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On double waves and wave-wave interaction in gasdynamics

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SPECIAL CLASSES of potential isentropic nonstationary flows in two space dimensions are considered. We demonstrate that locally these solutions can be understood as resulting from what we call *elastic* interaction (since no other waves are produced) of two Riemann waves (simple waves). It appears that this is a generic property of interactions of sound modes in gasdynamics. That is, two nonlinear (localised) sound waves propagating at any angle can cross each other without producing new waves – similarly as it happens in one space dimension.

1. Introduction

IN THIS PAPER we deal with certain classes of isentropic nonstationary flows of an ideal compressible fluid. Each of these classes depends on two arbitrary functions of one variable and may be obtained as a solution of a certain specialized system of two hyperbolic equations with two dependent and two independent variables. We call them hyperbolic double waves or, for short, double waves. It appears that they can be understood as resulting from the special type of interaction of two simple waves. This interaction can be called *elastic* since collision of two waves leads also to two waves, in contrast to the case of nonelastic interaction [18, 19]. This idea can be clearly explained in the case of one space dimension. One can take two localised perturbations in the form of two simple waves which are approaching each other. Then they begin to interact. Depending on the nature of the waves, different scenarios are possible:

1. In spite of nonlinear interaction, the waves can cross the region of interaction, the state of rest being there restored.

2. Due to nonlinear interaction, certain new disturbances are produced. This happens, for instance, when a sound wave is interacting with an entropy wave [18, 20].

In the Case 1 one can speak of elastic interaction. In a similar way, one can speak of elastic interaction in the case of waves crossing each other at a certain angle in many spatial dimensions. In this case one should restrict the considerations to the domain of determinacy of the solution. This will be explained later in Sec. 6.

Although this subject has a long history starting from Riemann (1858) [22, 5, 9, 23, 3, 4, 21, 12, 19, 20, 24, 6, 10, 11], it is still far from being exhausted. Like solitons, it contributes to the understanding of nonlinear phenomena. The most complete analysis of mathematical properties of such solutions, as well as the general theory of k-waves, is contained in [19]. A considerable part of this results can be found also in [20]. In this paper we present a simplified version of the theory, with application to nonstationary gasdynamics.

In Sec. 2 we define the Riemann (simple) waves and then the hyperbolic double waves for a general quasilinear system of the first order and derive the conditions of their existence. Section 3 contains the application of theory of Sec. 2 to the equations of nonstationary two-dimensional flows in gas-dynamics. We confined ourselves to the case of nonstationary two-dimensional flows, although, as it may be proved, a similar analysis can be made for three-dimensional flows. Having, however, a three-dimensional hodograph space (as there is for two-dimensional nonstationary or three-dimensional stationary flows) makes it possible to obtain a single equation i.e. Eq. (3.8) describing hodographs of double waves. In Secs. 4 and 5 specific classes of such hodographs and the corresponding double waves are considered. Then in Sec. 6 we discuss, in general, the interaction problem for sound modes in gasdynamics. We demonstrate there that for sufficiently small amplitudes (in order not to enter the elliptic region of Eq. (3.8) or Eq. (6.1)), the waves are subjected to an elastic interaction described by double waves.

Similar considerations can be performed for stationary supersonic flows. Also by using imaginary characteristic elements, one can generalize this procedure [19, 20] to the elliptic (or mixed) case of stationary supersonic (transonic) flows. Then one can prove that these generalized double waves can represent flows past three-dimensional profiles which are developable surfaces. This may be useful in searching for 3-D-developable airfoils, similarly as it was done in the two-dimensional case [1, 2, 8].

2. Simple and double waves

Let us consider a nonlinear system of PDE's

$$a_j^{s\nu}(u)\, u^j_{,x^\nu} = 0, \qquad \nu = 1,\ldots,n, \quad j = 1,\ldots,l, \quad s = 1,\ldots,m. \tag{2.1}$$

A solution $u(\cdot) : \Omega \to \mathbb{R}^l$, $\Omega \in \mathbb{R}^n$, of system (2.1) is of the simple wave-type if and only if the Jacobi matrix $du = (u^j{}_{,x^\nu})$ is of rank 1 in Ω.

Let us note that any $l \times n$ matrix of rank 1 can be represented as $\gamma \otimes \lambda = (\gamma^j \lambda_\nu)$, $j = 1,\ldots,l$, $\nu = 1,\ldots,n$. If $\gamma \otimes \lambda$ satisfies

$$a_j^{s\nu}(u)\, \gamma^j\, \lambda_\nu = 0, \tag{2.2}$$

then one speaks of the polarization vector $\gamma \in \mathbb{R}^l$ and the wave vector (or characteristic covector) $\lambda \in \mathbb{R}^n$. The matrix $\gamma \otimes \lambda$ satisfying (2.2) will be called the characteristic element at u. There are solutions of (2.1) associated with characteristic elements. These solutions, called simple waves (or Riemann waves), can be constructed by taking any parameterized C^1 curve $u = f(R)$, $R \in (a, b)$ in $\mathbb{R}^l$ and such that for every $R_0 \in (a, b)$, the tangent vector $\gamma := \dfrac{d}{dR} f_{|R=R_0}$ is a polarization vector at $u_0 = f(R_0)$. Let $\lambda(R)$ be the field of the corresponding wave vectors defined over this curve, i.e.

$$a_j^{s\nu}(f(R))\,\gamma^j(R)\,\lambda_\nu(R) = 0, \qquad R \in (a,b).$$

Then one implicitly defines a class of local simple wave-type solutions

$$u = f(R), \qquad R = \varphi(\lambda_\nu x^\nu), \tag{2.3}$$

where $\varphi:\ \mathbb{R}^1 \to (a,b)$ is an arbitrary C^1 function. One can also easily verify that the equations

$$\lambda_\nu(R)(x^\nu - y^\nu(R)) = 0, \qquad u = f(R) \tag{2.4}$$

define a simple wave solution in some neighbourhood of the curve $x = y(R)$, which takes values $f(R)$ at this curve, provided $\lambda_\nu \dfrac{\partial y^\nu}{\partial R} \neq 0$. If $\lambda \wedge \lambda_{'R} \neq 0$, then the limiting case $u = f(R)$ and $\lambda_\nu(R)(x^\nu - \underset{0}{x}{}^\nu) = 0$ also defines a simple wave.

Similarly, any solution $u(\cdot): \Omega \to \mathbb{R}^l$ is of a hyperbolic double wave-type if for every $x \in \Omega$, $du(x)$ is a sum of two characteristic elements

$$du = \underset{1}{\gamma} \otimes \overset{1}{\lambda} + \underset{2}{\gamma} \otimes \overset{2}{\lambda} \tag{2.5}$$

with linearly independent $\underset{1}{\gamma}, \underset{2}{\gamma}$ and $\overset{1}{\lambda}, \overset{2}{\lambda}$. Thus rank $du = 2$, and the range $u(\Omega)$ of $u(\cdot)$ is a two-dimensional surface in $\mathbb{R}^l$. The plane tangent to $u(\Omega)$ at $u(x)$ is spanned by $\underset{1}{\gamma}, \underset{2}{\gamma}$. As is known from differential geometry, given a two-dimensional surface with two independent vector fields $\underset{1}{\gamma}, \underset{2}{\gamma}$ defined on it, there exists a local system of coordinates on the surface, whose lines are tangent to $\underset{1}{\gamma}, \underset{2}{\gamma}$. In other words, there exists a local parameterization $u = f(R^1, R^2)$ of the surface, such that

$$\frac{\partial f}{\partial R^1} \sim \underset{1}{\gamma}, \qquad \frac{\partial f}{\partial R^2} \sim \underset{2}{\gamma}. \tag{2.6}$$

Therefore the double wave solution can be sought in the form $u = f(R^1(x), R^2(x))$. Inserting $u = f(R^1, R^2)$ into Eq. (2.5) one comes to the conclusion that the functions $R^1 = R^1(x)$, $R^2 = R^2(x)$ satisfy

$$dR^1 = \xi^1(x)\overset{1}{\lambda}(R^1, R^2), \qquad dR^2 = \xi^2(x)\overset{2}{\lambda}(R^1, R^2) \tag{2.7}$$

for some functions $\xi^1(x)$ and $\xi^2(x)$. Here $\overset{1}{\lambda}, \overset{2}{\lambda}$ are the wave vectors corresponding to the polarization vectors $\underset{1}{\gamma}, \underset{2}{\gamma}$, respectively.

The construction of double wave solutions is, however, much more involved than the one for simple waves. Since, in general, system (2.7) is overdetermined ($2n$ equations for four unknown functions R^1, R^2, ξ^1, ξ^2), it can have no solutions of rank 2. For this reason, not every two-dimensional surface parameterized according to (2.6) is in the range of a double wave solution. Additional restrictions which follow from the compatibility conditions of Eqs. (2.7) must be imposed. They require [15, 16, 20] the existence of functions $\alpha_2^1(R^1, R^2)$, $\alpha_1^2(R^1, R^2)$, $\beta_2^1(R^1, R^2)$, $\beta_1^2(R^1, R^2)$ such that

$$\overset{1}{\lambda}_{,R^2} = \alpha_2^1 \overset{2}{\lambda} + \beta_2^1 \overset{1}{\lambda}, \qquad \overset{2}{\lambda}_{,R^1} = \alpha_1^2 \overset{1}{\lambda} + \beta_1^2 \overset{2}{\lambda}. \tag{2.8}$$

The above conditions are equivalent to

$$\Delta_s^r := \overset{(r)}{\lambda} \wedge \overset{s}{\lambda} \wedge \overset{s}{\lambda}_{,R^r} = 0, \qquad r, s = 1, 2, \tag{2.9}$$

where $\wedge$ denotes an exterior product [7] and no summation over r is performed. If (2.8) is satisfied, then Eqs. (2.7) are compatible (involutive) and their general solution depends on two arbitrary functions of one variable [15, 16, 20]. In order to obtain a solution of Eqs. (2.7) several approaches may be applied. From the form of Eqs. (2.7) we see that the solution is constant over certain linear manifolds $\mathcal{M}_{\underset{0}{x}}$ of dimension $(n-2)$ (n is the dimension of the configuration space of $x^1, \ldots, x^n$ variables). $\mathcal{M}_{\underset{0}{x}}$ is given by the following equations for x

$$\overset{1}{\lambda}_\nu \Big(f(R^1(\underset{0}{x}),\ R^2(\underset{0}{x})) \Big) (x^\nu - \underset{0}{x}{}^\nu) = 0,$$
$$\overset{2}{\lambda}_\nu \Big(f(R^1(\underset{0}{x}),\ R^2(\underset{0}{x})) \Big) (x^\nu - \underset{0}{x}{}^\nu) = 0, \qquad \underset{0}{x} = (\underset{0}{x}{}^1, \ldots, \underset{0}{x}{}^n).$$

Therefore, at the beginning of the solution we may confine our attention to a two-dimensional plane in the configuration space, which has the property of intersecting each $\mathcal{M}_{\underset{0}{x}}$ at only one point. Suppose that the plane x^1, x^2 has this property. In such a case system (2.7) may be restricted to the plane x^1, x^2, to obtain

$$R^1_{,x^s} = \xi^1 \overset{1}{\lambda}_s, \qquad R^2_{,x^s} = \xi^2 \overset{2}{\lambda}_s, \qquad s = 1, 2. \tag{2.10}$$

Eliminating the variables ξ^1, ξ^2 we reduce Eq. (2.10) to the following hyperbolic system

$$C_1^s(R^1,\ R^2) R^1_{,x^s} = 0, \qquad C_2^s(R^1,\ R^2) R^2_{,x^s} = 0, \qquad s = 1, 2, \tag{2.11}$$

where $C_1 = (\overset{2}{\lambda}_2, -\overset{2}{\lambda}_1)$, $C_2 = (\overset{1}{\lambda}_2, -\overset{1}{\lambda}_1)$ are "tangent characteristic vectors" for system (2.11). This system can be treated by the method of characteristics.

Another possibility is to apply the hodograph transformation which converts Eq. (2.11) into a linear system. Indeed, multiplying Eqs. (2.10) by $\partial x^s/\partial R^2$ and by $\partial x^s/\partial R^1$ respectively, we obtain a linear homogeneous system in R^1, R^2

$$\overset{1}{\lambda}_s(f(R^1,\ R^2))\frac{\partial x^s}{\partial R^2} = 0, \qquad \overset{2}{\lambda}_s(f(R^1,\ R^2))\frac{\partial x^s}{\partial R^1} = 0,$$

which is equivalent to the previous one for nondegenerate solutions (rank $\|R^\alpha_{,s}\| = 2$).

Another approach can be also applied in which system (2.7) is also reduced to a linear hyperbolic system. This approach may sometimes be useful. Assuming that $\alpha^1_2(R^1, R^2)$, $\alpha^2_1(R^1, R^2)$, $\beta^1_2(R^1, R^2)$, $\beta^2_1(R^1, R^2)$ are the coefficients appearing in conditions (2.8), we have

THEOREM 1. *If ψ^1, ψ^2 is a solution of the linear equations*

$$\psi^1{}_{,R^2} = \alpha^1_2\psi^2 + \beta^1_2\psi^1, \qquad \psi^2{}_{,R^1} = \alpha^2_1\psi^1 + \beta^2_1\psi^2, \tag{2.12}$$

then the implicit formulae

$$\psi^1(R^1, R^2) = \overset{1}{\lambda}_\nu(R^1, R^2)x^\nu, \qquad \psi^2(R^1, R^2) = \overset{2}{\lambda}_\nu(R^1, R^2)x^\nu, \tag{2.13}$$

define a solution of Eqs. (2.1) *in some neighbourhood of* $(\underset{0}{x}, \underset{0}{R}{}^1, \underset{0}{R}{}^2)$ *provided that* $(\psi^1 - \overset{1}{\lambda}_\nu \underset{0}{x}{}^\nu)_{,R^1} \neq 0$ *and* $(\psi^2 - \overset{2}{\lambda}_\nu \underset{0}{x}{}^\nu)_{,R^2} \neq 0$ *at* $(\underset{0}{x}, \underset{0}{R}{}^1, \underset{0}{R}{}^2)$ *satisfying* (2.13).

Indeed, by differentiating the implicit formulae (2.13) we see that the gradient of R^1, R^2 is proportional to $\overset{1}{\lambda}(R^1, R^2)$, $\overset{2}{\lambda}(R^1, R^2)$, respectively.

Formulae (2.13) constitute an interesting generalization of a similar formula (Eq. (2.3)) for simple waves. Let us note that Eqs. (2.13) have always the trivial solution $\psi_1 \equiv \psi_2 \equiv 0$ which by Eqs. (2.13) defines a certain double wave (generalization of formula (2.4)). Theorem 1 can be generalized in an obvious way for k-waves by replacing indices 1 and 2 with $\alpha, \beta = 1, \ldots, k$, $\alpha \neq \beta$. Then the general solution depends on k-functions of one variable, e.g. defining the problem of waves entering the interaction (the formulation of the theorem in [10, 11] is erroneous).

In principle one can start from two independent characteristic elements in Eq. (2.5) expressed as some functions of u. Then the Frobenius theorem tells us that for any given point u_0, there exists a two-dimensional surface passing through u_0 and tangent at each of its points to the vector fields $\underset{1}{\gamma}$, $\underset{2}{\gamma}$ if and only if

$$\left[\underset{1}{\gamma}, \underset{2}{\gamma}\right] \in \mathrm{Lin}\{\underset{1}{\gamma}, \underset{2}{\gamma}\},$$

where $[X, Y] = X^i \frac{\partial}{\partial u^i} Y - Y^i \frac{\partial}{\partial u^i} X$ is the commutator of the vector fields X, Y and $\mathrm{Lin}\{X, Y\}$ denotes the linear combination of X, Y. In such a case differentiation with respect to R^1, R^2 in Eqs. (2.8) must be replaced by differentiation along $\underset{1}{\gamma}$, $\underset{2}{\gamma}$, e.g. $\frac{\partial}{\partial R^2} \overset{1}{\lambda} \to \underset{2}{\gamma}^i \frac{\partial}{\partial u^i} \overset{1}{\lambda}$.

Now we will demonstrate that the solution of Eq. (2.5) *can be interpreted locally as resulting from the interaction of two localised Riemann waves, in the sense that one wave is propagating across the other. By "localised" we mean here that the first derivative of* φ *in* (2.3) *is localised.*

The level sets $R^1(x) = \mathrm{const}$ and $R^2 = \mathrm{const}$ can be thought of as constant phase surfaces of the first and second wave, respectively. Since they are solutions of the Pfaff forms $\overset{1}{\lambda}_\nu dx^\nu = 0$ and $\overset{2}{\lambda}_\nu dx^\nu = 0$ respectively, they are orthogonal to their wave vectors, $\overset{1}{\lambda}$ or $\overset{2}{\lambda}$. For an unperturbed Riemann wave such a surface is a hyperplane. In general, however, the mutual interaction expressed in the nonlinearity of system (2.5) leads to local changes of the wave vectors. For brevity, in the following we confine our attention to the two-dimensional case, when the level sets of R^1 and R^2 define two families of curves which are characteristic curves of Eq. (2.5). In case of three dimensions t, x^1, x^2 one can think of the picture at any constant t.

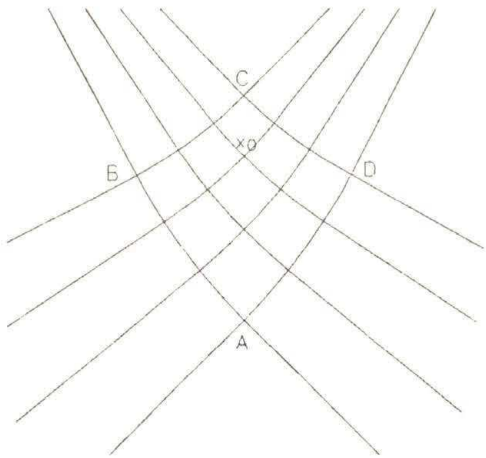

FIG. 1.

Now, suppose a solution of Eq. (2.5) of rank 2 is given in a region Ω containing a point x_0. Consider a neighbourhood of the point x_0 bounded by a curvilinear quadrangle A, B, C, D (Fig. 1), the sides of which are characteristic curves (i.e. these curves are perpendicular to $\overset{1}{\lambda}$ and $\overset{2}{\lambda}$, respectively). By what we call the circumvention procedure, we construct a new solution which takes the same values as the former one in the quadrangle (including its sides). Outside it we extend the solution in the following way: first we prolong any characteristics contained in

$ABCD$ by a straight half-line keeping the direction of the characteristic (Fig. 2). On this line we define the solution u to be a constant equal to the value it takes at the point where the line crosses the side of the quadrangle.

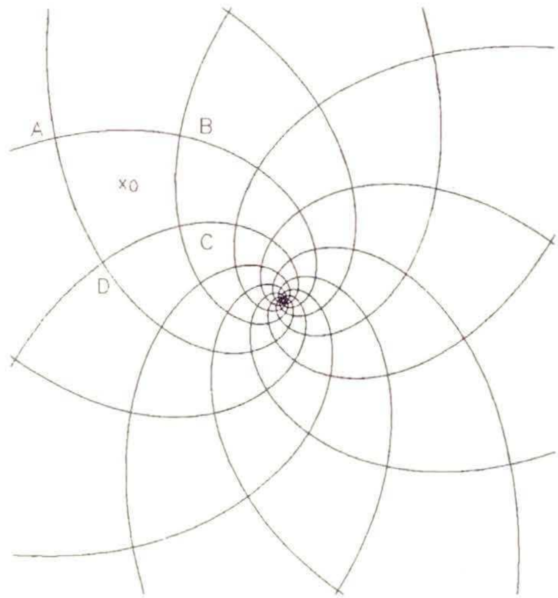

FIG. 2.

In the remaining domains which are the interiors of the four angles with the vertices A, B, C, D we put u equal to $u(A)$, $u(B)$, $u(C)$, $u(D)$, respectively. This procedure defines, in a certain neighbourhood of the quadrangle, a Lipschitz continuous mapping which is also differentiable in this neighbourhood, perhaps with the exception of characteristic curves passing through any of the points A, B, C, D. In this way weak discontinuity of a solution may occur. This mapping is the new solution of Eq. (2.5) which represents two interacting localized Riemann waves. The interior of the quadrangle is a domain of interaction, and outside it, by construction, the mapping is either a constant or is constant on the lines orthogonal to the respective wave vector λ, thus assuring that it is a Riemann wave-type solution.

3. The hodograph problem

We will consider equations of plane nonstationary isentropic flows [5, 13]

$$(3.1)\qquad \begin{aligned} (\partial_t + \mathbf{u}\cdot\nabla)a + \frac{\kappa-1}{2} a \operatorname{div}\mathbf{u} &= 0, \\ (\partial_t + \mathbf{u}\cdot\nabla)\mathbf{u} + \frac{2}{\kappa-1} a\nabla a &= 0, \end{aligned}$$

where $\mathbf{u} = (u^1, u^2)$, $\nabla = (\partial_x, \partial_y)$, a is the sound speed, $a^2 = dp/d\varrho$ and κ is the isentropy exponent, $1 < \kappa < 3$. The number of equations is equal to three, as is the number of unknown functions a, u^1, u^2 or the number of independent variables t, x^1, x^2.

According to Eq. (2.2), the characteristic directions are

$$
(3.2)\qquad
\begin{aligned}
&1)\quad \gamma = \left(\frac{\kappa-1}{2},\ e^1,\ e^2\right) \longrightarrow \lambda = (\langle \mathbf{u}|\gamma\rangle),\ -e^1,\ -e^2),\\
&2)\quad \gamma = (0,\ e^1,\ e^2) \qquad\quad\ \longrightarrow \lambda = (u^1e^2 - u^2e^1,\ -e^2,\ e^1),
\end{aligned}
$$

where $\langle \mathbf{u}|\gamma\rangle = [2/(\kappa-1)]a\gamma^0 + u^1\gamma^1 + u^2\gamma^2$ and $e = (e^1, e^2)$ is a two-dimensional unit vector [17]. Similarly to three-dimensional flows [17], the first type of simple elements generates potential flows (sound modes) and this case will be considered here. Note that the characteristic vector λ for the potential elements can be represented as a linear function of γ

$$\lambda = \mathbb{P}\gamma,$$

where

$$
\mathbb{P} = \begin{vmatrix} \dfrac{2}{\kappa-1}, & u^1, & u^2 \\ 0, & -1, & 0 \\ 0, & 0, & -1 \end{vmatrix}.
$$

Let us denote $f_{,\gamma} := \gamma^j \dfrac{\partial}{\partial u^j} f$, for simplicity. The exterior product of three vectors in three-dimensional space t, x^1, x^2 may be identified with the determinant, so that

$$
\Delta^s_r := \overset{s}{\lambda} \wedge \overset{r}{\lambda} \wedge \overset{s}{\lambda}_{,\underset{s}{\gamma}} = \left[\overset{s}{\lambda} \,|\, \overset{r}{\lambda} \,|\, \overset{s}{\lambda}_{,\underset{r}{\gamma}}\right] = \left[\mathbb{P}\underset{s}{\gamma} \,|\, \mathbb{P}\underset{r}{\gamma} \,|\, \mathbb{P}_{,\underset{r}{\gamma}}\underset{r}{\gamma}\right],
$$

where $[\alpha \,|\, \beta \,|\, \gamma] = \det \|\alpha, \beta, \gamma\|$.

By a well-known transformation property of determinants $[\mathbb{P}\alpha \,|\, \mathbb{P}\beta \,|\, \mathbb{P}\gamma] = \det \mathbb{P}[\alpha, \beta, \gamma]$, we can write the conditions of involutions (2.9) $\Delta^s_r = 0$ as follows

$$
\Delta^s_r = \langle \underset{s}{\gamma} \mid \underset{r}{\gamma} \rangle \cdot \left[\underset{s}{\gamma} \mid \underset{r}{\gamma} \mid q\right] + \frac{2}{\kappa-1}a\left[\underset{s}{\gamma} \mid \underset{r}{\gamma} \mid \underset{s}{\gamma}_{,\underset{r}{\gamma}}\right] = 0, \qquad q = (1,0,0),
$$

or

$$
(3.3)\qquad \Delta^r_s = (\underset{s}{\gamma} \times \underset{r}{\gamma}) \cdot (q\langle \underset{s}{\gamma} \mid \underset{r}{\gamma} \rangle + \frac{2}{\kappa-1}a\underset{s}{\gamma}_{,\underset{r}{\gamma}}) = 0,
$$

where the dot denotes the scalar product.

Suppose that the hodograph surface which will be denoted by G^2 is described by a C^1-function $F(a, u^1, u^2) = 0$, with a nonvanishing gradient. The gradient $\nabla_u F = (F_a, F_{u^1}, F_{u^2})$ is orthogonal to the surface G^2 and therefore it must be proportional to the vector product $\underset{1}{\gamma} \times \underset{2}{\gamma}$ of vectors $\underset{1}{\gamma}, \underset{2}{\gamma}$ which are tangent to G^2. Using this fact, we substitute $\nabla_u F$ for $\underset{1}{\gamma} \times \underset{2}{\gamma}$ in Eq. (3.3) to obtain

$$(3.4) \qquad \Delta_r^{\prime s} = \langle \underset{s}{\gamma} \mid \underset{r}{\gamma} \rangle F_{,a} + \frac{2}{\kappa - 1} a F_{,u^j} (\underset{s}{\gamma}{}^j_{,\underset{r}{\gamma}}) = 0, \quad s \neq r, \quad a = u^0, \quad j = 0, 1, 2.$$

On the other hand, differentiation of a self-evident relation $F_{,u^j} \underset{r}{\gamma}{}^j = 0$ with respect to u^i and multiplication by $\underset{r}{\gamma}{}^i$ yields

$$F_{,u^j} (\underset{s}{\gamma}{}^j_{,\underset{r}{\gamma}}) = -F_{,u^i u^j} \underset{s}{\gamma}{}^i \underset{r}{\gamma}{}^j,$$

which allows us to transform Eq. (3.4) to the following symmetric form

$$\Delta_r^{\prime s} = \langle \underset{s}{\gamma} \mid \underset{r}{\gamma} \rangle F_{,a} - \frac{2}{\kappa - 1} a F_{,u^i u^j} \underset{s}{\gamma}{}^j \underset{r}{\gamma}{}^j.$$

Therefore, the set of conditions (3.4) is reduced to the following one

$$(3.5) \qquad \langle \underset{1}{\gamma} \mid \underset{2}{\gamma} \rangle F_{,a} - \frac{2}{\kappa - 1} a F_{,u^j u^i} \underset{1}{\gamma}{}^j \underset{2}{\gamma}{}^i = 0.$$

In accordance with 1) of (3.2) we can assume

$$\underset{s}{\gamma} = \left(\frac{\kappa - 1}{2}, \; \cos \varphi_s, \; \sin \varphi_s \right).$$

Then we have $\langle \underset{1}{\gamma} \mid \underset{2}{\gamma} \rangle = (\kappa - 1)/2 + \cos 2\delta$, where $\delta = \varphi_1 - \varphi_2$ and the vectors $\underset{1}{\gamma}$, $\underset{2}{\gamma}$ can be obtained from the relations

$$(3.6) \qquad F_a \gamma^0 + F_{u^1} \gamma^1 + F_{u^2} \gamma^2 = 0.$$

Further on, $\cos 2\delta$ may be expressed by the derivatives of F to obtain

$$(3.7) \qquad \cos^2 \delta = \frac{F_a^2}{F_{u^1}^2 + F_{u^2}^2} \left(\frac{\kappa - 1}{2} \right)^2$$

from which it follows that in order to have a real δ, we must satisfy the following inequality

$$F_a^2 < \left(\frac{2}{\kappa - 1} \right)^2 \left(F_{u^1}^2 + F_{u^2}^2 \right).$$

Introducing $\tilde{a} = [2/(\kappa - 1)]a$ in Eq. (3.5) and expressing $F_{u^i u^j}\gamma^i\gamma^j$ in terms of derivatives of F, we arrive at the following, rather complicated, equation which must be satisfied on surface G^2

$$(3.8)\qquad \left(\frac{\kappa-3}{2} + \frac{2F_{\tilde{a}}^2}{F_u^2 + F_v^2}\right)\frac{2F_{\tilde{a}}}{\kappa - 1} - \tilde{a}\left(F_{\tilde{a}\tilde{a}} + \frac{1}{2}(F_{uu} - F_{vv})\frac{F_u^2 - F_v^2}{F_u^2 + F_v^2} + 2F_uF_v\frac{F_{uv}}{F_u^2 + F_v^2} + \frac{1}{2}\frac{F_{uu} + F_{vv}}{F_u^2 + F_v^2}\left(2F_{\tilde{a}}^2 - F_u^2 - F_v^2\right) + \frac{2F_{\tilde{a}}}{F_u^2 + F_v^2}(F_{\tilde{a}u} + F_{\tilde{a}v}F_v)\right) = 0,$$

where u, v denote u^1, u^2.

By specialization $F \equiv \varphi(\tilde{a}) - f(u, v)$ we may remove the terms involving the derivatives $F_{\tilde{a}u}$ and $F_{\tilde{a}v}$. This form of F can be assumed without much loss of generality. As follows from the Sard theorem [14], such a representation of F can cease to hold only on a set of measure zero. A particular case $F \equiv \psi(\tilde{a}) - \frac{1}{2}(u^2 + v^2)$, leads to rotational hodograph surfaces, described by ordinary differential equations

$$(3.9)\qquad \left(\frac{\kappa - 3}{2} + \frac{\psi'^2}{\psi}\right)\frac{2\psi'}{\kappa - 1} - \tilde{a}\left(\psi'' - \frac{\psi'^2}{\psi} + 1\right) = 0.$$

Here "$'$" denotes differentiation with respect to $\tilde{a}$. The solution of Eqs. (3.9) depends on two arbitrary constants both of which have a physical meaning, i.e. they cannot be retransformed by Galilean transformation. Therefore both constants determine the shape of the rotational surface. It may be checked that the functions

$$\text{a)}\quad \psi(\tilde{a}) \equiv \frac{1}{4}\tilde{a}^2, \qquad \text{and} \qquad \text{b)}\quad \psi(\tilde{a}) \equiv -\frac{\kappa - 1}{4}\tilde{a}^2 + c$$

are the particular solutions of Eqs. (3.9). Both solutions lead to hodographs which are the quadratic surfaces:

a. $\frac{1}{2}\tilde{a}^2 - u^2 - v^2 = 0$ – the hodograph is a cone; its equation in variables a, u, v takes the form $\left(\frac{a\sqrt{2}}{\kappa - 1}\right)^2 - u^2 - v^2 = 0$. From relation (3.7) we have $\cos^2\delta = 1/2$; we can put $\delta = \pi/4$ since other cases provide nothing new.

b. $\frac{2}{\kappa - 1}a^2 + u^2 + v^2 = c$ – describes a family of ellipsoids and physically expresses Bernoulli's law. Therefore, solutions associated with such hodographs are the well known two-dimensional stationary hypersonic flows (if $u^2 + v^2 > a^2$).

4. Flows with constant Mach number

We direct our attention to the conical hodograph surface (Case a). The curves tangent to $\underset{1}{\gamma}$ and $\underset{2}{\gamma}$ are spirals and their projections on the (u,v)-plane are logarithmic spirals. In the parametrical form with parameters φ, ϱ, the cone may be represented by the following expressions

$$(4.1) \qquad a = \frac{\kappa - 1}{\sqrt{2}}\varrho, \qquad u = \varrho\cos\varphi, \qquad v = \varrho\sin\varphi.$$

The mentioned spirals on the cone are given by Eqs. (4.1) with an additional relation between ϱ and φ

$$(4.2) \qquad \varrho = e^{\cot\delta(\varphi + 2R^2)},$$

with constant R^2 on the curves of the first family; or

$$(4.3) \qquad \varrho = e^{-\cot\delta(\varphi - 2R^1)}$$

with constant R^1 on the spirals of the second family.

By eliminating ϱ and φ from relations (4.2) and (4.3) we may take R^1, R^2 as the coordinates. Thus, remembering that $\cot\delta = 1$, ϱ, φ become functions of R^1, R^2

$$(4.4) \qquad \varrho = e^{-(R^1+R^2)}, \qquad \varphi = R^1 - R^2.$$

Since $\underset{s}{\gamma} \simeq \dfrac{\partial}{\partial R^s}(a,u,v)$ then, utilizing Eqs. (4.1), (4.4) and (3.2), we arrive at the characteristic elements ($\underset{1}{\gamma}$ corresponds to $-$ and $\underset{2}{\gamma}$ to $+$)

$$\begin{aligned} \underset{1}{\gamma}, \underset{2}{\gamma} &= \left(\frac{\kappa - 1}{2}, \cos\left(\varphi \mp \frac{\pi}{4}\right), \sin\left(\varphi \mp \frac{\pi}{4}\right)\right), \\ \overset{1}{\lambda}, \overset{2}{\lambda} &= \left(\frac{\kappa}{\sqrt{2}}\varrho, -\cos\left(\varphi \mp \frac{\pi}{4}\right), -\sin\left(\varphi \mp \frac{\pi}{4}\right)\right) \end{aligned}$$

and the vector $\sigma \sim \overset{1}{\lambda} \times \overset{2}{\lambda}$. As we know from Sec. 1, the solution is constant along the directions σ orthogonal to $\overset{1}{\lambda}$, $\overset{2}{\lambda}$. Thus $\sigma = \overset{1}{\lambda} \times \overset{2}{\lambda} = (1, \kappa\varrho\cos\varphi, \kappa\varrho\sin\varphi) = (1, \kappa u, \kappa v)$. By using this property, the Pfaff equations (3.8) which in this case take the form

$$(4.5) \qquad \begin{aligned} dR^1 &= \xi^1\left(\frac{\kappa}{\sqrt{2}}\varrho\, dt - \cos\left(\varphi - \frac{\pi}{4}\right)dx^1 - \sin\left(\varphi - \frac{\pi}{4}\right)dx^2\right), \\ dR^2 &= \xi^2\left(\frac{\kappa}{\sqrt{2}}\varrho\, dt - \cos\left(\varphi + \frac{\pi}{4}\right)dx^1 - \sin\left(\varphi + \frac{\pi}{4}\right)dx^2\right) \end{aligned}$$

can be restricted to the plane t = const. Then their solution $R_0^1(x^1,x^2)$, $R_0^2(x^1,x^2)$ may be extended to a certain neighborhood of the plane t = const in such a way that this extended solution is constant along the straight lines defined in each point $(0,x^1,x^2)$ by the vector $\sigma(R^1(x^1,x^2),R^2(x^1,x^2))$. Let us note that σ is never parallel to the (x^1,x^2)-plane. Putting $dt=0$ and eliminating ξ^1,ξ^2 from Eqs. (4.5) we arrive at an equivalent hyperbolic system

$$R^1_{0,x^1} = R^1_{0,x^2}\cot\left(R_0^1 - R_0^2 - \frac{\pi}{4}\right), \qquad R^2_{0,x^1} = R^2_{0,x^2}\cot\left(R_0^1 - R_0^2 + \frac{\pi}{4}\right), \tag{4.6}$$

which is treatable by the method of characteristics. In the case of nondegenerate solutions of Eqs. (4.6) one can also apply the hodograph transformation which, by exchanging the role of dependent and independent variables, leads to the following linear system

$$\begin{aligned} x^1_{,R^2}\cos\left(R^1 - R^2 - \frac{\pi}{4}\right) + x^2_{,R^2}\sin\left(R^1 - R^2 - \frac{\pi}{4}\right) &= 0,\\ x^2_{,R^1}\cos\left(R^1 - R^2 + \frac{\pi}{4}\right) + x^2_{,R^1}\sin\left(R^1 - R^2 + \frac{\pi}{4}\right) &= 0. \end{aligned} \tag{4.7}$$

Equations (4.5) may be also reduced to the telegraphic equation by introducing new variables ψ^1, ψ^2 according to Theorem 1. Then from (2.12) one obtains an equivalent form of Eqs. (4.5)

$$\psi^1_{,R^2} - \psi^2 = 0, \qquad \psi^2_{,R^1} - \psi^1 = 0. \tag{4.8}$$

Elimination of one of the unknown functions, say ψ^2, reduces Eqs. (4.7) to the telegraphic equation

$$\psi^1_{,R^1R^2} - \psi^1 = 0.$$

The solution is then determined by

$$\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \begin{pmatrix}\cos\left(R^1 - R^2 - \frac{\pi}{4}\right), & \sin\left(R^1 - R^2 - \frac{\pi}{4}\right)\\ \cos\left(R^1 - R^2 + \frac{\pi}{4}\right), & \sin\left(R^1 - R^2 + \frac{\pi}{4}\right)\end{pmatrix}\begin{pmatrix}x^1\\ x^2\end{pmatrix} - \frac{\kappa}{\sqrt{2}}\varrho t\begin{pmatrix}1\\ 1\end{pmatrix} \tag{4.9}$$

and by Eq. (4.4) in a parametric form, R^1, R^2 are the parameters. Let us note that the considered solutions describe nonstationary flows with the constant Mach number $M = (u^2+v^2)^{1/2}/a = \sqrt{2}/(\kappa - 1)$.

The matrix in (4.9), call it $S(\varphi-\pi/4)$, is an orthogonal matrix of the clockwise rotation by the angle $R^1 - R^2 - \pi/4 = \varphi - \pi/4$. The coordinates (x^1, x^2) can be easily expressed as

$$(x^1, x^2)^T = S\left(\varphi - \frac{\pi}{4}\right)(\psi^1, \psi^2)^T + \frac{\kappa}{2}\varrho t S(\varphi)(1,0)^T,$$

where T denotes the matrix transposition. In the trivial case $\psi^1 \equiv \psi^2 \equiv 0$ which obviously satisfies (4.8), we arrive at

$$\mathbf{u} = \frac{\sqrt{2}}{\kappa}\frac{\mathbf{x}}{t}. \tag{4.10}$$

Then according to (4.1) $a = (\kappa-1)/\kappa|\mathbf{x}/t|$. In Fig. 1 the projection on u^1, u^2 plane of the curves $R^1 = \text{const}$ and $R^2 = \text{const}$, respectively, defined by Eq. (4.4) are shown. By Eq. (4.10), up to the scaling factor $\sqrt{2}/\kappa$, they are also characteristic curves on $\mathbf{x}/t$-plane.

As it was pointed out in Sec. 1, every double wave can be locally interpreted as resulting from the interaction of two localised simple waves. In this way, a somewhat trivial solution defined by Eq. (4.4) may give rise to interesting interactions of simple waves. Inside the quadrangle $ABCD$ (Fig. 2) on the $(x^1/t, x^2/t)$ plane, the solution is defined by (4.10). Outside it we have simple waves (or constant states in the corners A, B, C, D). The sides of the quadrangle are lines defined by

$$\frac{\mathbf{x}}{t} = \frac{\kappa}{\sqrt{2}}e^{R^1+R^2}\left\{\cos(R^1 - R^2),\ \sin(R^1 - R^2)\right\},$$

where R^1 (respectively R^2) take appropriate constant values. The straight lines emanating from the quadrangle are the lines of constant phase of the corresponding simple waves. In accordance with Sec. 1 these waves are defined analytically by Eq. (2.4).

5. Cylindrical hodograph

Now we may ask whether the hodographs which are cylindrical surfaces exist. The case $F = F(u, v)$ is an immediate generalization of one-dimensional nonstationary flows for which the hodograph is $v \equiv 0$. Of course, by adjusting the system of coordinates we may at least locally restrict our attention to functions F of the following form $F(u, v) \equiv u - \psi(v)$ (a consequence of implicit function theorem) which, substituted into equation (3.8) yields

$$-\frac{1}{2}\psi''\frac{1-\psi'^2}{1+\psi'^2} + \frac{1}{2}\psi'' = 0.$$

This may be split into two alternative conditions

$$\psi'' = 0, \qquad \text{or} \qquad \frac{1-\psi'^2}{1+\psi'^2} = 1.$$

In both cases we obtain nothing more than linear functions (hence defined globally) which by an appropriate Galilean transformation can be transformed into a one-dimensional case $v = 0$.

Thus, in addition to the planes described by the above linear functions, there are no hodographs of the form $F(u, v) = 0$.

We shall see, however, that there are other hodographs generated by a curve moving along the constant direction in the space of a, u, v – each cylindrical surface is generated by an appropriate curve moving in a constant direction. As an example, consider the hodograph given by the relation $u - \psi(\widetilde{a}) = 0$. Substituting it to Eq. (3.8) we obtain the following ordinary differential equation

$$\frac{2}{\kappa-1}\left(\frac{\kappa-3}{2} + 2\psi'^2\right)\psi' - \widetilde{a}\psi'' = 0 \tag{5.1}$$

with separable variables. Hence we have

$$\frac{d\psi'}{\left(\frac{\kappa-3}{2} + 2\psi'^2\right)\psi'} = \frac{2}{\kappa-1}\,\frac{d\widetilde{a}}{\widetilde{a}}, \tag{5.2}$$

which may be integrated to yield for $1 < \kappa < 3$

$$\psi(\widetilde{a}) = \pm\frac{\sqrt{3-\kappa}}{2}\int\left(1 - C\widetilde{a}^{2(3-\kappa)/(\kappa-1)}\right)^{-1/2} d\widetilde{a} + C_1, \tag{5.3}$$

where C, C_1 are arbitrary constants. This solution depends on two arbitrary constants but one of them, C_1, may be retransformed by the Galilean transformation.

As follows from a closer analysis, constant C can also be retransformed to obtain one of the three values $C = -1,\ 0,\ 1$. This can be achieved by using the following transformation $(a, u) \to (\mu a, \mu u)$, $(t, x) \to (\beta t, \mu\beta x)$ which transforms solutions of Eqs. (3.1) into other solutions, μ and β are arbitrary nonzero constants. The case $C = 0$ leads to the plane hodograph of noninteracting waves which were found in [17]. These represent a certain interesting feature of the gasdynamic system: nonlinear waves can be subjected to a linear interference. Therefore we may restrict ourselves to the cases $C = \pm 1$. Equation (3.4), Case 1, leads to the following expressions

$$\underset{1}{\gamma} = \left(1,\ \psi'(\widetilde{a}),\ \sqrt{1-\psi'^2(\widetilde{a})}\right), \qquad \underset{2}{\gamma} = \left(1,\ \psi'(\widetilde{a}),\ -\sqrt{1-\psi'^2(\widetilde{a})}\right)$$

for the polarization vectors in the hodograph space of $(\widetilde{a}, u, v)$ variables.

Calculating the characteristic vectors $\overset{1}{\lambda}, \overset{2}{\lambda}$ corresponding to $\underset{1}{\gamma}, \underset{2}{\gamma}$ respectively, we have (+ for $\overset{1}{\lambda}$ and $-$ for $\overset{2}{\lambda}$)

$$\overset{1}{\lambda}, \overset{2}{\lambda} = \left(\frac{\kappa-1}{2}\widetilde{a} + \psi'(\widetilde{a})\psi(\widetilde{a}) \pm v\sqrt{1-\psi'^2(\widetilde{a})},\ -\psi'(\widetilde{a}),\ \pm(-1)\sqrt{1-\psi'(\widetilde{a})}\right),$$

and the direction $\boldsymbol{\sigma} = \overset{1}{\lambda} \times \overset{2}{\lambda}$ on which the solution must be constant

$$\boldsymbol{\sigma} = \left(1,\ \frac{\kappa-1}{2\psi'}\widetilde{a} + u,\ v\right).$$

In the case $\kappa = 2$ integration in formula (5.3) may be done explicitly and we obtain

$$\psi(\widetilde{a}) = \frac{1}{2}\int \frac{d\widetilde{a}}{\sqrt{1+C\widetilde{a}^2}} = \pm\begin{cases} \dfrac{1}{2}\operatorname{arsh}\widetilde{a} & \text{if} \quad C = 1, \\ -\dfrac{1}{2}\arcsin\widetilde{a} & \text{if} \quad C = -1. \end{cases}$$

The hodograph surfaces may be given by relations

$$\text{(5.4)} \qquad 1)\ \ \widetilde{a} - \sinh 2u = 0, \qquad 2)\ \ \widetilde{a} - \sin 2u = 0.$$

Taking advantage of the fact that the operator

$$\sigma^0\frac{\partial}{\partial t} + \sigma^1\frac{\partial}{\partial x^1} + \sigma^2\frac{\partial}{\partial x^2} \qquad \left(= \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla + \frac{\kappa-1}{2\psi'(\widetilde{a})}\widetilde{a}\frac{\partial}{\partial x^1}\right)$$

vanishes on the solutions considered, one can replace $\partial_t + \mathbf{u}\cdot\nabla$ in Eqs. (3.1) by $-(\kappa-1)(2\psi'(\widetilde{a}))^{-1}\widetilde{a}\partial/\partial x^1$. Expressing then $\widetilde{a}$ in terms of u according to Eq. (5.3) (e.g. $\widetilde{a} = \sinh 2u$ or $\widetilde{a} = \sin 2u$, for $\kappa = 2$), one immediately obtains two equations defining double waves in terms of the original variables u, v

$$\begin{aligned} \left(\widetilde{a}^2_{,u} - 1\right)u_{x^1} - v_{x^2} &= 0, \\ v_{x^1} - u_{x^2} &= 0. \end{aligned}$$

Their solutions define double waves taken at some constant t, therefore they must be extended over the space t, x^1, x^2 in such a way that they are constant along the directions of $\boldsymbol{\sigma}(u, v)$. This illustrates another procedure of obtaining the equations defining double waves, probably the fastest since no additional variables R^1, R^2 are required. In the Case 1 of (5.4) this system is hyperbolic for any value of u, whereas in the Case 2 it changes its type on lines $u = \pm\pi/6$, (i.e. $\cos 4u = -1/2$). For $\cos 4u < -1/2$ it becomes elliptic. One can demonstrate, however, that its solutions are still solutions of the basic system (3.1) [19, 20]. To conclude, let us also note that the most general cylindrical surface can be locally represented by $u - \psi(\widetilde{a} - \alpha v) = 0$, α is a constant. This leads, however, to a more complicated equation than Eq. (5.1).

6. A general remark on the problem of wave-wave interaction

By substitution $F = \tilde{a} - \psi(u,v)$, Eq. (3.8) may be transformed to the following form

$$(6.1)\qquad \frac{2}{\kappa-1}\left[\frac{\kappa-1}{2}\left(\psi_u^2+\psi_v^2\right)+2\right]-\psi\Big\{\left(\psi_v^2-1\right)\psi_{uu} + \left(\psi_u^2-1\right)\psi_{vv}-2\psi_u\psi_v\psi_{uv}\Big\}=0.$$

This specialization excludes the case when $F = F(u,v)$ but, as we know from previous consideration, only one-dimensional nonstationary flows are of this form. Equation (6.1) is of the second order and its characteristics $(u(s), v(s))$ are given by

$$(6.2)\qquad (\psi_u^2-1)u'^2+(\psi_v^2-1)v'^2+2\psi_u\psi_v\, u'\, v'=0.$$

However, as follows from Eq. (3.2), any C^1 curve $(\tilde{a}(s), u(s), v(s))$ is a range of an irrotational simple wave if its tangent vector $(\tilde{a}, u', v')$ is a polarization vector, i.e. if it satisfies

$$(6.3)\qquad (\tilde{a}')^2-u'^2-v'^2=0.$$

Suppose that this curve lies on the surface $\tilde{a} - \psi(u,v) = 0$. Differentiating this relation with respect to s we have $a' = \psi_u u' + \psi_v v'$. When applied to Eq. (6.3) it converts it into Eq. (6.2). Thus, characteristic lines of Eq. (6.1) are also projections of characteristics of Eq. (3.8) on the (u,v) plane. However, according to our considerations in Sec. 2, the characteristics of Eq. (3.8) are the images of simple waves. As can be seen from (6.2), Eq. (6.1) becomes elliptic if $\psi_u^2 + \psi_v^2 < 1$.

Let us now consider the initial condition for the flow equations (3.1) which is defined in a circle on the (x,y) plane for $t = 0$, and which represents two localized simple waves approaching each other, separated by a constant state $U_0 = (a_0, u_0, v_0)$ (Fig. 3). If the amplitudes and their derivatives are not too large, then, in the "conical" region of t, x^1, x^2, where the solution is uniquely determined, the solution exists and it represents two simple waves crossing each other for some larger t (similarly as in Fig. 2).

In order to demonstrate this, let us notice that the image (the set of values) of the initial conditions consists of two pieces of characteristic curves Γ_1, Γ_2 in the hodograph space passing through point U_0 (Fig. 4). Let us now take Γ_1, Γ_2 as the Darboux problem for Eq. (6.1). The value of ψ is then given on two intersecting characteristics of Eq. (6.1). Let Γ_1, Γ_2 be given in a parametrical form

$$\Gamma_1 = \Big(\tilde{a} = \alpha_0(s),\ u = \alpha_1(s),\ v = \alpha_2(s)\Big), \qquad b_1 \le s \le b_2,$$

$$\Gamma_2 = \Big(\tilde{a} = \beta_0(\tau),\ u = \beta_1(\tau),\ v = \beta_2(\tau)\Big), \qquad c_1 \le \tau \le c_2.$$

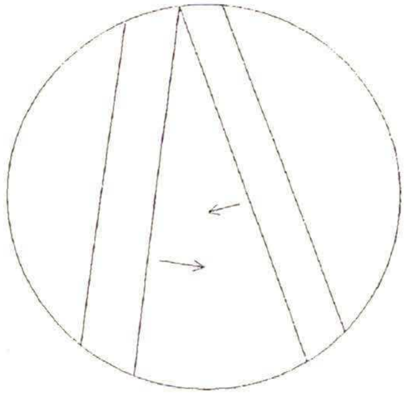

FIG. 3.

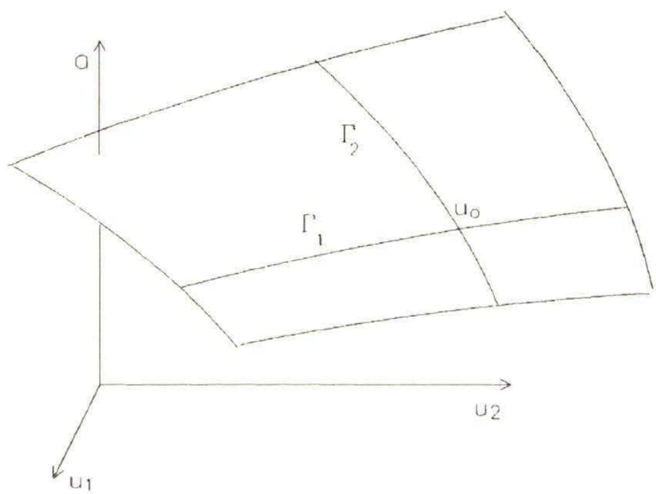

FIG. 4.

Then the projection $\widetilde{\Gamma}_1 = (\alpha_1(s),\ \alpha_2(s))$, $\widetilde{\Gamma}_2 = (\beta_1(\tau),\ \beta_2(\tau))$ are the characteristics of Eq. (6.1) and we may set

$$\psi_{|_{\widetilde{\Gamma}_1}} = \alpha_0(s), \qquad \psi_{|_{\widetilde{\Gamma}_2}} = \beta_0(\tau).$$

Let us also assume that $\alpha_0(s),\ \beta_0(\tau) > 0$, otherwise we would have a singularity in Eq. (6.1). Indeed, $a^2 = dp/d\varrho$, $a = 0$ corresponds to the vacuum.

In conclusion, we state the following theorem

THEOREM 2. *Let Γ_1, Γ_2 be two characteristic curves of class C^2 in the hodograph space passing through the point $U_0 = (a_0, u_0, v_0)$. If $a_0 > 0$, then*

1) *there exists a unique solution of the above Darboux problem provided that Γ_1, Γ_2 are of sufficiently small length and have a curvature small enough*;

2) *the surface $\widetilde{a} = \psi(u, v)$ representing the solution is covered by two families of characteristic curves in the hodograph space, and its boundary consists of pieces of characteristic curves passing through the end points of Γ_1 and Γ_2, thus forming a curvilinear quadrangle* (Fig. 3).

The proof can be obtained by representing the solution of the linearized problem in the form of a double integral along the characteristics. Then, using successive approximations and the Banach contraction principle one can prove the convergence. Let us note, that we work in the region of hyperbolicity of Eq. (6.1), since there are two characteristics Γ_1, Γ_2 at U_0 and this must be true in some neighborhood of the Darboux data.

We do not present the details of the proof since the proof for a general case of interacting waves can be found in [20].

Suppose now, that the considered solution $a = \psi(u, v)$ of Eq. (6.1), representing the surface as shown in Fig. 3, is given. At this moment one can assume a new system of coordinates on the surface, whose lines are the two families of characteristic curves from which the surface is "weaved". Vectors tangent to these lines are the polarization vector-fields and thus λ^1, λ^2 can be determined according to (3.1) and expressed in terms of R^1, R^2. Having λ^1, λ^2, the equations for $R^1(t, x, y)$, $R^2(t, x, y)$ can be solved by assuming for $R^1(0, x, y)$, $R^2(0, x, y)$ the profiles of the waves specified by the initial conditions.

If the amplitudes of the waves are large, the solution of Eq. (6.1) can enter the ellipticity region (e.g. Case 2 in Sec. 5). Similarly, if the profiles of initial waves are too steep, the solution can develop the singularities (gradient catastrophe) before the interaction is fully developed.

The form of Eq. (6.1) suggests the possibility of a geometric interpretation. The term

$$(\psi_v^2 - 1)\psi_{uu} + (\psi_u^2 - 1)\psi_{vv} - 2\psi_u\psi_v\psi_{uv}$$

in Eq. (6.1) is proportional to the mean curvature of the surface $\widetilde{a} = \psi(u, v)$ when the surface is considered as embedded in the hodograph space endowed with the Minkowskian metric $(1, -1, -1)$. But this is the form (6.3) defining the polarization vectors. One can also verify that the first term in Eq. (6.1) vanishes when computed for the surface representing the range of two noninteracting waves [7]. This also suggests that the first term in Eq. (6.1) measures in some way the strength of interaction. This line of reasoning which appears to be also useful in the proof of existence, was developed in [19, 20].

Equation (6.1) was obtained also in [21], where the authors were searching for solutions with degenerate hodograph. They did not relate these solutions to interacting waves. It seems that Yanenko was also aware (private conversation) of the connection between Eq. (6.1) and the curvature of the hodograph surface. This justifies to call Eq. (6.1) the Yanenko equation.

In conclusion, we should emphasize, however, that the property of elastic interaction that irrotational modes exhibit when subjected to a nonlinear interaction, is rather exceptional. If one of these two interacting waves were the shear wave or the entropic wave [18], then the picture would be more complicated due to the production of new waves (e.g. reflected waves) in the process of interaction. For this reason, in such cases as represented by the two potential modes considered here, we propose to speak of an elastic interaction.

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References

1. F. BAUER, R. GARABEDIAN and D. KORN, *Supercritical wing sections*, Lectures Notes in Economics and Mathematical Systems – 66, Springer Verlag, New York 1972.
2. F. BAUER, R. GARABEDIAN, D. KORN and A. JAMESON, *Supercritical wing sections II*, Lectures Notes in Economics and Mathematical Systems – 108, Springer Verlag, New York 1978.
3. M. BURNAT, *Hyperbolic double waves*, Bull. Acad. Polon. Sci., Série Sci. Techn., **16**, 1, 867, 1968.
4. M. BURNAT, *The method of Riemann invariants for multidimensional nonelliptic system*, Bull. Acad. Polon. Sci., Série Sci. Techn., **17**, 97, 1019, 1969.
5. R. CAOURANT and K.O. FRIEDRICHS, *Supersonic flow and shock waves*, Interscience Publishers, New York 1948.
6. L. DINU, *Some remarks concerning the Riemann invariance. Burnat – Peradzyński and Martin approaches*, Rev. Roumaine de Math. Pures et Appl., **35**, 203, 1990.
7. H. FLANDERS, *Differential forms with applications to the physical sciences*, Academic Press, New York 1963.
8. R. GARABEDIAN and D. KORN, *Numerical design of transonic airfoils. Numerical solution of partial differential equations* – II, Academic Press, New York 1971.
9. J.H. GIESE, *Compressible flow with degenerate hodographs*, Quarterly Appl. Math., **9**, 237, 1951.
10. A.M. GRUNDLAND and P.J. VASSILIOU, *On the solvability of the Cauchy problem for Riemann double-waves by the Monge–Darboux method*, Analysis, **11**, 221, 1991.
11. A.M. GRUNDLAND and R. ŻELAZNY, *Simple waves in quasilinear hyperbolic systems*. Part I and II., J. Math. Phys., **24**, 1983.
12. P. KUCHARCZYK, Z. PERADZYŃSKI and E. ZAWISTOWSKA, *Unsteady multi-dimensional isentropic flow described by linear Riemann invariants*, Arch. Mech., **25**, 319–350, 1973.
13. R. VON MISES, *Mathematical theory of compressible fluid flow*, Academic Press, New York 1958.
14. L. NIRENBERG, *Lectures on nonlinear functional analysis* [in Russian], Seria Matematika, Mir, Moskva 1977.
15. Z. PERADZYŃSKI, *Nonlinear plane k-waves and Riemann invariants*, Bull. Acad. Polon. Sci., Série Sci. Techn., **19**, 9, 59–66, 1971.
16. Z. PERADZYŃSKI, *Riemann invariants for the nonplanar k-waves*, Bull. Acad. Polon. Sci., Série Sci. Techn., **19**, 10, 59–74, 1971.
17. Z. PERADZYŃSKI, *On certain classes of exact solutions for gasdynamics equations*, Arch. Mech., **24**, 287–303, 1972.
18. Z. PERADZYŃSKI, *Asymptotic decay of solutions of hyperbolic system into simple waves*, Bull. Acad. Polon. Sci., Série Sci. Techn., **26**, 12, 513–519, 1078.
19. Z. PERADZYŃSKI, *Geometry of nonlinear interaction in partial differential equations* [in Polish], IFTR Reports, Warszawa 1981.

20. Z. PERADZYŃSKI, *Geometry of Riemann waves*, [in:] Advances in Nonlinear Waves, L. DEBNATH [Ed.], Pitman, 244–285, 1985.

21. YU. YA. POGODIN, V.A. SUTCHKOV and N.N. YANENKO, *Equations of wave propagation in dynamics of gases* [in Russian], DAN SSSR, **119**, 3, 443, 1958.

22. G.B. RIEMANN, *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Abhandl. Ges. Wiss., Göttingen Math.–Physik., Kl. 8, 43, 1958/59.

23. B.L. ROZHDESTVENSKI and N.N. YANENKO, *Systems of quasi-linear equations and their applications to dynamics of gases* [in Russian], Nauka, Moskva 1968.

24. A.F. SIDOROV, V.P. CHAPEEV and N.N. YANENKO, *The method of differential constraints and its application in gas-dynamics* [in Russian], Nauka, Moskva 1984.

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