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An approach to gauge potentials in the non-Abelian *ISO*(3)-gauge model of defects in solids

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A METHOD is proposed to reduce the Cartan structure equations and the Bianchi identities of the non-Abelian $ISO(3)$-gauge model of defects in solids to the appropriate relations of the theory of disclinations considered as the Abelian $\mathfrak{iso}(3)$-gauge model. As the result, the possibility arises to identify the $ISO(3)$-gauge potentials in terms of the defect loop densities.

1. Introduction

IN THE REFS. [1, 2] it has been shown that both the field equations and the continuity equations of the theory of disclinations [3] can be rewritten in the form inherent to Abelian gauge models. The Lie algebra $\mathfrak{iso}(3)$ of the group $ISO(3) = T(3) \otimes SO(3)$ ($\otimes$ – semi-direct product of groups) of 3D rigid body motions can be considered as additive Abelian group (as a vector space $\mathbb{R}^6$, in fact) and it plays the role of non-compact gauge group in the picture revealed in [1, 2]. In Refs. [1, 2] the possibility is suggested to apply a special exterior calculus where the role of exterior differential is played by the SCHAEFER'S differential [4]. Once the additive action of $\mathfrak{iso}(3)$ is non-homogeneous (i.e. coordinate-dependent) under the Schaefer's differential, both the disclination and dislocation densities appear as components of the $\mathfrak{iso}(3)$-valued gauge field strength. Eventually, the field equations of the defect theory [3] acquire the form of Cartan structure equation, while the continuity equations – of Bianchi identity of a certain Abelian gauge model.

On the other hand, the idea is widely known (cf. [5, 6]) to use $ISO(3)$ to formulate a geometrically nonlinear dynamical theory of defects in solids as a classical model of Yang-Mills type (that is $ISO(3)$ is attempted to be gauged as internal symmetry in [5, 6]). The algebra $\mathfrak{iso}(3)$ plays an important role in [5, 6] thus rising the question: is it possible to reduce certain relations of this general model so that the corresponding ones of the theory [3] will appear just in the $\mathfrak{iso}(3)$-representation found in [1, 2]? Such reduction would deserve consideration because, linear as it is, the approach [3] (R. deWit emphasizes that the linear assumption promotes complete analytical computations) provides a reliable scheme for a number of calculations concerning both the isolated defects and their distributions (see [7, 8] and numerous refs. therein).

The question proposed has been asked at the end of [1], and the present paper will point out a possible strategy to answer it. That is, a way to map the geometric relations of the $ISO(3)$-gauge model to the appropriate relations of [3] is presented below. Namely, it is proposed to reduce by linearization the $ISO(3)$ Cartan equations to the "Cartan equation" (i.e. to the field equations written in the $\mathfrak{iso}(3)$-representation) of the theory [3]. The same is true for the Bianchi identities. Provided a correspondence between the gauge transformation groups is accounted for, a special restriction arises for the $\mathfrak{iso}(3)$-gauge parameters. It is demonstrated that this restriction is fulfilled by a finite loop defect. As the result, the correspondence established allows for a "mechanical" interpretation for the $ISO(3)$-fields. Though a different one has been proposed in [5, 6], we hope to argue the naturalness of that suggested here.

The paper is organized as follows. Section 2 reminds briefly [1, 2] to specify the meaning of the $\mathfrak{iso}(3)$-gauge fields. Section 3 contains both the Cartan structure equations and the Bianchi identities of the $ISO(3)$-gauge model and the truncation prescription for them. Section 4 concerns the mapping between the two sets of relations. Discussion in the Sec. 5 completes the paper.

We establish the following conventions. Our consideration is time-independent and all indices run from 1 to 3, the repeated ones imply summation. The Lie algebra $\mathfrak{so}(3)$ consists of real skew matrices of third order and the matrix elements of its three generators l_a coincide with the permutation symbol components. Therefore we shall represent $\lambda = \lambda_a l_a \in \mathfrak{so}(3)$ as 3-vectors $\boldsymbol{\lambda}$ and matrix action of λ as vector multiplication. For elements of $\mathfrak{iso}(3)$ two notations are equivalent: $\begin{pmatrix}\boldsymbol{\lambda}\\ \boldsymbol{\eta}\end{pmatrix}$ (by R. VON MISES [9]) and $\boldsymbol{\eta} \oplus \boldsymbol{\lambda}$ (semi-direct sum), where $\boldsymbol{\eta} \in \mathfrak{t}(3)$ and $\boldsymbol{\lambda} \in \mathfrak{so}(3)$. For shortness $G \equiv ISO(3)$ and $\mathfrak{g} \equiv \mathfrak{iso}(3)$.

2. The theory of disclinations as the Abelian $\mathfrak{iso}(3)$-gauge model

In this section we sketch some basic relations obtained in [1, 2]. Let M be a flat three-dimensional manifold and T^*M be a cotangent bundle over M. The objects we need here are sections

$$\omega^{(n)} \in \Omega^n(M) \equiv C^\infty(M, \mathfrak{g} \otimes \wedge^n(T^*M)) \tag{2.1}$$

of sheaves [10] of smooth differential n-forms taking their values in $\mathfrak{g}$. In (2.1) $\wedge^n(T^*M)$ means exterior product of n copies of T^*M (n does not exceed 3, clearly). According to the "6-vector" structure of $\mathfrak{g}$, we shall put $\omega^{(n)}$ (2.1) as $\overset{(n)}{\boldsymbol{\eta}} \oplus \overset{(n)}{\boldsymbol{\lambda}}$, where both the vectors are referred to a frame $\{\mathbf{e}_a\}$ in a covering U. That is to say that as the frame $\{\mathbf{e}_a\}$ is transformed, the vectors $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$ are transformed too.

Apart from the standard exterior differential d, there exists a homeomorphism d^{Sh} on $\Omega^n(M)$ such that n is increased by one:

$$d^{Sh} : \Omega^n(M) \to \Omega^{n+1}(M) . \tag{2.2}$$

The corresponding operator is known as the Schaefer's differential [4] and with respect to $\{\mathbf{e}_a\}$ it takes the form:

$$d^{Sh}\overset{(n)}{\omega} = \begin{pmatrix} d\overset{(n)}{\boldsymbol{\lambda}} \\ d\overset{(n)}{\boldsymbol{\eta}} + d\mathbf{x} \overset{\times}{\wedge} \overset{(n)}{\boldsymbol{\lambda}} \end{pmatrix}, \tag{2.3}$$

where $\overset{\times}{\wedge}$ implies that the differential forms are multiplied externally while their coefficients as 3-vectors. It can be verified that d^{Sh} (2.3) is nilpotent, i.e. $d^{Sh} \circ d^{Sh} = 0$ [4, 1]. The definition (2.3) gives us the corresponding partial differentiation operator $P_a \equiv \partial_a^{Sh}$ which is one of the two generators of $\mathfrak{g}$. The second one $M_a \equiv (\mathbf{x} \times \boldsymbol{\partial})_a^{Sh}$ has been found in [1, 2] so that P_a and M_a fulfil together the fundamental brackets which display $\mathfrak{g}$ as a semi-direct sum of $\mathfrak{t}(3)$ and $\mathfrak{so}(3)$.

Using the definitions (2.1) – (2.3) one gets the following relations in the theory of disclinations [3]. Let us consider $\mathcal{A}^{Sh} \in \Omega^1(M)$

$$\mathcal{A}^{Sh} = \boldsymbol{\gamma} \mathbin{\dot{+}} \boldsymbol{\zeta}, \tag{2.4}$$

and $\mathcal{F}^{Sh} \in \Omega^2(M)$ in the form

$$\mathcal{F}^{Sh} \equiv \boldsymbol{\mathcal{F}}_m \mathbin{\dot{+}} \boldsymbol{\mathcal{F}}_v = d^{Sh}\mathcal{A}^{Sh}, \tag{2.5}$$

that is

$$\begin{aligned} \boldsymbol{\mathcal{F}}_v &= d\boldsymbol{\zeta}, \\ \boldsymbol{\mathcal{F}}_m &= d\boldsymbol{\gamma} + \boldsymbol{\zeta} \overset{\times}{\wedge} d\mathbf{x}. \end{aligned} \tag{2.6}$$

Defining duals to the coefficients of $\boldsymbol{\mathcal{F}}_v$ and $\boldsymbol{\mathcal{F}}_m$ by

$$\theta_{qp} = \frac{1}{2}\epsilon^{qab}(\mathcal{F}_v)^p_{ab}, \qquad \alpha_{qp} = \frac{1}{2}\epsilon^{qab}(\mathcal{F}_m)^p_{ab}, \tag{2.7}$$

one has (2.6) written in components:

$$\begin{aligned} \theta_{qp} &= \epsilon^{qab}\partial_a\zeta_{bp}, \\ \alpha_{qp} &= \epsilon^{qab}(\partial_a\gamma_{bp} + \epsilon^{bpc}\zeta_{ac}). \end{aligned} \tag{2.8}$$

The formulas (2.8) are nothing but the fundamental relationships between disclination and dislocation densities θ_{qp}, α_{qp} and disclination and dislocation loop densities ζ_{bp}, γ_{bp} just in the sense of [3]. In other words, the coefficients of $\mathcal{A}^{Sh}$ (2.4) can be viewed as the defect loop densities, once the L.H.S. of (2.5) is considered as the motor of the defect densities. Note that we call Eqs. (2.8) the field equations to distinguish them from the Eqs. (2.10) below.

Let us also introduce the vector-valued 1-form $\boldsymbol{\varphi}$ by the equation

$$\boldsymbol{\zeta} = \boldsymbol{\varphi} - \frac{1}{2} d\,(\mathbf{e}_c \times \boldsymbol{\gamma}_c)\,, \tag{2.9}$$

where $\boldsymbol{\gamma}_c$ are the coefficients of $\boldsymbol{\gamma}$ (2.4) and $\{\mathbf{e}_c\}$ is the frame. Then (2.8) reads:

$$\begin{aligned} \theta_{qp} &= \epsilon^{qab}\partial_a \varphi_{bp}\,, \\ \alpha_{qp} &= \epsilon^{qab}(\partial_a \gamma_{(bp)} + \epsilon^{bpc}\varphi_{ac})\,, \end{aligned} \tag{2.10}$$

where $\gamma_{(bp)}$ implies symmetrization $(1/2)(\gamma_{bp} + \gamma_{pb})$. It seems that the reasons to write (2.9) are independent from the matter considered here, i.e. they have nothing to do with the algebra and gauging at hands below and therefore [3] contains more information about them. Once we introduce $-e^P_{bp}$ instead of $\gamma_{(bp)}$ and $-\kappa^P_{bp}$ instead of φ_{bp}, (2.10) express the defect densities through the basic plastic fields of strain (e^P_{bp}) and bend-twist (κ^P_{bp}) [3]. Recall that in order to extend the theory of dislocations so that both translations and rotations would be no longer integrable, R. deWit has proposed to postulate basic plastic fields of strain and bend-twist instead of plastic distortion (which does not exist with disclinations) to describe static distributions of defects.

By nilpotency of d^{Sh} it is seen that integrability for the Eq. (2.5) is expressed by

$$d^{Sh}\mathcal{F}^{Sh} = 0\,, \tag{2.11}$$

or

$$\begin{aligned} d\boldsymbol{\mathcal{F}}_v &= 0, \\ d\boldsymbol{\mathcal{F}}_m - \boldsymbol{\mathcal{F}}_v \overset{\times}{\wedge} d\mathbf{x} &= 0. \end{aligned} \tag{2.12}$$

It is straightforward to verify that the Eqs. (2.12) imply the standard continuity equations for α_{qp}, θ_{qp} provided (2.7) holds [1, 2]. Though (2.4)–(2.6), (2.11), (2.12) has already appeared in [1, 2], their interpretation is more transparent here. The Eqs. (2.9), (2.10) are useful connecting [1] and [3].

To conclude the section, the 2-form $\mathcal{F}^{Sh}$ (2.5) (gauge field strength) is invariant under the shift

$$\mathcal{A}^{Sh} \to \mathcal{A}^{Sh} - \delta\mathcal{A}^{Sh}\,, \qquad \delta\mathcal{A}^{Sh} = d^{Sh}\omega^{(0)} \tag{2.13}$$

for any $\omega^{(0)} \in \Omega^0(M)$. Therefore it is seen that the Eqs. (2.5) ("Cartan structure equation"), (2.11) ("Bianchi identity") and (2.13) (gauge transformation group) display us the theory of disclinations [3] as the Abelian gauge model with the additive gauge group $\mathfrak{g} \approx \mathbb{R}^6$ [1, 2].

3. Cartan structure equations and Bianchi identities of the $ISO(3)$-gauge model

Now let us have a look at the geometric relations in affine gauge models, i.e. in the models using principal fiber bundles of affine frames as geometric background. To this end we shall follow [11] (see also [12]) but more rigorous details on bundles of affine and linear frames can be found in [13, 14].

Let us start with the bundles $\mathbb{A}(M)$ of affine frames and $\mathbb{L}(M)$ of linear frames over an arbitrary manifold M. Our gauge group is G. We shall denote by ε the homeomorphism of $\mathbb{L}(M)$ to $\mathbb{A}(M)$ induced by the injection $\mathbb{L}(M) \to \boldsymbol{O} \otimes \mathbb{L}(M) \subset \mathbb{A}(M)$ ($\boldsymbol{O}$ is a "zero" vector). Let $\tilde{\mathbb{A}}$ be a generalized affine connection 1-form on $\mathbb{A}(M)$. The conjugated homeomorphism (pullback) ε^* maps it in a $\mathfrak{g}$-valued 1-form on $\mathbb{L}(M)$ which is split as follows:

$$\varepsilon^* \tilde{\mathbb{A}} = \boldsymbol{\phi} \dotplus \mathbf{A}, \tag{3.1}$$

where $\mathbf{A}$ and $\boldsymbol{\phi}$ are $\mathbb{R}^3$-valued differential 1-forms on $\mathbb{L}(M)$. The corresponding affine curvature 2-form on $\mathbb{A}(M)$ is $\tilde{\mathbb{F}}$ and it also is split into the translation and linear parts $\mathbf{\Phi}$ and $\mathbf{F}$:

$$\varepsilon^* \tilde{\mathbb{F}} = \mathbf{\Phi} \dotplus \mathbf{F}. \tag{3.2}$$

The 1-form $\mathbf{A}$ can be referred to as a linear connection on M, while $\mathbf{F}$ as its curvature 2-form (both are $\mathfrak{so}(3)$-valued, in fact). The couple of structure equations holds for the objects in the R.H.S. of (3.1), (3.2):

$$\begin{aligned} d\mathbf{A} + (1/2)\mathbf{A} \overset{\times}{\wedge} \mathbf{A} &= \mathbf{F}, \\ d\boldsymbol{\phi} + \mathbf{A} \overset{\times}{\wedge} \boldsymbol{\phi} &= \mathbf{\Phi}, \end{aligned} \tag{3.3}$$

where $\overset{\times}{\wedge}$ is defined in the Sec. 2.

It is well known that the translation part $\mathbf{\Phi}$ of the affine curvature $\tilde{\mathbb{F}}$ is transformed non-homogeneously under the infinitesimal affine gauge transformation

$$\begin{aligned} \mathbf{A} &\to \mathbf{A} - \mathbf{A} \times \boldsymbol{\lambda} - d\boldsymbol{\lambda}, \\ \boldsymbol{\phi} &\to \boldsymbol{\phi} + \boldsymbol{\lambda} \times \boldsymbol{\phi} - \mathbf{A} \times \boldsymbol{\eta} - d\boldsymbol{\eta}, \end{aligned} \tag{3.4}$$

where $\boldsymbol{\eta} \dotplus \boldsymbol{\lambda} \in \Omega^0$ are the group parameters, and therefore it is impossible to consider it as the torsion of the linear connection $\mathbf{A}$ (3.1) though the Eq. $(3.3)_2$ looks properly [14]. In order to "extract" from $\mathbf{\Phi}$ the contribution which is transformed under (3.4) appropriately, let us define the vector-valued zero-form $\boldsymbol{\chi}$ ("affine Higgs" field, following [11]) which is a local cross-section of an associated vector bundle and its gauge transformation is

$$\boldsymbol{\chi} \to \boldsymbol{\chi} + \boldsymbol{\lambda} \times \boldsymbol{\chi} + \boldsymbol{\eta}.$$

Then

$$\boldsymbol{\vartheta} = \boldsymbol{\Phi} + \mathbf{F} \times \boldsymbol{\chi} \tag{3.5}$$

is transformed as required:

$$\boldsymbol{\vartheta} \to \boldsymbol{\vartheta} + \boldsymbol{\lambda} \times \boldsymbol{\vartheta}.$$

Namely $\boldsymbol{\vartheta}$ (3.5) can be referred to as the torsion 2-form of the linear connection $\mathbf{A}$, while $\mathbf{F}$ is its curvature [11, 14]. Now the Eq. (3.5) may be rewritten as the corresponding Cartan structure equation

$$d\boldsymbol{B} + \mathbf{A} \overset{\times}{\wedge} \boldsymbol{B} = \boldsymbol{\vartheta}, \tag{3.6}$$

where $\boldsymbol{B} = \boldsymbol{\phi} + d\boldsymbol{\chi} + \mathbf{A} \times \boldsymbol{\chi}$ can be thought as a canonical ("soldering") 1-form [11, 12, 14–16]. Therefore $(3.3)_1$ and (3.6) give us the couple of Cartan structure equations where all the ingredients are transformed appropriately. The corresponding Bianchi identities appear straightforwardly:

$$\begin{aligned} d\mathbf{F} &= \mathbf{F} \overset{\times}{\wedge} \mathbf{A}, \\ d\boldsymbol{\vartheta} &= \mathbf{F} \overset{\times}{\wedge} \boldsymbol{B} - \mathbf{A} \overset{\times}{\wedge} \boldsymbol{\vartheta}. \end{aligned} \tag{3.7}$$

As to the matter at hands, the picture suggested in [5, 6] seems to be geometrically very close to affine gauge models with the group G because the fundamental Eqs. (3.3), (3.6), (3.7) have been extensively used there. The basic "fields" $\mathbf{A}$, $\boldsymbol{\phi}$, $\boldsymbol{\chi}$ have been supplied in [5, 6] with space and time dependence (i.e. dim M is four) to be considered as dynamical variables describing media with continuously distributed defects. As it might be understood from [12] (the Chapter 3), whenever affine gauge models are concerned, the "affine Higgs" field ($\boldsymbol{\chi}$ in our context) gets an appropriate problem-motivated interpretation. In the monograph [6] considerable attention has been paid to motivate $\boldsymbol{\chi}$ in the framework of the dynamical model of defects. Loosely speaking one can say that $\boldsymbol{\chi}$ has been put there as a field of current configurations $\boldsymbol{\chi} = \mathbf{x} + \mathbf{u}(\mathbf{x}, t)$, so that $\mathbf{u}(\mathbf{x}, t)$ implies a displacement corresponding to point $\mathbf{x}$ in a reference configuration at time t.

The truncated version of $(3.3)_1$, (3.6), (3.7) we are interested in relies drastically on the decomposition $\boldsymbol{\chi} = \mathbf{x} + \mathbf{u}$, and can easily be obtained. To this end let us consider $\mathbf{u}$, functions parametrizing the differential forms $\mathbf{A}$, $\boldsymbol{\phi}$, and their derivatives to be small so that all their products can be neglected. It means, for instance, that (3.4) are simply shifts by exact 1-forms $d\boldsymbol{\lambda}$ and $d\boldsymbol{\eta}$. Linearizing the L.H.S. of Eqs. (3.3) one obtains for $\mathbf{F}$ $(3.3)_1$ and $\boldsymbol{\vartheta}$ (3.5):

$$\begin{aligned} d\mathbf{A} &= \mathbf{F}, \\ d\boldsymbol{\phi} + d\mathbf{A} \times \mathbf{x} &= \boldsymbol{\vartheta}. \end{aligned} \tag{3.8}$$

Further, simplifying analogously the R.H.S. of (3.7) one can see that the resulting equations

$$\begin{aligned} d\mathbf{F} &= 0, \\ d\boldsymbol{\vartheta} &= \mathbf{F} \overset{\times}{\wedge} d\mathbf{x}, \end{aligned} \tag{3.9}$$

are the integrability conditions for (3.8). It has to be noticed that the prescription alleged to drop out the products would imply in fact not spacially global but rather local (being valid, say, only for certain regions) weakness of some concrete field configurations which display a chosen geometry by means of the set of Eqs. (3.3), (3.6), (3.7).

4. The mapping

At last let us establish the correspondence between the Eqs. (2.6), (2.12) of the $\mathfrak{iso}(3)$-model and the Eqs. $(3.3)_1$, (3.6), (3.7) of the $ISO(3)$-gauge model of defects in solids. For as it has already been stressed, we shall do it by comparing the first group of equations with the truncated ones (3.8), (3.9).

It is indeed seen that (2.6) and (3.8) being written as

$$d^{Sh}\begin{pmatrix} \mathbf{A} \\ \boldsymbol{\phi} + \mathbf{A} \times \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \boldsymbol{\vartheta} \end{pmatrix} \tag{4.1}$$

look similar and lead us to the following basic identification: the connection 1-form $\mathbf{A}$ would be the 1-form of disclination loop densities $\boldsymbol{\zeta}$ while the 1-form $\mathbf{A} \times \mathbf{x} + \boldsymbol{\phi}$ would be the 1-form of dislocation loop densities $\boldsymbol{\gamma}$. Once this interpretation is accepted, it becomes natural to say that the curvature and the torsion 2-forms $\mathbf{F}$ and $\boldsymbol{\vartheta}$ in the R.H.S. of (3.8) acquire the sense of $\mathcal{F}_v$ and $\mathcal{F}_m$, accordingly, i.e. of the disclination and dislocation densities 2-forms. It has to be reminded that the idea to identify dislocation density as a differential geometric torsion is not new at all [17]. Finally, the continuity equations (2.12) and (3.9) are fairly identical upon the identifications proposed. However, the correspondence of the gauge transformations requires some attention. Besides, the fact that the continuity equations for α_{pq}, θ_{pq} result from the linearized Bianchi identities, has been discussed also in [18] but in the framework of a metric-torsion gauge approach to the continuum defects.

In view of (3.4) the gauge variation of $\boldsymbol{\phi} + \mathbf{A} \times \mathbf{x} \oplus \mathbf{A}$ reads:

$$\delta\begin{pmatrix} \mathbf{A} \\ \boldsymbol{\phi} + \mathbf{A} \times \mathbf{x} \end{pmatrix} = \begin{pmatrix} d\boldsymbol{\lambda} \\ d\boldsymbol{\eta} + d\boldsymbol{\lambda} \times \mathbf{x} \end{pmatrix}, \tag{4.2}$$

where the R.H.S. is not a d^{Sh}-differential as in (2.13):

$$\delta \mathcal{A}^{Sh} = d^{Sh}\begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} d\boldsymbol{\omega} \\ d\mathbf{u} - \boldsymbol{\omega} \times d\mathbf{x} \end{pmatrix}. \tag{4.3}$$

However, in both the cases $\vartheta \oplus \mathbf{F}$ (4.1) and $\mathcal{F}^{Sh}$ (2.5) are d^{Sh}-differentials invariant under the variations (4.2) and (4.3), respectively. The case here is simple: the relation

$$d(d\mathbf{v} \times \mathbf{x}) = -d\mathbf{v} \overset{\times}{\wedge} d\mathbf{x} = d(-\mathbf{v} \times d\mathbf{x}) \tag{4.4}$$

holds for any 3-vector $\mathbf{v}$ and it is why $\mathcal{F}^{Sh}$ and $\vartheta \oplus \mathbf{F}$ are both invariant.

The transformation parameters $\boldsymbol{\lambda}$ and $\boldsymbol{\eta}$ are small independent 3-vector functions in the $ISO(3)$ approach. Let us put the R.H.S. of (4.2) as a complete d^{Sh}-differential:

$$\begin{pmatrix} d\boldsymbol{\lambda} \\ d(\boldsymbol{\eta} + \boldsymbol{\lambda} \times \mathbf{x}) - \boldsymbol{\lambda} \times d\mathbf{x} \end{pmatrix}. \tag{4.5}$$

After this it is suggestive to state that $\boldsymbol{\lambda}$ (4.5) would correspond to $\boldsymbol{\omega}$ (4.3) and $\boldsymbol{\eta} + \boldsymbol{\lambda} \times \mathbf{x}$ (4.5) to $\mathbf{u}$ (4.3). This observation means that the gauge parameters in (4.3) have not to be considered as independent to get one-to-one correspondence (at the linearized level) with the $ISO(3)$-gauge geometry. Precisely, $\boldsymbol{\omega}$ as rotation becomes related to the displacement $\mathbf{u}$ by $\frac{1}{2}\boldsymbol{\partial} \times \mathbf{u}$ at constant $\boldsymbol{\lambda}$, $\boldsymbol{\eta}$. In this way, the Eqs. $(3.3)_1$, (3.6) and (3.7) supplied with the gauge transformation rules indeed result in the $\mathfrak{iso}(3)$-represented relations of the theory of disclinations under the truncation prescribed.

In order to argue the correspondence presented, it is worth to find concrete non-trivial gauge transformation in the theory [3] which fulfils the restriction found, thus confirming its meaning. Fortunately, such example is given by an isolated defect loop

$$\begin{aligned} \boldsymbol{\zeta}_k &= \delta_k(S)\,\boldsymbol{\Omega}, \\ \boldsymbol{\gamma}_k &= \delta_k(S)\,(\mathbf{b} + \boldsymbol{\Omega} \times \mathbf{x}), \end{aligned} \tag{4.6}$$

where $\boldsymbol{\Omega}$ and $\mathbf{b}$ are the Frank and the Burgers vectors, and $\delta_k(S)$ is the singular function concentrated on the surface S [19, 3]. Here S is an open surface (so-called jump surface) which is bounded by closed defect line $L = \partial S$. It is easily seen that (4.6) respects our basic identification if we put $\mathbf{A}_k = \delta_k(S)\boldsymbol{\Omega}$ and $\boldsymbol{\phi}_k = \delta_k(S)\mathbf{b}$.

Let us define another open surface $\widetilde{S}$ which is also bounded by L but oriented with respect to L oppositely to S so that $S \cup \widetilde{S}$ is a closed smooth surface enclosing the volume V. Further, the variation (4.3) with the parameters

$$\begin{aligned} \boldsymbol{\omega} &= \delta(V)\,\boldsymbol{\Omega}, \\ \mathbf{u} &= \delta(V)\,(\mathbf{b} + \boldsymbol{\Omega} \times \mathbf{x}) \end{aligned} \tag{4.7}$$

acts on (4.6) as follows:

$$\begin{aligned} &\boldsymbol{\zeta}_k \to \boldsymbol{\zeta}_k + \partial_k\,(\delta(V))\,\boldsymbol{\Omega}\,, \\ &\boldsymbol{\gamma}_k \to \boldsymbol{\gamma}_k + \partial_k\,[\delta(V)\,(\mathbf{b} + \boldsymbol{\Omega} \times \mathbf{x})] - \delta(V)\boldsymbol{\Omega} \times \mathbf{e}_k \\ &\qquad = \boldsymbol{\gamma}_k + \partial_k\,(\delta(V))\,(\mathbf{b} + \boldsymbol{\Omega} \times \mathbf{x})\,. \end{aligned} \tag{4.8}$$

Taking into account

$$\partial_k\,(\delta(V)) = -\delta_k(S \cup \widetilde{S})$$

and the formal equation

$$\delta_k(S) - \delta_k(S \cup \widetilde{S}) = -\delta_k(\widetilde{S}) = \delta_k(-\widetilde{S})$$

($-\widetilde{S}$ and $\widetilde{S}$ are of opposite orientation), again one obtains from (4.8) the loop (4.6) with the jump surface $-\widetilde{S}$. Therefore, the transformation given by (4.3), (4.7) is nothing but an orientation-preserving change of non-physical jump surface of the defect loop. Besides, the R.H.S. of (4.8) looks like the R.H.S. of (4.2) thus confirming the coincidence of the two sets of relations. For (4.6) the continuity equations (2.12) are satisfied too and so, the solution found indeed behaves properly. The defect loop (4.6) serves in [3] as the source which allows to obtain, for instance, the complete set of relations characterizing straight dislocation and disclination in an isotropic infinite body.

Before concluding the section let us try to make the correspondence found more transparent. Indeed, the defect loop definition (4.6) turns out to be a complete d^{Sh}-differential if one admits the surface S to be closed:

$$\begin{pmatrix}\boldsymbol{\zeta}\\ \boldsymbol{\gamma}\end{pmatrix} = -\begin{pmatrix} d\boldsymbol{\mathcal{V}} \\ d\boldsymbol{\mathcal{U}} + d\boldsymbol{\mathcal{V}}\times\mathbf{x}\end{pmatrix} = -\,d^{Sh}\begin{pmatrix}\boldsymbol{\mathcal{V}}\\ \boldsymbol{\mathcal{U}} + \boldsymbol{\mathcal{V}}\times\mathbf{x}\end{pmatrix}, \tag{4.9}$$

where $\boldsymbol{\mathcal{V}} = \delta(W)\,\boldsymbol{\Omega}, \boldsymbol{\mathcal{U}} = \delta(W)\,\mathbf{b}$, and W is volume inside S. This is because the components $\partial_k S$ become the derivatives $-\partial_k\,(\delta(W))$ for closed S. One simply has to replace $d\boldsymbol{\mathcal{V}}$ and $d\boldsymbol{\mathcal{U}}$ by certain 1-forms $\mathbf{A}$ and $\boldsymbol{\phi}$ which are not exact, to break the d^{Sh}-exactness of $\boldsymbol{\gamma} \oplus \boldsymbol{\zeta}$ (4.9) and to obtain (4.6). It is just the way how the definition (4.6) appears for unclosed surface bounded by defect line. From a more general point of view, the Eqs. (4.6), (4.7) and (4.9) are particular manifestations of the sequence of homeomorphisms (2.2) being considered for elements $\mathcal{P}^{(n)} \in \Omega^n(M)$ of the following form:

$$\mathcal{P}^{(n)} = \begin{pmatrix}\boldsymbol{\mathcal{V}}\\ \boldsymbol{\mathcal{U}} + \boldsymbol{\mathcal{V}}\times\mathbf{x}\end{pmatrix} \xrightarrow{d^{Sh}} \begin{pmatrix} d\boldsymbol{\mathcal{V}} \\ d\boldsymbol{\mathcal{U}} + d\boldsymbol{\mathcal{V}}\times\mathbf{x}\end{pmatrix}, \tag{4.10}$$

where $\boldsymbol{\mathcal{V}}$ and $\boldsymbol{\mathcal{U}}$ imply now vector-valued n-forms. This special choice of $\mathcal{P}^{(n)}$ ensures that they are motors under coordinate shift $\mathbf{x} \to \mathbf{x} + \mathbf{y}$. Now it is seen that (4.9) gives the action of d^{Sh} on $\mathcal{P}^{(0)}$ being a rigid body displacement of inclusion W in motor representation. The coefficients (4.6) correspond to the defect loop 1-form $\mathcal{P}^{(1)}$ which is not d^{Sh}-exact. Differentiating $\mathcal{P}^{(1)}$ (4.6) one obtains the defect densities 2-form $\mathcal{P}^{(2)}$ (compare with (2.5), (2.6)) where the density components $\boldsymbol{\mathcal{V}}_{kl} = \epsilon_{klm}\,\delta_m(L)\,\boldsymbol{\Omega}, \boldsymbol{\mathcal{U}}_{kl} = \epsilon_{klm}\,\delta_m(L)\,\mathbf{b}$ are singular on the line L (see [19, 3] about $\delta_m(L)$). Moreover, one can see that gauge variations $\delta\mathcal{P}^{(1)}$ preserving any d^{Sh}-exact defect density $\mathcal{P}^{(2)}$ are not of the general form (2.13), (4.3) but rather of the form (4.2).

Realizing the differential complex (4.10) to contain all the relevant gauge-geometric information concerning [3], it is straightforward to try to relate it to the $ISO(3)$ geometric relations which would generalize the Abelian ones. As it has been stressed, the underlying algebra $\mathfrak{iso}(3)$ is of importance here. From (3.8) and (4.10) at $n = 1$ it is especially easy to understand the above key identification: after truncation in the structure equations $(3.3)_1$, (3.5), the gauge potentials $\mathbf{A}$ and $\boldsymbol{\phi}$ coincide with the corresponding elements of the defect loop densities 1-form $\mathcal{P}^{(1)}$ and the R.H.S. of (3.8) acquires the sense of motor of the defect densities. The Bianchi identities do not impose extra restrictions and are fairly identical. Therefore the reduction proposed both points out natural interpretation for the $ISO(3)$-gauge fields and shows definitely that the reduced gauge transformation can be combined as that corresponding to (4.10).

5. Discussion

We have described the reduction of the Cartan structure equations and the Bianchi identities of the $ISO(3)$-gauge model of defects in solids to the field equations and the continuity equations of the theory of disclinations being considered as the Abelian $\mathfrak{iso}(3)$-gauge model. It is the basis for both the cases underlying Lie algebra $\mathfrak{iso}(3)$ which prompts the idea to do this reduction by linearizing the non-Abelian geometric relations. Requiring additionally that the gauge parameters (rotations and translations) of the $\mathfrak{iso}(3)$-model are not independent, it is possible to display a certain correspondence between the two sets of relations.

Special example in the Sec. 4 fulfils the restriction found for the gauge parameters thus seemingly testifying on behalf of the chosen strategy. Besides, it is known that point-like sources are forbidden for non-compact Abelian gauge group, i.e. only sourceless strings might appear as solutions to the corresponding equations. So it is attractive to encounter the densities of the defect loops (closed strings) in our investigation. It is hopeful that such loop solutions should find generalization in non-Abelian situation.

The way proposed to identify the $ISO(3)$-gauge fields differs from that in [5, 6], and it is worth to pay some more attention to this fact. It is crucial that the pair of equations analogous to the Eqs. (2.12) has been written in [5, 6] as

$$(5.1)\qquad \begin{aligned} d\boldsymbol{\Omega} &= 0, \\ d\mathbf{D} - \boldsymbol{\Omega} &= 0, \end{aligned}$$

(time is fixed) so that $\mathbf{D}$ and $\boldsymbol{\Omega}$ $(5.1)_2$ are the vector-valued differential forms corresponding to $\mathcal{F}_m$ and $\mathcal{F}_v \overset{\times}{\wedge} d\mathbf{x}$ $(2.12)_2$, accordingly. Because the disclination density $\boldsymbol{\Omega}$ has been referred to as a 3-form in [5, 6], the Eq. $(5.1)_1$ holds identically for the three-dimensionality of d, though it looks like integrability condition for $(5.1)_2$.

After this the following key identification has been made in [5, 6] to pass to the non-Abelian case: the kinematic equation

$$d\mathbf{B} + \mathbf{K} = \mathbf{D}$$

(it is reminiscent of $(2.6)_2$ in our notation, see also $(2.10)_2$ and the comment after that), where **B** is distortion 1-form and **K** is bend-twist 2-form, would correspond to the Cartan equation (3.6) itself (not to its linearization $(3.8)_2$ as we do) so that $\boldsymbol{B}$, $\mathbf{A} \overset{\times}{\wedge} \boldsymbol{B}$, and $\boldsymbol{\vartheta}$ would be **B**, **K**, and **D**, respectively. It is for these reasons that the Bianchi equation $(3.7)_2$ has been considered as the generalization of $(5.1)_2$ so that just the 3-form $\mathbf{F} \overset{\times}{\wedge} \boldsymbol{B} - \mathbf{A} \overset{\times}{\wedge} \boldsymbol{\vartheta}$ acquires the sense of the source $\boldsymbol{\Omega}$ in the nonlinear situation. Obviously, such generalized "disclinations" will be also sourceless.

Conversely, in our approach both the disclination and dislocation densities are the vector and moment parts of the $\mathfrak{iso}(3)$-valued 2-form $\mathcal{F}^{Sh}$ which appears owing to the use of d^{Sh}. Therefore both the couples of the $ISO(3)$ Cartan structure equations and Bianchi identities after linearization are also considered as vector and moment parts of certain $\mathfrak{iso}(3)$-valued equations. Further, the 1-form $\mathbf{A}\times\mathbf{x}+\boldsymbol{\phi}$ (up to exact contribution it is just the reduced "soldering" 1-form) is identified here as the dislocation loop densities 1-form while in [5, 6] it is a distortion 1-form. The linear connection 1-form **A** corresponds here to the disclination loop densities 1-form.

The distinctive suggestion of the given approach is that that the curvature 2-form **F** $(3.3)_1$ should be treated as a generalization of the disclination density $\mathcal{F}_v$. For the truncated case it is "almost" as $d\mathbf{A} \overset{\times}{\wedge} d\mathbf{x}$ which arises from $\boldsymbol{\Omega} = \mathbf{F} \overset{\times}{\wedge} \boldsymbol{B} - \mathbf{A} \overset{\times}{\wedge} \boldsymbol{\vartheta}$ [5, 6] (see [1]), but the general situation is different because $d\mathbf{F}$ is not zero by $(3.7)_1$ and therefore nonlinearity of the $ISO(3)$ model can result in sources for "disclinations". Besides, the way how the group of the $ISO(3)$-gauge transformations includes that of the disclination theory is also different in [5, 6], i.e. it is not in the sense of one-to-one correspondence as in the Sec. 4.

To conclude, the given approach seems to show that the way adopted in [5, 6] to connect the affine gauge model with the classical defect model is not the only one possible. The main disagreement between the two viewpoints is clear. The present treatment proposes that only after reduction (e.g. asymptotically), the non-Abelian $ISO(3)$ relations could be identified in the framework of the theory of disclinations [3], whereas in [5, 6] the idea is that the form of the appropriate equations of the defect theory remains externally unchanged but their ingredients become complicated for $ISO(3)$. But surely the decisive conclusion would be drawn only by explicit $ISO(3)$ stringy solutions which allow simple loops, like that found above as a limiting case. Besides, having in mind a successive descent from affine frames to orthogonal ones [11, 13], it would be interesting to make contact with [17, 18, 20] where similar problems have been treated in a metric approach. Anyway, further considerations seem to be needed.

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