

In some important aspects, the vortex dynamics in the vicinity of a single surface is different from the dynamics described by Schwarz in narrow channels. First of all, vortices can not be spanned between perpendicular or opposite walls, and second, there is no opposite wall to annihilate the growing vortices.

Consider at the beginning the simple example of a vortex pinned to $z = 0$ plane and subjected to the applied velocity $v_s$ in the $\hat{x}$ direction. Assume that initially the vortex filament having the shape of a half circle of radius $R$ lies in plane $x = 0$ (i.e. plane perpendicular to the applied velocity and the boundary plane). If the driving velocity is equal in the value but opposite in direction to the self-induced velocity

$$(2) \qquad\qquad v_i = \frac{\beta}{R} := v_{cr},$$

the configuration is stationary. The higher applied velocity bends the vortex stream-wise and stretches it out. At some critical angle of declination (i.e. depinning angle dependent on the size of pinning site) the ends of the vortex depin. If the driving velocity is smaller than the critical one, the vortex bows against the flow and decreases. The vortex oriented in another direction, so that the driving velocity adds to the self-induced one, will bow with the flow, but then the self-induced velocity directs it to the boundary, where it annihilates. The numerical simulations done by the author confirm the above considerations.

Statistically, when the driving velocity is applied (or the boundary starts moving), roughly a half of the pinned vortices has a chance to grow, other will annihilate. It means that the motion of the boundary introduces some order, and it is easy to check that the orientation of the remaining vortices is such, that close to the boundary the superfluid is moving in the same direction as the wall.

As it was stated above, the end of vortex depins at some critical angle, dependent on the size of the pinning site, and then moves freely till the next bump. The two end points of a filament may encounter various irregularities and consequently must depin simultaneously. Hence, one can conclude, from the preceding analysis that a "well oriented" big enough vortex loop will grow any time, when pinned, while other loops will decrease. The situation simplifies when the small protrusions occur so densely, that the pinning and release events are so frequent, that the intermittent motion of vortex end points may be approximated by a continuous motion with friction. When the friction is present, the moving end of vortex is bowed to the boundary at such angle that tangent component of tension

force equals the friction force $f_s$. Namely

$$(3) \qquad\qquad \sin \Theta = \frac{f_s}{f_t},$$

where $\Theta$ is the angle between the vortex at its end points and the normal to the boundary, and $f_t$ is the value of tension force. The angle $\Theta$, corresponding to the average angle of declination, may be considered as the material constant depending on the density and the size of boundary irregularities. It means that for normally "smooth" surface $\sin \theta$ is small when compared to unity.

Consider then the following example:

Let the boundary plane $z = 0$ be moving with constant velocity $V_d = (V_d, 0, 0)$ with respect to the fluid. Consider the dynamics of a vortex which at the initial time has the shape of a half circle symmetrically placed with respect to plane $y = 0$, and the driving velocity. Assume that the self-induced velocity $V_i$ is smaller than the driving velocity $V_d$, and that the vortex loop is moving so that the angle between the vortex at its end points and the normal to the boundary is $\Theta$. The self-induced velocity (in LIA), at the given point of the vortex, is binormal to the vortex line at that point. Hence, at the ends of the vortex, the angle between the self-induced velocity and the wall is $\theta$, but for one (*positive*) orientation of the vortex the self-induced velocity is directed out from the wall, and for another (*negative*) orientation it is directed to the boundary. For the positive orientation the vortex loop will be growing. The rate of growth may be calculated as follows: let $a$ be a point moving with the vortex and $p$ be a vortex end point (Fig. 1).
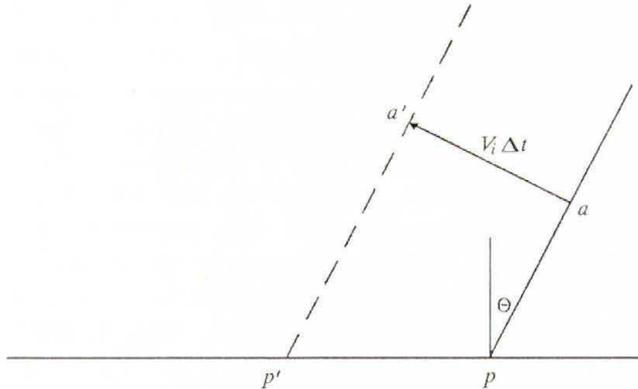


FIG. 1. Fragment of the vortex close to the boundary. The vortex moves so that point $a$ goes to $a'$ while end point $p$ goes to $p'$. Vector $\overrightarrow{ap}$ is parallel to the vector $\overrightarrow{a'p'}$.

In a short time $\Delta t$ the point $a$ moves to $a' = a + \Delta t V_i$. In the same time the end point moves to such a point $p'$ that $\overrightarrow{a'p'}$ is parallel to $\overrightarrow{ap}$. If so, the initial fragment of vortex line grows up by $\Delta t V_i \sin \Theta$. Because there are two ends, the

vortex growing rate $\partial l/\partial t$ satisfies the condition

(4)
$$\frac{\partial l^+}{\partial t} = 2V_i(t)\sin\Theta\,,$$

where + means the growing vortex. The oppositely oriented vortex is decreasing at the rate

(5)
$$\frac{\partial l^-}{\partial t} = -2V_i(t)\sin\Theta\,,$$

where sign − means a shrinking vortex.
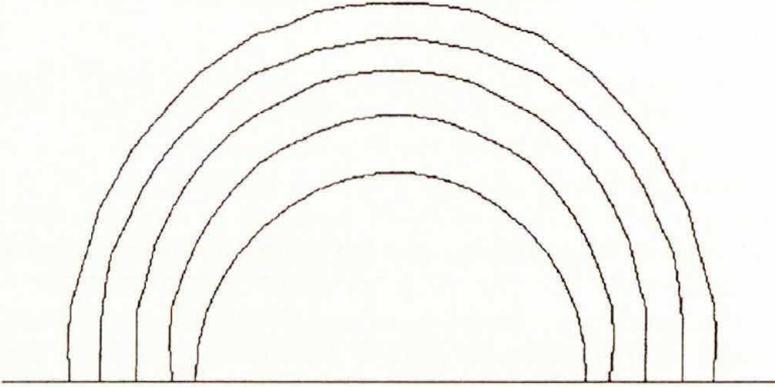
FIG. 2. Motion of a vortex when friction is present. The figure shows the projection of vortex filament on the plane perpendicular to the boundary and the driving velocity.

It may look curious that the driving velocity does not appear in Eqs. (4), (5). However it plays an important role: only these vortex loops for which the self-induced velocity is smaller than the imposed one, can grow. So for bigger driving velocity another smaller loops can grow, and as one can see from the last equation, smaller loops grow faster. It should be said also, that Eqs. (4), (5) are not valid for the driving velocities only slightly bigger than the critical one. When there is no friction (i.e. for $\Theta = 0$), the vortex maintains its shape of a half circle. Also for small declination angles ($\sin\Theta < 0.3$), numerical simulations (Fig. 2) show that the vortex grows maintaining approximately the shape of a half circle. For bigger declination angles, however, the vortex becomes elongated. The instant radius of uniformly growing loop is $R = l/\pi$, while its self-induced velocity is $v_i = \beta/R$. Those relations put into Eqs. (4), (5) give the equation for $R(t)$:

(6)
$$\frac{\partial R^\pm}{\partial t} = \pm\frac{2\beta\sin\Theta}{\pi R}$$

leading to

(7)
$$R^\pm = \left(\pm\frac{4\beta t\sin\Theta}{\pi} + R_0^2\right)^{1/2}.$$

Consider now the array of vortices with end points on the boundary. If at time $t = 0$ the boundary wall starts moving with some constant velocity $V_b$, some vortices will grow and some will decrease. As a result, a layer of superfluid close to the boundary starts moving in the same direction as the wall.

To consider this mechanism in more detail, assume that, at the time $t = 0$:

- all vortices have the shape of the half circle of the same radius $R_0$;
- the vortex loops form a regular pattern where half of the loops have positive orientation (positive vortices) and another half has the negative orientation (negative vortices);
- the driving velocity $V_b$ is bigger than the initial critical one so the positive vortices can grow (i.e. $V_b > \beta / R_0$);
- the declination $\Theta$ is small, so as was stated above, the vortex loops maintain their shape of a half circle;
- there are $n$ growing vortices and $n$ decreasing.

Then Eq. (7) allows to calculate the average velocity at a given distance from the wall $z$. The velocities $V^+$ and $V^-$ generated by positive and negative vortices, respectively, will be calculated separately. The resultant velocity is $V = V^+ - V^-$.

The average velocity $V^+$ (at a given distance $z$) may be calculated from the Ampere principle. Let $F$ be the surface $z = z_0$. For $z_0 < R^+ \cos \Theta$ that surface is pierced twice by every vortex loop. The distance between the piercing points, (or the diameter of "the cut-off" loop segment) is:

$$(8) \qquad s^+ = 2 \left( (R^+)^2 - z^2 / \cos^2 \Theta \right)^{1/2} \qquad \text{for} \quad z < R^+ \cos \Theta.$$

Above the large square lying on the surface $F$ there are $N_s = nA^2$ positive loops (where $A$ is the side of the square). It means that in average, above a line with length $A$ lying on $F$ along the $\hat{x}$ axis, there are $N_a = s^+ An$ vortex segments. Then from the Ampere principle, which states that the circulation of the velocity field around a closed path is equal to the flux of the vorticity linked through this path, the average induced velocity is

$$(9) \qquad V^+ = s^+ \kappa n.$$

The $V^-$ velocity may be calculated similarly. In the explicit form both velocities read:

$$(10) \qquad V^\pm(z, t) = \kappa n \left( \pm \frac{4\beta t \sin \Theta}{\pi} + R_0^2 - \frac{z^2}{\cos^2 \Theta} \right)^{1/2}.$$

At the time $T_c = \pi R_0 / (4\beta \sin \Theta)$ the decreasing loops vanish, and so, the velocity $V^-$ becomes zero everywhere.

For times $t < T_s$ the induced velocity $V$ is:

$$
(11) \qquad V = \begin{cases} 0 & \text{for} \quad z > R^+ \cos\theta, \\ V^+ & \text{for} \quad z \in (R^- \cos\Theta, R^+ \cos\theta), \\ V^+ - V^- & \text{for} \quad z \in (0, R^- \cos\Theta). \end{cases}
$$

For times $t > T_s$, $V$ is:

$$
(12) \qquad V = \begin{cases} 0 & \text{for} \quad z > R^+ \cos\theta, \\ V^+ & \text{for} \quad z \in (0, R^+ \cos\theta), \end{cases}
$$

since $V^- \equiv 0$.

One can see from Eq. (12) that the thickness of the boundary layer (i.e. that layer where the velocity $V > 0$) grows as $R_+ \cos\theta$ proportionally to $\sqrt{\beta t}\sin\Theta$. The induced velocity has the same direction as the velocity of the boundary, so it reduces the relative velocity between the superfluid and the boundary.

Recall that velocity of a viscous fluid in the boundary layer appearing when the wall starts moving with some constant velocity $v_b$, is described by the diffusion equation:

$$
(13) \qquad \frac{\partial v}{\partial t} - \nu \frac{\partial^2 v}{\partial z^2} = 0.
$$

The usual assumption that there is no slip, leads to the boundary condition $v(0, t) = v_b$. Then with the initial condition $v(0, t) = v_b$ the equation leads to

$$
(14) \qquad v(z, t) = v_b \Phi\left(\frac{z}{2\sqrt{\nu t}}\right),
$$

where

$$
(15) \qquad \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\alpha^2}\, d\alpha.
$$

One can see that in a viscous fluid, the characteristic thickness of the boundary layer is $\sqrt{\nu t}$.

In the considered "vortex boundary layer", if there are few vortices at the beginning, there is a slip i.e. the fluid velocity at the boundary is different from the velocity of the wall. The role of the viscosity is played here by the parameter $\beta \sin\Theta$ – proportional to the quantum of circulation and surface roughness. The fact that the "nonsmooth" velocity profiles were obtained is due to the assumption, that at the beginning all vortex loops were identical.

In conclusion, a consideration of vortex friction on microscopic boundary roughness leads to the mechanisms of the origin of specific vortex boundary layer in some aspects similar to the boundary layer in viscous fluids. It is still interesting, however, to consider that problem under more general assumptions, namely when there are different vortices and they interact with each other.

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