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Necking in steady-state drawing of polymer fibres

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THE CONCEPT of non-uniform extensional motions of materially non-uniform simple locally isotropic solids is used to discuss some general properties of fibre drawing processes. In the approach presented, axial and radial temperature and structure variations can be taken into account as some kind of non-uniformity. For steady cold-drawing processes various necking conditions are considered in general and particular cases. The effects of particular force components, i.e. inertial and rheological, are discussed in greater detail for S-shaped velocity profiles.

1. Introduction

DRAWING IS THE OPERATION which changes the textile characteristics of man-made fibres, improving, in particular, initially low tenacity, high irreversible deformation, low moduli, etc. It consists of irreversible elongation in the solid state from 20 to 20000 % of the original length. Such a process with coexistent undrawn and drawn parts, exhibiting the necking phenomenon, is often called the "cold drawing" although it may be realized at pretty high temperatures of baths or heaters. The most exhaustive information on drawing of polymer fibres can be found in the monograph by ZIABICKI [1].

From the rheological point of view, drawing of a long filament can be considered as a non-uniform and frequently non-isothermal quasi-elongational motion. As compared to melt-spinning processes, a relevant analysis is much more difficult since usually for deformed solids the dissipation energy cannot be neglected, leading to an additional increase of temperature. Also the nonlinear viscoelastic behaviour of solid polymers is an essential factor of the process considered, and neither Newtonian nor linearly elastic approximations can be applied at all.

In the present paper we use our previous concept of non-uniform extensional motions (NUEM) of materially non-uniform simple locally isotropic solids [2] to discuss some general properties of fibre-drawing processes without applying any particular models. This approach enables taking into account temperature and structure variations along and across the filament, replaced by some kind of spatial non-uniformity. The corresponding constitutive equations used for description of steady quasi-elongational motions involve the stretch ratios (axial velocities) as well as their derivatives in the direction of the axis (axial velocity gradients). An explicit dependence of material properties on the radial and axial coordinates can also be introduced. To satisfy the boundry conditions in stresses at the free surface of a filament, the assumptions very similar to those made in the case of flows with dominating extension (FDE) may be used like in [3, 4].

In Sec. 2 we discuss the drawing process as a steady non-uniform quasi-elongational motion, using the constitutive equations of a materially non-uniform simple locally isotropic solid. Next Sec. 3 is entirely devoted to various necking conditions expressed either in velocities or stresses. In Sec. 4 the effect of rheological force on the conventional stresses along the filament is discussed in greater detail. Last Sec. 5 summarizes our previous results in the form of several conclusions.

2. Drawing as a steady non-uniform motion of materially non-uniform solids

In the paper [2] it was proved that a steady non-uniform drawing process can be described by the following stress-components difference, resulting from the more general constitutive equations of materially non-uniform simple solids:

$$T^{33} - T^{11} = \sigma(V, V'; r, z) = \sigma_1(\lambda, \lambda'; r, z) = \sigma_2(\varepsilon, \dot{\varepsilon}, r, z), \tag{2.1}$$

where $V(z)$ is the axial velocity under a quasi-elongational approximation and the primes denote the corresponding derivatives with respect to the axial coordinate z (Fig. 1). The stretch ratio λ, the strain ε and their derivatives are defined as follows:

$$\lambda = \frac{V(z)}{V_0}, \qquad \frac{\lambda'}{\lambda} = \frac{V'}{V}, \tag{2.2}$$

$$\varepsilon = \ln \lambda, \qquad \dot{\varepsilon} = \frac{\lambda'}{\lambda} V = V', \tag{2.3}$$

where the dot denotes the corresponding time-rate and V_0 – the feeding velocity (at the first pair of godets, Fig. 1). Denoting by V_L the take-up velocity, we can

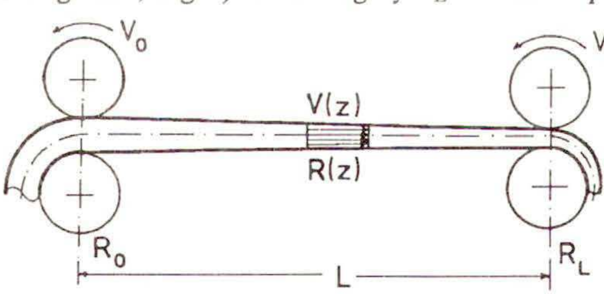

FIG. 1. Scheme of drawing process.

define the draw ratio $\Re$ (see Ref. [1]) as

$$\Re = V_L/V_0, \tag{2.4}$$

and the conventional or normal stress (related to the original cross-section) as

$$\sigma^0 = \sigma/\lambda, \tag{2.5}$$

if the process is steady-state, the mass flow rate remains constant, i.e.

$$W = \varrho \pi R^2 V = \varrho \pi R_0^2 V_0 = \varrho \pi R_L^2 V_L = \text{const.} \tag{2.6}$$

We should emphasize that Eqs. (2.1) or (2.5) can describe many types of non-linear visco-elastic behaviour, in particular, that shown for the polystyrene [5] in Fig. 2. The rate- and temperature-dependent stress-strain characteristics with stress overshoot before yielding usually lead to the necking phenomenon (see Ref. [1]).

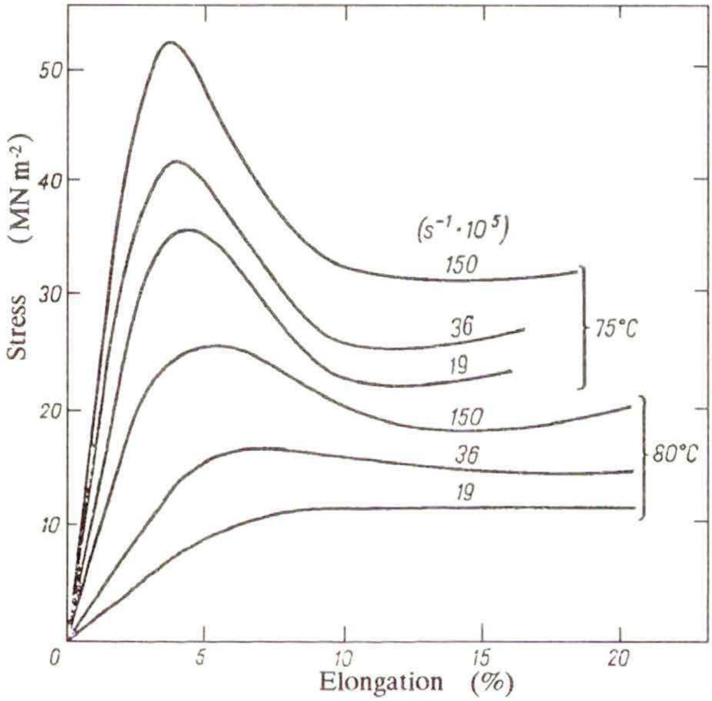

FIG. 2. Deformation characteristics for polystyrene samples after [5]. Temperatures and deformation rates indicated.

The balance of forces acting in a drawn filament can be written as follows (see Refs. [1] and [6]):

$$F(z) = F_{\text{ext}}(z) = F_{\text{rh}}(z) + F_{\text{in}}(z) + F_{\text{ad}}(z) + F_{\text{st}}(z) - F_{\text{gr}}(z), \tag{2.7}$$

where the subscripts rh, in, ad, st and gr denote rheological, inertial, air-drag, surface-tension and gravitational components, respectively.

Moreover, introducing a simplifying model assumption that (see Ref. [4])

$$\sigma^0 = \gamma(V, V'; z)\varphi(r), \tag{2.8}$$

where the function $\varphi(r)$ describes the radial dependence on r, the same for all z, we arrive at

$$F(z) = \gamma\lambda \int_0^R 2\pi r\varphi(r)\, dr = \gamma\lambda\pi\Phi, \tag{2.9}$$

where

$$\Phi = \int_0^R 2r\varphi(r)\, dr, \qquad \Phi = R^2 \qquad \text{for} \quad \varphi(r) \equiv 1. \tag{2.10}$$

If, in particular, we apply the parabolic approximation according to the first term in KASE'S [7] expansion:

$$\varphi(r) = 1 + ar^2, \qquad a > 0, \tag{2.11}$$

we obtain

$$\Phi = R^2 \left(1 + \frac{1}{2} aR^2\right). \tag{2.12}$$

Differentiating with respect to z the relation for λ resulting from Eq. (2.9), and taking into account Eqs. (2.2), (2.3), and that

$$\gamma = \frac{F}{\pi R_0^2}, \qquad \frac{\gamma'}{\gamma} = \frac{F'}{F}, \qquad \frac{\lambda'}{\lambda} = -2\frac{R'}{R}, \tag{2.13}$$

we can calculate the first and second derivatives of the radius R with respect to z in the following forms:

$$R' = -\frac{\varrho\pi V' R^3}{2W} \frac{\left(1 + \frac{1}{2} aR^2\right)^2}{(1 + aR^2)}, \tag{2.14}$$

and

$$R'' = -\frac{\varrho\pi R^3}{2W} \frac{\left(1 + \frac{1}{2} aR^2\right)^2}{(1 + aR^2)} \left[V'' - \frac{3}{2} \frac{V'^2}{V} \frac{\left(1 + \frac{1}{2} aR^2\right)^2}{(1 + aR^2)} + \frac{aR^2 V'^2}{V} \frac{\left(1 + \frac{1}{2} aR^2\right)^2}{(1 + aR^2)^2} - \frac{aR^2 V'^2}{V} \frac{\left(1 + \frac{1}{2} aR^2\right)}{(1 + aR^2)} \right], \tag{2.15}$$

respectively.

3. Necking conditions expressed in velocities or stresses

If we assume that the necessary condition of necking, characteristic for cold-drawing processes, is connected with a change of sign of the corresponding curvature (see Refs. [8] and [9]), we may use the following condition:

$$R'' \leq 0, \tag{3.1}$$

where negative values describe convex parts of the filament profile, and the equality defines an inflexion point. The above condition, after taking into account Eq. (2.15), leads to

$$V''V \geq \frac{3}{2}V'^2 D, \tag{3.2}$$

where

$$D = \frac{\left(1 + \frac{1}{2}aR^2\right)^2}{(1 + aR^2)} \left[1 + \frac{1}{3}\frac{(aR^2)^2}{(1 + aR^2)\left(1 + \frac{1}{2}aR^2\right)}\right] \tag{3.3}$$

denotes the function which is identically equal to 1, if there is no radial variation of properties. In this particular case the necessary condition of necking simplifies to the form:

$$V''V \geq \frac{3}{2}V'^2, \tag{3.4}$$

which is exclusively of kinematic character, independently of the form of the constitutive equations considered! (1)

It is noteworthy that for fluids ZIABICKI [10] attempted to establish a "necking intensity" of the kinematic character, but the introduced quantity was not related to any necking condition (see Refs. [8] and [11]).

The solution of the differential equation resulting from Eq. (3.4):

$$f = \overline{V}''\,\overline{V} - \frac{3}{2}\overline{V}'^2 = 0, \tag{3.5}$$

with the following boundary conditions: $\overline{V}(0) = V_0$ and $\overline{V}(L) = V_L$, amounts to

$$\overline{V} = V_0\frac{1}{(1 - \overline{c}z)^2}, \tag{3.6}$$

(1) It can be shown that the above inequality is also valid for fluids.

where the overbars in Eq. (3.5) emphasize the solution corresponding to the equality in Eq. (3.4), and

$$(3.7)\qquad \bar{c}L = 1 - \sqrt{\frac{V_0}{V_L}} = 1 - \frac{1}{\sqrt{\Re}}.$$

The above solution is schematically shown by a solid line in Fig. 3. For such a velocity distribution the curvature of the filament profile is always equal to zero. This means that the dependence of the radius $R(z)$ on z is linear.

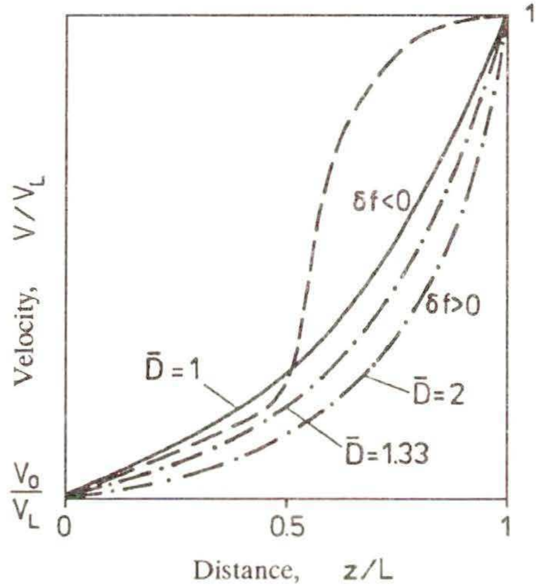

FIG. 3. Velocity profile along the filament. Solid-line – the profile for vanishing curvature; broken-line – the S-shaped profile characteristic for necking; dotted lines – the profiles for vanishing curvatures in the case of radial variations of material properties.

In more general cases, when $D \neq 1$, introducing the notion of mean value $\overline{D}$ of the function D along the length L (the parameter a is a constant and the radius R does not vary so much), we obtain the simplified differential equation:

$$(3.8)\qquad f_1 = \overline{V}''\,\overline{V} - \frac{3}{2}\overline{V}'^2\overline{D} = 0,$$

the solution of which amounts to

$$(3.9)\qquad \overline{V} = V_0 \frac{1}{(1 - \bar{c}_1 z)^n},$$

where

$$(3.10)\qquad \bar{c}_1 L = 1 - \left(\frac{V_0}{V_L}\right)^{1/n} = 1 - \frac{1}{(\Re)^{1/n}}, \qquad n = \frac{2}{3\overline{D} - 2}.$$

The corresponding curves for $\overline{D} = 1.33$ and $\overline{D} = 2$ are also shown by dotted lines in Fig. 3. Thus, we may conclude that any radial variation of the conventional stresses lowers the graphs describing solutions for zero curvatures.

3.1. S-shaped velocity profiles required for the existence of necking

It is commonly accepted that the existence of necking during cold-drawing processes is connected with the type of stress-strain curves shown in Fig. 3, leading to the well-known instability conditions (see Ref. [1]). A similar result can be inferred on the basis of our further considerations.

Since the differential expression f described by Eq. (3.5) is continuous at the profile (3.6) in the sense of proximity of the 2-nd order, we can prove that the variation:

$$\delta f = \overline{V}\delta V'' - 3\overline{V}'\delta V' + \overline{V}''\delta V, \tag{3.11}$$

where the quantities V, V' and V'' are defined in the Appendix, changes its sign depending on the values of the small parameter m (see the Appendix) viz.

$$\begin{array}{lll} \delta f > 0 & \text{if} & m < 0, \\ \delta f = 0 & \text{if} & m = 0, \\ \delta f < 0 & \text{if} & m > 0. \end{array} \tag{3.12}$$

Thus, any S-shaped velocity profile starting at $z = 0$ slightly below the curve described by Eq. (3.6) leads to negative values of the curvature R''. The inflexion point $V'' = 0$ on the velocity profile may be situated either below or above the curve (3.6). In the case of $\overline{D} \neq 1$, the region in which negative values of R'' can be expected is seriously diminished and the curves corresponding to $R'' = 0$ for $\overline{D} \neq 1$ always lie below the curve (3.6).

It should be noted, however, that S-shaped velocity profiles leading to two inflexion points on the filament profile are possible either below or above the curve described by Eq. (3.6). Then the filament curvatures take positive, negative and again positive values corresponding to the "bottle-like" shape of the drawn fibre.

As an illustration, consider the following velocity profile (S-shaped for particular $g(z)$)

$$V = V_0 \exp\left\{ \ln\frac{V_L}{V_0} \frac{\int\limits_0^z g(z)\,dz}{\int\limits_0^L g(z)\,dz} \right\}, \tag{3.13}$$

satisfying automatically the relevant boundary conditions. The condition (3.4) leads to

$$g' \geq \frac{1}{2} \ln \frac{V_L}{V_0} \frac{1}{\int\limits_0^L g(z)\, dz} g^2. \tag{3.14}$$

In the case of exponential viscosity function, used for the description of Newtonian behaviour (see Ref. [9]), i.e. for $g = \exp(-Az)$, we obtain e.g. $A = -0.64$ if $V_L/V_0 = 4$, and the velocity profile corresponding to Eq. (3.13) at $z = 0$ is tangent to the curve (3.6).

In a more general case of $\overline{D} \neq 1$, negative curvatures R'' are possible only for

$$z \leq -\frac{1}{A} \ln \left\{ \frac{1}{\ln \dfrac{V_L}{V_0} \left(\dfrac{3}{2}\overline{D} - 1 \right)} [\exp(-AL) - 1] \right\}, \tag{3.15}$$

i.e. for negative values the parameter A or positive viscosity gradients.

3.2. Purely inertial and isothermal cases

If only rheological and inertial terms occurring in the force balance (2.7) are retained and F_0 denotes the rheological force F_{rh} at $z = 0$, we obtain

$$F(z) = F_0 + W(V(z) - V_0), \qquad W = \varrho \pi R_0^2 V_0 \tag{3.16}$$

or

$$\gamma(z) = \varrho V_0 \left[\frac{F_0}{W} - V_0 + V(z) \right]. \tag{3.17}$$

On the basis of the inequality (3.4), we arrive at

$$\gamma'' \left[\gamma - \varrho V_0 \left(\frac{F_0}{W} - V_0 \right) \right] \geq \frac{3}{2} \gamma'^2. \tag{3.18}$$

The solution of the differential equation resulting from the equality in Eq. (3.18) amounts to

$$\gamma = 6V_0 K + \frac{1}{(c_0 - c_1 z)^2}, \qquad K = \frac{\varrho}{6} \left(\frac{F_0}{W} - V_0 \right), \tag{3.19}$$

where

$$c_0 = \frac{1}{(\gamma_0 - 6V_0 K)^{1/2}}, \qquad c_1 L = \frac{1}{(\gamma_0 - 6V_0 K)^{1/2}} - \frac{1}{(\gamma_L - 6V_0 L)^{1/2}}; \tag{3.20}$$

γ_0 and γ_L denote the conventional stresses at the exit and take-up end, respectively.

The graphs illustrating the functions (3.19) are very similar to those shown in Fig. 3. Our previous remarks concerning S-shaped velocity profiles remain valid in the case considered.

In a purely inertial case, if also $a = 0$, Eq. (2.15) can be replaced by

$$\tag{3.21} R''\left[1 - \frac{6V_0}{\gamma R^2}(KR^2 + M)\right] = -\frac{3V_0}{\gamma R}\left\{\left(\frac{\gamma'}{\gamma} + \frac{R'}{R}\right) \times \left[(KR^2 + M)\frac{\gamma'}{\gamma} + 2(KR^2 + 2M)\frac{R'}{R}\right] - 2KRR'\frac{\gamma'}{\gamma} - (KR^2 + M)\left(\frac{\gamma''}{\gamma} - \frac{\gamma'^2}{\gamma^2}\right) - 4KR'^2 + 2(KR^2 + 2M)\frac{R'^2}{R}\right\},$$

where $M = W/6\pi$.

If we assume, moreover, that in an isothermal case the appropriate constitutive equations can be approximated by the following power-law equations:

$$\tag{3.22} \gamma = \gamma_0\lambda^n, \qquad n > 0,$$

where γ_0 does not depend on z, Eq. (3.21) takes the simplified form:

$$\tag{3.23} R''\left[\frac{\varrho V^2}{\lambda\gamma} - n\right] = \frac{12V_0}{\gamma R}\left\{(n-1)KR'^2 + (n-1)(n-2)M\frac{R'^2}{R^2}\right\}.$$

An inspection of the above equality shows that the necking condition (3.1) can be satisfied only for n belonging to the open domain $(0, 1)$. This is the case, in particular, for

$$\tag{3.24} \begin{aligned} n &= 2/5 \quad \text{if} \quad \frac{3}{2}V_L \le \frac{F_0}{W} - V_0 \le 4V_0, \\ n &= 1/2 \quad \text{if} \quad V_L \le \frac{F_0}{W} - V_0 \le 3V_0, \\ n &= 3/5 \quad \text{if} \quad \frac{2}{3}V_L \le \frac{F_0}{W} - V_0 \le \frac{7}{3}V_0. \end{aligned}$$

If $n = 1$, we have $R'' = 0$, what means that for a linear dependence of the conventional stress γ on λ, the radius of filament varies also linearly.

It is noteworthy that the conditions (3.24) can be satisfied for small inertia region defined as follows (see Ref. [7]):

$$\tag{3.25} \frac{KR_0^2}{M} \ge \Re, \qquad \text{or} \qquad \frac{F_0}{W} \ge V_0 + V_L.$$

4. Temperature-dependent rheological force

If we assume that the only force acting in the filament is the rheological force $F_{\text{rh}} = F_{\text{ext}}$, we conclude that such a force has to be constant with respect to z. On the other hand, if only thermal effects are considered, the dependence $\gamma(V)$ may be an increasing or decreasing function of V like in Fig. 4. In particular, for the processes close to adiabatic ones, the plot of the conventional stress γ versus the velocity V (or the draw-ratio $\Re$) may be a decreasing function because of dissipative effects (see Ref. [1]).

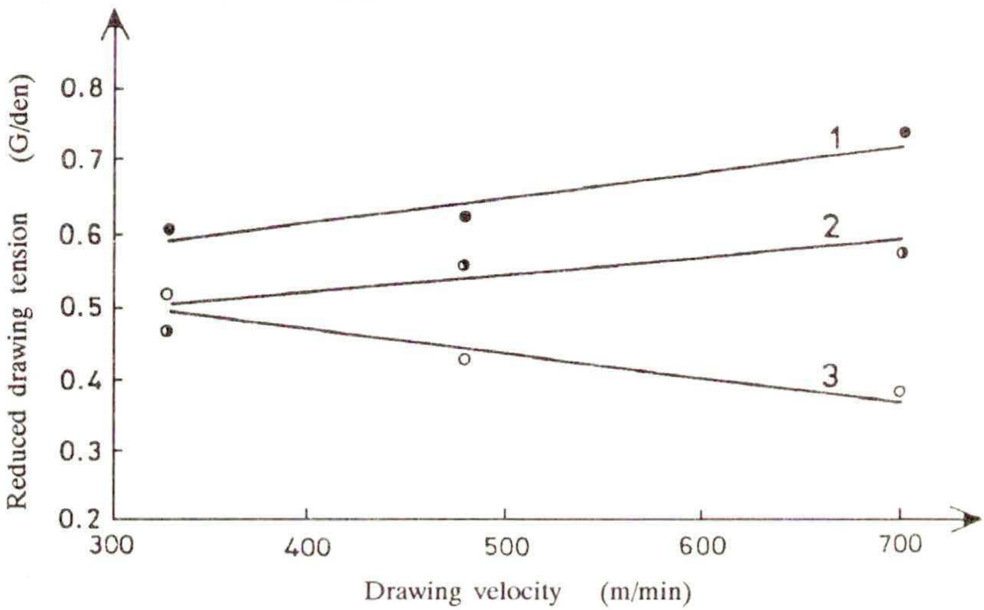

FIG. 4. Reduced drawing tension vs. drawing velocity from [1]. 1, nylon-6, temperature 80° C; 2, nylon-6, temperature 20° C; 3, polyethylene tetraphtalate, temperature 80° C.

Since *a priori* we know very little on how the function $\gamma(V)$ as well as the profile $V(z)$ look like, we assume, for simplicity, that the conventional stress can formally be described by

$$\gamma = \gamma_0(V^n/V_0^n) = \exp\left[n \ln(V_L/V_0)\frac{\int_0^z g(z)\,dz}{\int_0^L g(z)\,dz}\right], \tag{4.1}$$

with $g = \exp(-Az)$, in particular. Such an approach enables taking into account any increasing ($n > 0$) or decreasing ($n < 0$) functions $\gamma(V)$ as well as an S-shaped character of the velocity profile $V(z)$. A linear dependence $\gamma(V)$, like that shown in Fig. 4, corresponds to $n = 1$.

The velocity profile in the form of Eq. (3.13), introduced into the condition (3.4) or the more general Eq. (3.2), proves that the necking phenomenon is possible for

$$z \leq -\frac{1}{A}\ln\left\{\frac{1}{n}\,\frac{1}{\ln\dfrac{V_L}{V_0}\left(\dfrac{3}{2}\overline{D}-1\right)}\Big[\exp(-AL)-1\Big]\right\}, \tag{4.2}$$

where $\overline{D}$ denotes the mean value of the parameter defined in Eq. (3.3).

It results from Eq. (4.2) that for $n > 0$ positive values of z can be obtained for any $A > 0$ and some $A < 0$. In other words, the above result means that necking is possible in the concave and for the S-shaped velocity profiles only for increasing functions $\gamma(V)$.

5. Final conclusions

On the basis of our previous considerations we may formulate the following conclusions:

1. The concept of non-uniform extensional motions (NUEM) of materially non-uniform simple locally isotropic solids is useful to discuss effectively the case of steady drawing of polymer fibres and to investigate some properties of general character, without assuming any particular constitutive equations.

2. The necessary condition of necking, characteristic for cold-drawing processes, can be formulated in terms of purely kinematic quantities: the velocities and their first and second derivatives with respect to the axial coordinate.

3. The necking phenomenon for an S-shaped velocity profile is possible, in principle, if its initial part is situated slightly below the concave velocity profiles obtained for the case of zero curvatures of the filament profile.

4. The existence of radial variation (Kase's type) of the conventional stress changes the regions in which negative curvatures of the filament profiles are possible for S-shaped velocity profiles.

5. The previous remarks do not exclude the cases in which two inflexion points on the filament profile are possible leading to the "bottle-like" shape of the drawn fibre.

6. In the cases of purely inertial and isothermal effects, necking is possible for small inertia regions.

7. In the case in which the only force is the temperature-dependent rheological force, the necking phenomenon can be observed for increasing as well as decreasing (caused by energy dissipation effects) dependence of the conventional stress on the velocity of drawing. For S-shaped velocity profiles necking may appear only if the conventional stresses increase with the velocity.

Appendix

A differential expression $f = f(V(z))$, involving derivatives of V at most of the k-th order, is continuous at the function $\overline{V} = \overline{V}(z)$ in the sense of proximity of the k-th order, if for any small positive ε there exists such a positive δ that

$$|f(V) - f(\overline{V})| < \varepsilon, \tag{A.1}$$

at

$$\begin{aligned} |V(z) - \overline{V}(z)| &< \delta, \\ |V'(z) - \overline{V}'(z)| &< \delta, \\ |V^{(k)}(z) - \overline{V}^{(k)}(z)| &< \delta. \end{aligned} \tag{A.2}$$

To prove that the differential expression f in Eq. (3.5) is continuous at $\overline{V}$ defined by Eqs. (3.6), (3.7) in the sense of proximity of the 2-nd order, it is sufficient to take the following functions:

$$\begin{gathered} V = V_0 \frac{1}{(1 - cz)^{2+m}}, \qquad V' = V_0 \frac{c(2+m)}{(1 - cz)^{3+m}}, \\ V'' = V_0 \frac{c^2(2+m)(3+m)}{(1 - cz)^{4+m}}, \end{gathered} \tag{A.3}$$

where m is a small parameter (positive or negative) and

$$cL = 1 - \left(\frac{V_0}{V_L}\right)^{1/2} \left(\frac{V_0}{V_L}\right)^{-m/4}, \tag{A.4}$$

and pass to the corresponding limits for m tending to zero.

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