Arch. Mech., 48, 6, pp. 1047–1067, Warszawa 1996

Integral equations for disturbance propagation in linearized Vlasov plasmas Numerical results

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SPACE-TIME responses of linearized Vlasov plasmas on the basis of multiple integral equations are considered. An initial-value problem for Vlasov–Poisson/Ampere equations is reduced to one integral equation and the solution is expressed in terms of a forcing function and its space-time convolution with the resolvent kernel. The forcing function is responsible for the initial disturbance and the resolvent is responsible for the equilibrium velocity distribution. For Maxwellian equilibrium distribution, a closed-form solution of the resolvent kernel equation is still unknown but the equation is eligible for computer calculations. Three types of exact analytical solutions of the space-time resolvent equations are shown to relate them to Maxwellian plasmas. Numerical calculations reveal the nature of the plasma response as a compound of a diffusive transition, being essentially a plasma oscillation mode with plasma frequency, a Gaussian type of amplitude profiles, and also a damped dispersive wave mode. The plasma response appears immediately in the whole space of x and zeros (nodes) travel according to the diffusion law, at least for long times. By use of the resolvent equations, time-reversibility and space-reflexivity can be revealed. The step-density disturbance of electron Maxwellian plasmas appears to be the electric current forcing function, which is proportional to the Maxwellian plasma kernel; hence the resolvent is the plasma response to the step-density disturbance. From inspections of the series representations of Maxwellian resolvent and its Fourier transform, a symmetry property with respect to the transformation is found. It is used for constructing approximate formulae for the resolvent kernels.

1. Introduction

THIS ARTICLE contains a unified treatment of disturbance propagation in linearized Vlasov plasmas, based on the space-time convolution integral equations. Although there already exists a vast literature on the subject, a complete and coherent discussion of space-time plasma response in relation to equilibrium distributions of particles, especially the Maxwellian equilibrium, is still lacking. Most of the papers are dealing with dispersion relations, approximate Fourier transforms of disturbances and asymptotic evolution of time-dependent stationary waves. However, papers dealing with integral equation presentation of wave propagation in linearized plasmas appear rather seldom, see the recently published paper [1].

The problem is of a linear nature but can be considered in relation to nonlinear Langmuir waves and solitary wave excitations, where we need space-time solutions, but under simplified assumptions concerning equilibrium of plasmas and the so-called "far field" approximations, which allow us to reduce the problem to model equations, e.g. NLS, KdV, Boussinesq, see [2]. To be more specific and at the same time, to present the general issue in the simplest way, let us consider the ion-sound solitary waves in Vlasov plasmas. It can be shown [3] that assuming a delta-Dirac velocity distribution for cold ions, "square distribution" for hot electrons, "far field" dependence of space-time in the form $x - vt$ and nonlinearity of the second order, we arrive at a Boussinesq equation for space-time propagation. The equation can be exactly solved leading to nonlinear oscillations or solitary waves. The seemingly simpler case of linear plasma of hot electrons and cold ions has no exact solution to an initial-value problem for linearized Vlasov–Poisson eqations. In Sec. 3 of the paper, we present the exact solution for the response function, but only for one-component electron plasma with square equilibrium distribution. The solution is a Riemann function for a wave equation with dispersion. The other exact solution in the case of the Lorentz equilibrium distribution of electrons is presented and the solution demonstrates "diffusive transition of oscillations". The space-time response for Maxwellian equilibrium is very important but a closed-form solution is still unknown (e.g., see [4]). The problem is easily analysed by computer calculations. There is another point of a general nature that deserves mentioning, namely, the way in which the disturbance propagation behaves. The question arises whether we are faced with diffusive transition of oscillations or with wave propagation. We shall especially focus on two distinctive features of the disturbance of the Maxwellian equilibrium. The first is that a step-density disturbance response is proportional to the resolvent kernel of our space-time convolution equations, that is a unique property of Maxwellian electron plasmas. The second feature is that the resolvent kernel is invariant with respect to the Fourier transform since the original and its transform are expandable in symmetric Hermite orthogonal series.

The article is organized as follows. In Sec. 2 analytical initial-value and one-point boundary-value problems of linearized Vlasov–Poisson/Ampere equations are reduced to equivalent two-dimensional integral equations to demonstrate the analytical approach to real functions in real space-time as compared with the Fourier-transform techniques. Section 3 is devoted to the main features of the integral equations in relation to plasma responses, dispersion relations and a presentation of exact solutions. Section 4 constitutes the main body of the article and contains a complete description of the Maxwellian plasma response based on orthogonal Hermite series presentations of the response. The computer-calculated characteristics are discussed on the basis of approximate formulae and compared with the exact solution of the "square" equilibrium case. Diffusive transition of resolvent zeros (nodes) is revealed for long time range. The final section contains the general discussion and conclusions.

2. Convolution equations for electric field, potential, current and charge density

We investigate the Vlasov–Ampere/Poisson system of equations for multicomponent plasmas, i.e.

$$(2.1)\qquad \left[\partial_t + u\partial_x + \frac{q_\alpha}{m_\alpha}\mathsf{E}(x,t)\partial_u\right]\mathsf{F}_\alpha(u,x,t) = 0, \qquad \partial/\partial u \equiv \partial_u \qquad \text{(Vlasov)},$$

$$(2.2)\qquad \varepsilon_0\partial_t\mathsf{E} + \sum_\alpha q_\alpha \int_{-\infty}^{\infty} u\mathsf{F}_\alpha\, du = 0, \qquad \partial/\partial x \equiv \partial_x\,, \quad \partial/\partial t \equiv \partial_t \quad \text{(Ampere)},$$

$$(2.3)\qquad \varepsilon_0\partial_x\mathsf{E} - \sum_\alpha q_\alpha \int_{-\infty}^{\infty} \mathsf{F}_\alpha\, du = 0, \qquad \mathsf{E} = -\partial_x\phi, \qquad \text{(Poisson)},$$

where x, u and t are independent variables of one-dimensional space, velocity and time, respectively. $\mathsf{E} = \mathsf{E}(x,t)$, $\phi = \phi(x,t)$, $\mathsf{F}_\alpha = \mathsf{F}_\alpha(u,x,t)$, q_α and m_α are electric field, potential, velocity distribution, charge and mass of α-particles, respectively. In view of (2.1), equations (2.2) and (2.3) are equivalent if appropriate constrains are applied to initial conditions for F_α. We emphasize that in order to derive the Vlasov equation, one must assume that F_α is analytic in its variables. This assumption of analyticity is reasonable since F_α is a physically measurable quantity, see [5].

Let us assume

$$(2.4)\qquad \mathsf{F}_\alpha(u,x,t) \cong N_0^\alpha \mathsf{F}_{0\alpha}(u) + \mathsf{F}_{1\alpha}(u,x,t),$$

where N_0^α, $\mathsf{F}_{0\alpha}(u)$ are the equilibrium particle concentration and velocity distribution for $\mathsf{E} = 0$, and $\mathsf{F}_{1\alpha}$ is of the order E.

Substituting (2.4) into (2.1), we derive the well-known linear equation

$$(2.5)\qquad (\partial_t + u\partial_x)\,\mathsf{F}_{1\alpha} = -(N_0^\alpha q_\alpha/m_\alpha)\mathsf{E}\partial_u\mathsf{F}_{0\alpha}\,.$$

For the initial-value problem

$$(2.6)\qquad \mathsf{F}_{1\alpha}(u,x,0) = g_\alpha(u,x), \qquad g_\alpha(u,x=\pm\infty) = 0 \qquad \text{and} \qquad \mathsf{E}(x,t) = 0 \quad \text{for} \quad t \le 0$$

we write the solution of Eq. (2.5)

$$(2.7)\qquad \mathsf{F}_{1\alpha} = -(N_0^\alpha q_\alpha/m_\alpha)\mathsf{F}'_{0\alpha}(u)\int_0^t \mathsf{E}(x-ut_1, t-t_1)\,dt_1 + g_\alpha(u, x-ut).$$

Substituting into (2.2), we have

$$(2.8)\qquad \partial_t\mathsf{E} = \sum_\alpha \omega_\alpha^2 \int_{-\infty}^{\infty} u\mathsf{F}'_{0\alpha}(u)\int_0^t \mathsf{E}(x-ut_1, t-t_1)\,dt_1\,du - \sum_\alpha (q_\alpha/\varepsilon_0)\int_{-\infty}^{\infty} u g_\alpha(u, x-ut)\,du$$

where $\omega_\alpha^2 \equiv N_0^\alpha q_\alpha^2/\varepsilon_0 m_\alpha$, and changing variables of integration in Eq. (2.8) as follows: $tu = \xi$, $t\,du = d\xi$, then integrating by parts, we obtain

$$\mathsf{E}(x,t) = \int_0^t dt_1 \int_{-\infty}^{\infty} \mathsf{K}(x,\xi)\mathsf{E}(x-\xi, t-t_1)\,d\xi\,dt_1 + G(x,t), \tag{2.9}$$

where

$$G(x,t) = -\sum_\alpha (q_\alpha/\varepsilon_0) \int_0^t \int_{-\infty}^{\infty} u g_\alpha(u, x - ut_1)\,du\,dt_1\,, \qquad \text{for} \quad t \geq 0,$$

and

$$\mathsf{K}(x,t) = -\sum_\alpha \omega_\alpha^2 \mathsf{F}_{0\alpha}(x/t).$$

More detailed derivation of Eq. (2.9) can be found in [6–8].

It is worth noting that the charge density, electric current and electric potential satisfy the same equations with the same kernel $\mathsf{K}(x,t)$ but with the respective forcing functions.

In the same way, we can derive the following integral equation, see [7].

$$\overline{\mathsf{E}}(x,t) = \int_0^x d\xi \int_{-\infty}^{\infty} \overline{\mathsf{K}}(\xi, t_1)\overline{\mathsf{E}}(x-\xi, t-t_1)\,dt_1 + \overline{G}(x,t) \tag{2.10}$$

for the one-point boundary value-problem

$$\mathsf{F}_{1\alpha}(u,x,t) = \overline{g}_\alpha(u,t) \quad \text{for} \quad x = 0, \qquad \overline{\mathsf{E}}(x,t) = 0 \quad \text{for} \quad x \leq 0,$$

where

$$\overline{\mathsf{K}}(x,t) = -\sum_\alpha \omega_\alpha^2 \left[\mathsf{F}_{0\alpha}\left(\frac{x}{t}\right) - \mathsf{F}_{0\alpha}(0)\right]$$

and

$$\overline{\mathsf{G}}(x,t) = -\sum_\alpha \frac{q_\alpha}{\varepsilon_0} \int_0^x d\xi \int_{-\infty}^{\infty} \overline{g}\left(u, t - \frac{\xi}{u}\right) du.$$

Taking space-Fourier transform of (2.9) one can derive one-dimensional Volterra integral equations for plasma density and plasma in an external electric field obtained in [1], where complex space-Fourier components are assumed. Similarly, time-Fourier transform of (2.10) leads to the planar case of the forced oscillatons investigated in [1]. We have derived here equations (2.9) and (2.10) analytically, without use of the Fourier–Laplace transform technique. It guarantees analyticity, existence and uniqueness of the solutions.

The existence and uniqueness of an analytic solution of Eq. (2.5) is determined by $g_\alpha(u,x)$ alone. The fact that we are given an independent function $\overline{g}_\alpha(u,t)$ does not contradict this statement since the solution is not necessarily analytic along the characteristic $x = ut$.

3. Properties of convolution equations in plasma context

Space-time convolution equations (2.9) can be solved by use of resolvent (reciprocal) kernels $\mathsf{R}(x,t)$. We write the solution in the form

$$\mathsf{E}(x,t) = G(x,t) + \int_0^t dt_1 \int_{-\infty}^{\infty} \mathsf{R}(x-\xi, t-t_1) G(\xi, t_1)\, d\xi, \tag{3.1}$$

where $G(x,t)$ is a forcing function and $\mathsf{R}(x,t)$ satisfies the following resolvent equation

$$\mathsf{R}(x,t) = \mathsf{K}(x,t) + \int_0^t dt_1 \int_{-\infty}^{\infty} \mathsf{K}(x-\xi, t-t_1) \mathsf{R}(\xi, t_1)\, d\xi. \tag{3.2}$$

The last equation describes plasma dynamic response $\mathsf{R}(x,t)$ and its functional dependence of the plasma equilibrium state only. We note that for the infinite support $x \in (-\infty, \infty)$ of a kernel $\mathsf{K}(x,t)$, the resolvent $\mathsf{R}(x,t)$ also possesses the infinite support $x \in (-\infty,\infty)$. The physical consequence of the property is that the plasma response to any disturbance, even if with a limited support, appears immediately in the full space $x \in (-\infty,\infty)$. On the ground of Eq. (3.2) we note, that for $\mathsf{K}(x,t) = \mathsf{K}(x,-t)$ it follows that $\mathsf{R}(x,t) = \mathsf{R}(x,-t)$ and for $\mathsf{K}(x,t) = \mathsf{K}(-x,t)$ we have $\mathsf{R}(x,t) = \mathsf{R}(-x,t)$. The property is reversible with respect to $\mathsf{R}(x,t)$ and $\mathsf{K}(x,t)$. It is called time reversibility and space reflexivity.

3.1. Dynamic response of Maxwellian plasmas to step-density disturbances

It is obvious, that the resolvent kernel can be considered as a response to the Dirac-delta disturbances $\delta(t)\delta(x)$ and sometimes the resolvent kernel is misnamed a Green function.

We show that a step-density disturbance of Maxwellian plasma will now become proportional to the kernel $\mathsf{K}(x,t)$ and according to Eqs. (3.1) and (3.2), it leads to plasma response being the resolvent. Considering the electric current forcing disturbance

$$G_{\mathsf{J}}(x,t) \equiv \mathsf{J}_0(x,t) \equiv \sum_\alpha q_\alpha \int_{-\infty}^{\infty} u g_\alpha(u, x-ut)\, du \tag{3.3}$$

for multi-component plasmas, we have the following step-density disturbance

$$\sum_\alpha g_\alpha(u,x) = \sum_\alpha \Delta N_\alpha \mathsf{F}_{0\alpha}(u) H(x), \tag{3.4}$$

where $H(x)$ is the Heaviside unit-step function, and

$$\mathsf{F}_{0\alpha}(u) = a_\alpha \pi^{-1/2} \exp(-u^2 a_\alpha^2), \qquad \alpha = e, i, \tag{3.5}$$

where $\langle u_\alpha^2 \rangle = 1/2a_\alpha^2$.

Roughly speaking, the disturbance can be realized in double- or triple- plasma devices.

According to (3.3), we have

$$\mathsf{J}_0(x,t) = \sum_\alpha q_\alpha \int_{-\infty}^{\infty} u\mathsf{F}_{0\alpha}(u)H(x-ut)\,du$$

and by virtue of

$$(3.6)\qquad u\mathsf{F}_{0\alpha}(u) = -\mathsf{F}'_{0\alpha}(u)/2a_\alpha^2\,, \qquad \text{since} \qquad \mathsf{F}_\alpha(u) \sim \exp(-a_\alpha^2 u^2)$$

we have

$$\int_{-\infty}^{\infty} u\mathsf{F}_{0\alpha}(u)H(x-ut)\,du = -(1/2a_\alpha^2)\mathsf{F}_{0\alpha}\left(\frac{x}{t}\right)$$

and

$$\mathsf{J}_0(x,t) = -\sum_\alpha A_\alpha \mathsf{F}_{0\alpha}(x/t) \simeq -A_e\mathsf{F}_{0e}(x/t),$$

where $A_\alpha = \Delta N_\alpha q_\alpha/2a_\alpha^2$. Neglecting the ion contribution to the electron plasma oscillations in view of the equation

$$\mathsf{J}(x,t) = \mathsf{J}_0(x,t) + \int_0^t dt_1 \int_{-\infty}^{\infty} \mathsf{R}(x-\xi,t-t_1)\mathsf{J}_0(\xi,t_1)\,d\xi$$

and Eq. (3.2), we have

$$\mathsf{J}(x,t) \simeq (A_e/\omega_e^2)\mathsf{R}(x,t).$$

The dynamic response of electron plasmas to the step-density disturbance is proportional to the resolvent $\mathsf{R}(x,t)$. It takes place uniquely only for Maxwelian plasmas because of relation (3.6). In order to obey linearization assumptions, the step-density ΔN must be small enough in relation to N_0.

3.2. Exact solutions

The advantage of the integral equation treatment of Vlasov plasmas consists in obtaining the solutions separately composed of the forcing function $G(z,t)$ resulting from the initial value disturbance $g(u,t)$, and the resolvent kernel depending only on the plasma equilibrium $\sum_\alpha \mathsf{F}_{0\alpha}(u)$. It opens up new possibilities for computer calculations. One may expect readily available computer program, say for PC, calculating and graphically illustrating resolvents, forcing functions and convolutions of these functions for real time and space. First of all, we review exact and approximate solutions for resolvent kernels and compare them with numerical results.

Assuming the hot electron plasma with the so-called "square" electron equilibrium velocity distribution

$$\mathsf{F}_{0e}(u) = \left[\mathsf{H}(u+\alpha) - \mathsf{H}(u-\alpha)\right]/2\alpha,$$

we have

$$\mathsf{K}(x,t) = -\omega_0^2\left[\mathsf{H}(x+\alpha t) - \mathsf{H}(x-\alpha t)\right]/2\alpha,$$

and the transforms of the kernel are

$$\begin{aligned} \mathsf{K}(k;t) &= -(\omega_0^2/\alpha k)\sin(k\alpha t), \\ \mathsf{K}(k;s) &= -\omega_0^2/(s^2+k^2\alpha^2). \end{aligned} \tag{3.8}$$

The resolvent kernel can be readily calculated as follows:

$$\begin{aligned} \mathsf{R}(k;t) &= -\frac{\omega_0^2}{(\omega_0^2+k^2\alpha^2)^{1/2}}\sin\left[(\omega_0^2+k^2\alpha^2)^{1/2}t\right], \\ \mathsf{R}(k;s) &= -\frac{\omega_0^2}{s^2\omega_0^2+k^2\alpha^2}, \end{aligned} \tag{3.9}$$

and

$$\mathsf{R}(x,t) = \begin{cases} -(\omega_0^2/2\alpha)\mathsf{J}_0\left[\omega_0(t^2-x^2/\alpha^2)^{1/2}\right] & \text{for} \quad t^2 \ge x^2/\alpha^2, \\ 0 & \text{elsewhere}. \end{cases} \tag{3.10}$$

The dispersion relation takes the form

$$\mathsf{D}(k;s) \equiv 1 - \mathsf{K}(k;s) = (s^2+\omega_0^2+k^2\alpha^2)/(s^2+k^2\alpha^2) = 0.$$

Substituting $s = -i\omega$ and since $\langle u^2\rangle = \alpha^2/3$, we have the well-known Bohm–Gross dispersion relation, see also [2],

$$\omega^2 \simeq \omega_0^2 + 3\langle u^2\rangle k^2.$$

We note, that $\mathsf{K}(x,t)$ and $\mathsf{R}(x,t)$ are time reversible and x-space reflexive and the resolvent is an undamped dispersive wave, i.e. the Riemann function of the following dispersive wave equation

$$\left(\alpha^2\partial_{xx} - \partial_{tt} + \omega_0^2\right)\mathsf{R}(x,t) = 0. \tag{3.11}$$

The asymptotic expansion of the function is

$$\mathsf{R}(x,t) \simeq -\omega_0(4\pi Dt)^{-1/2}\sin(\omega_0 t + \pi/4), \qquad t \to \infty, \tag{3.12}$$

where $D = 3\langle u^2\rangle/2\omega_0$. It appears that the asymptotic formula is common for all resolvents in cases of equilibrium velocity distributions possessing all moments and the mean-square velocity being $\langle u^2\rangle$. We do not present here the proof of these properties.

The next exact solution known to us is the resolvent for the Lorentz electron plasma. The equilibrium distribution is

$$\mathsf{F}_0(u) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + u^2}$$

where λ is a positive parameter. The distribution has some unrealistic features, for instance, infinite mean-square velocity, but many authors consider it to be of interest. A generalized Lorentzian distribution (possessing a finite number of moments) is useful for modelling plasma with a high-energy tail that typically occurs in space [9].

We quote results of papers [6, 7] presenting kernels

$$\begin{aligned} \mathsf{K}(x,t) &= -(\omega_0^2/\pi)\,\lambda/(\lambda^2+u^2)\Big|_{u=x/t}, \\ \mathsf{K}(k;t) &= -\omega_0^2 t\exp(-|k|\lambda t), \end{aligned} \tag{3.13}$$

and resolvents

$$\begin{aligned} \mathsf{R}(x,t) &= -(\omega_0/t)\mathsf{F}_0(x/t)\sin(\omega_0 t), \\ \mathsf{R}(k;t) &= -\omega_0\Big[\exp(-|k|\lambda t)\Big]\sin(\omega_0 t). \end{aligned} \tag{3.14}$$

The resolvent is drastically different from the previous one. It does not exhibit wave propagation and there is no dispersion relation. We observe a rather "diffusive transition" of oscillations. The amplitude $t^{-1}\cdot\mathsf{F}_0(x/t)$ of oscillations does obey the Chapman - Kolmogoroff equation (see Eq. (4.12) and [6]). Wave damping has no meaning, but time reversibility and space reflexivity are preserved.

Let us note that for the kernels

$$\begin{aligned} \mathsf{K}(x,t) &= -\omega_0^2(t/4\pi D)^{1/2}\exp(-x^2/4Dt), \\ \mathsf{R}(x,t) &= -\omega_0(4\pi Dt)^{-1/2}\Big[\exp(-x^2/4Dt)\Big]\sin(\omega_0 t), \end{aligned} \tag{3.15}$$

Equation (3.2) is satisfied. The example exhibits a pure diffusive transition of oscillations. However there is no equilibrium velocity distribution $\mathsf{F}_0(u)$, which could be regained from the kernel (3.15), and there is no time reversibility.

4. Maxwellian plasmas

Maxwellian equilibrium distribution (3.5) is considered to be most appropriate but analytically almost intractable. In this section, Maxwellian plasmas are analysed by means of approximate formulae and computer diagram presentations.

For numerical calculations we introduce dimensionless variables, based on the following characteristic quantities; $\omega_0 = 2\pi f_0$ [1/s]-plasma frequency, $\langle u^2 \rangle = 1/2a^2$ [m^2/s^2]-square of thermal velocity, $\lambda_D = 2\pi/k_D = 2\pi\langle u^2\rangle/\omega_0$ [m]-Debye length.

We scale the independent and dependent variables as follows:

$$X = k_D x = 2\pi x/\lambda_D \ [\pi], \qquad T = \omega_0 t \ [\pi], \qquad K = k/k_D, \qquad \tau_o = 1/f_0,$$

$$\mathsf{K}(X,T) = (1/a)\mathsf{K}(x,t), \qquad \mathsf{R} = (1/a)\mathsf{R}(x,t).$$

Before commenting on the computer plots we would like to remind the reader that the amplitudes of all physical quantities are arbitrary, as in all linear theories.

Following [7], we may write

$$(4.1) \qquad \mathsf{R}(k;t) \simeq -\omega_0 \sin(\omega_0(1 + 3k^2/4a_e^2\omega_0^2)t)\exp(-s_L(k)t), \qquad k \to 0,$$

where $s_L(k)$ is the Landau damping [10] and by virtue of the method of stationary phase, the asymptotic expansion takes the form

$$(4.2) \qquad \mathsf{R}(x,t) \simeq -\omega_0(4\pi Dt)^{-1/2}\sin(\omega_0 t + \pi/4) \qquad \text{for} \quad t \to \infty$$

and $D = 3/4a_e^2\omega_0 = 3\langle u^2\rangle/2\omega_0$. We observe that the Landau damping has no influence on the asymptotic formula (4.2) since $s_L(k)$ and all its derivatives disappear as $k \to 0$ and, according to stationary phase method, it does not appear in Eq. (4.2), which is identical with that of undamped waves (3.12).

According to our numerical results, the effect of Landau damping is insignificant up to $K = 0.2$ but for $K = 0.25$ the damping rate reduces the amplitude of $\mathsf{R}(K,T)$ to approximately one half for each $50\tau_0$-interval, so that for $150\tau_0$, the amplitude is smaller a little less than 8 times. In the case of $K = 0.3$ the damping rate is drastically increased and the amplitude decreases 50, 70 and 90 times for the succesive intervals of $50\tau_0$, that is about $3 \cdot 10^5$ times for the whole $150\tau_0$ interval.

The properties of the damping phenomena of the resolvent F-transforms are summarized in Fig. 1. It refers to the behaviour of the resolvent as a function of K for fixed values of dimensionless T. We observe that in the vicinity of $K = 0.2$, a rapid increase of the damping rate starts. The distributions of zeros (nodes) of $\mathsf{R}(K,T)$ is in general agreement with the approximate formula (see Eq. (4.1)). The last feature should be emphasized as it also takes place for $\mathsf{R}(x,t)$, see formula (4.9). For comparison, the resolvent $\mathsf{R}(K,T)$ of the undamped dispersive wave, Eq. (3.9), is shown in Fig. 2.

Figures 3 and 4 refer to the behaviour of the Maxwellian resolvent $\mathsf{R}(X,T)$ versus time T for fixed values of dimensionless X. To comment on the diagrams we recall Eq. (3.15). According to the graphs of Figs. 3 and 4, we do not observe the wave fronts, which could be distinguished like in Fig. 5, where the $\mathsf{R}(X,T)$ of

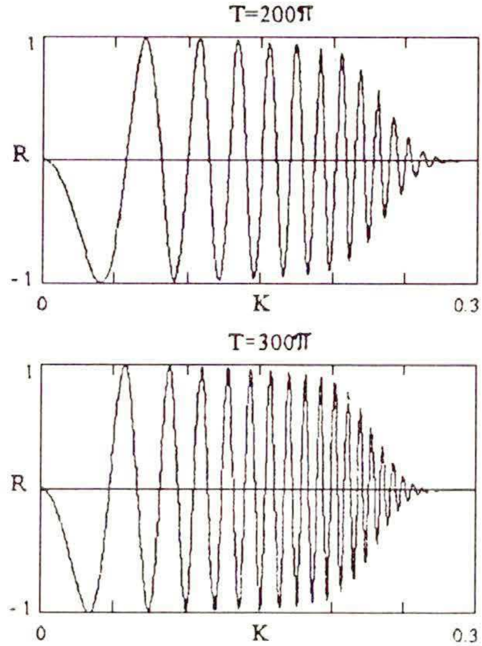

FIG. 1. Maxwellian plasma Fourier transforms $\mathsf{R}(K, T)$ vs K for $T = 200\pi$, 300π.

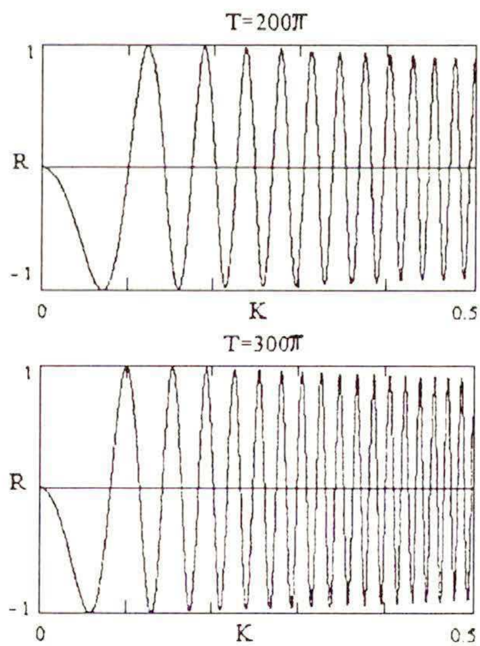

FIG. 2. Fourier transforms of Riemann's functions $\mathsf{R}(K, T) = -(\sin(T(1 + K^2)))/(1 + K^2)^{1/2}$ vs K for $T = 200\pi$, and 300π.

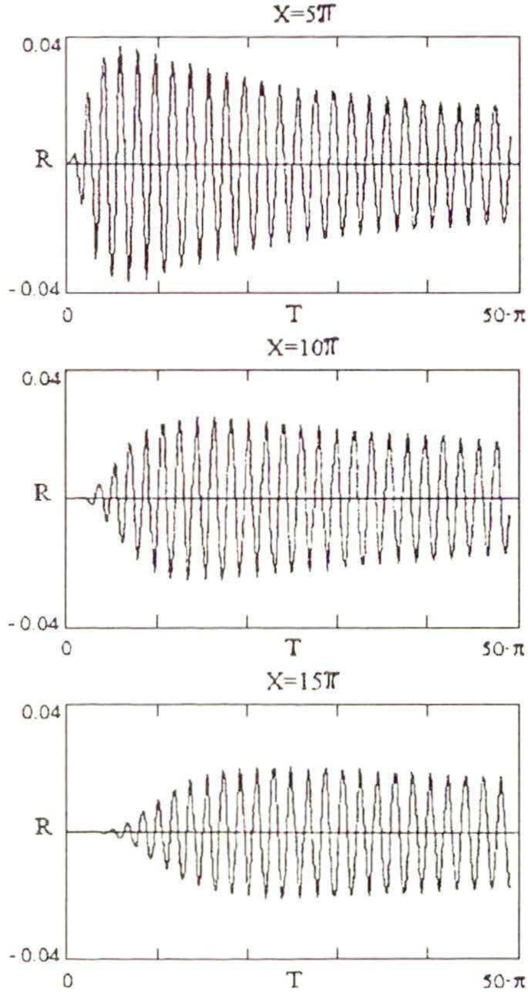

FIG. 3. Resolvent kernels of Maxwellian plasma $\mathrm{R}(X,T)$ vs T for $X = 5\pi$, 10π, and 15π.

"square" equilibrium is exhibited. However, there are two characteristic features of the Maxwellian resolvent profiles. The time period is slightly less than the electron plasma period τ_0 at the begining of time scale, but later on is equal to the period with computed accuracy. The second feature is that the profiles of amplitude envelopes behave according to the Gaussian distribution, that is like $A_x T^{-1/2}\exp(-B_x/T)$, where A_x and B_x are constant values for fixed values of X. These features are in agreement with the formula (3.15).

To discuss the remaining diagrams we need to use the formulae, which could explain the $\mathrm{R}(X,T)$ characteristics versus X for fixed values of T_n. One can note the striking resemblance between the $\mathrm{R}(K,T)$ and $\mathrm{R}(X,T)$ characteristics for fixed values of T_n.

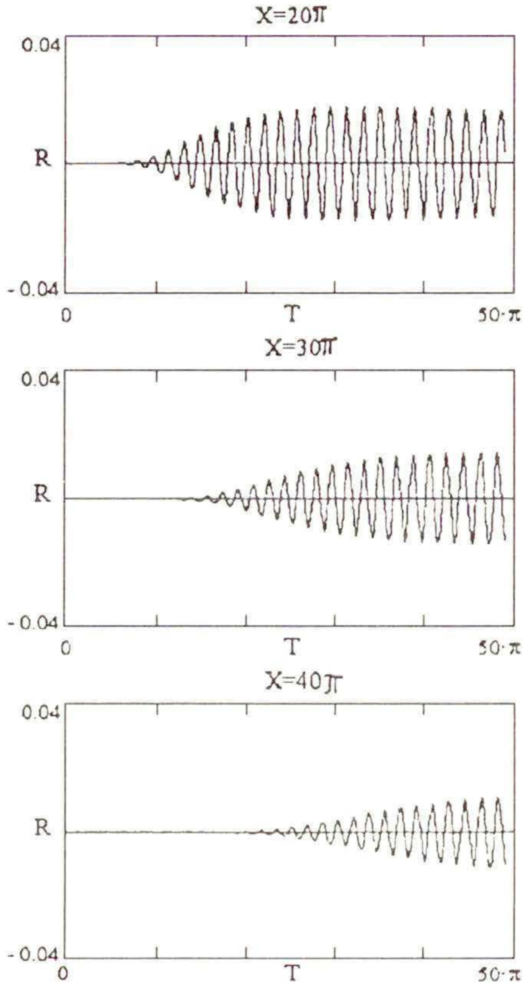

FIG. 4. Resolvent kernels of Maxwellian plasma $\mathrm{R}(X,T)$ vs T for $X = 20\pi$, 30π, and 40π.

The Maxwellian kernel can be expanded in the following Taylor series

$$(4.3)\qquad \mathsf{K}(x-\xi) = \mathsf{K}(x) + \xi\mathsf{K}'(x) + (\xi^2/2!)\mathsf{K}''(x) + \ldots = \sum_{l=0}^{\infty}(\xi^l/l!)\mathsf{K}^{(l)}(x).$$

We note, that

$$\mathsf{K}(x) = -\omega_0^2 a\pi^{-1/2} H_0(Z)\exp(-Z^2),$$
$$\mathsf{K}^{(l)}(x) = -\omega_0^2 a\pi^{-1/2} H_l(Z)\exp(-Z^2),$$

where $Z = ax/t$ and Hermite polynomials $H_l(Z)$ are determined by the formula

$$H_l(x) = (-1)^l e^{x^2}\frac{d^l}{dx^l}e^{-x^2}.$$

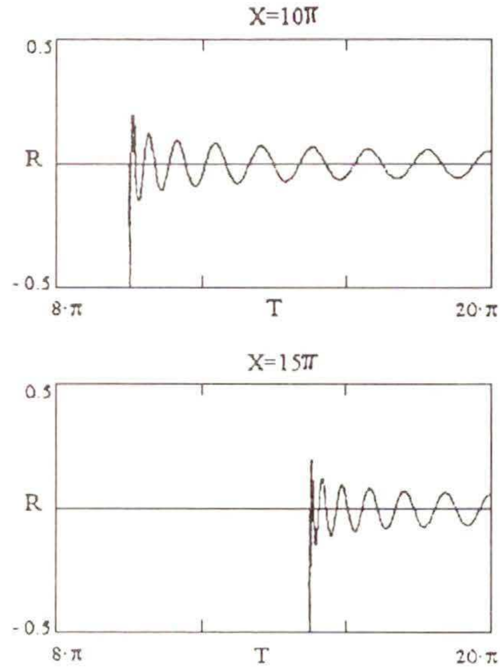

FIG. 5. Resolvent kernels for "square" velocity equilibrium; $\mathsf{R}(X,T) = -0.5\mathrm{J}_0((T^2 - X^2)^{1/2})$ vs T for $X = 10\pi$ and 15π.

Substituting (4.3) into Eq. (3.1), we have

$$\mathsf{R}(x,t) = -\omega_0^2 a\pi^{-1/2}\left[e^{-Z^2} + \sum_{n=0}^{\infty}\frac{1}{(2n)!}\int_0^t r_{2n}(t-t_1)\left(\frac{a}{t_1}\right)^{2n} \times H_{2n}(Z_1)e^{-Z_1^2}\,dt_1\right], \tag{4.4}$$

where

$$Z_1 = ax/t_1\,,$$

$$r_{2n}(t) = \int_{-\infty}^{\infty} x^{2n}\mathsf{R}(x,t)\,dx,$$

$$\int_{-\infty}^{\infty} x^m\mathsf{R}(x,t)\,dx = \int_{-\infty}^{\infty} x^m\mathsf{K}(x,t)\,dx = 0 \qquad \text{for odd } m.$$

Equations determining $r_{2n}(t)$ can be derived by multiplying Eq. (3.2) by x^{2n} and integrating it with respect to x. The first two solutions are

$$r_0(t) = -\omega_0\sin(\omega_0 t), \qquad r_2(t) = -2\omega_0 D\left[\sin(\omega_0 t) - \omega_0 t\cos(\omega_0 t)\right].$$

The Fourier transform of Eq. (3.2) takes the form

$$(4.5)\qquad \mathsf{R}(k;t) = \mathsf{K}(k,t) + \int_0^t \mathsf{K}(k,t-t_1)\mathsf{R}(k;t_1)\,dt_1\,,$$

where

$$\mathsf{K}(k;t) = -\omega_0^2 t \exp(-p^2t^2), \qquad p^2 = k^2/4a^2,$$

and proceeding like in the previous case, i.e substituting the Taylor series for $\mathsf{K}(k;t-t_1)$ into Eq. (4.5), we obtain

$$(4.6)\qquad \mathsf{R}(k;t) = -\omega_0^2\left[te^{-p^2t^2}+\sum_{n=0}^{\infty}\frac{1}{(2n)!}\int_0^t r_{2n}(t-t_1)\left(\frac{a}{t_1}\right)t_1H_{2n}(pt_1)e^{-p^2t_1^2}d_1\right].$$

Equations (4.4) and (4.6) are symmetric and invariant with respect to the Fourier transform, due to the Hermite function properties. The following changes of variables lead from $\mathsf{R}(k;t)$ to $\mathsf{R}(x,t)$ and conversely,

$$(4.7)\qquad \begin{aligned} te^{-p^2t^2}H_{2n}(pt) &\rightleftarrows (a/\pi^{1/2})e^{-Z^2}H_{2n}(Z),\\ \frac{a}{\pi^{1/2}}e^{-Z^2} &\rightleftarrows te^{-p^2t^2}. \end{aligned}$$

From relation (4.5) it is evident that $R(k,t)/p$ is a function of pt only for a fixed value of ω_0^2/p^2, and this property was also exhibited by the numerically calculated plots in [1]. The property of (4.7) will be exploited to derive an approximate formula for $\mathsf{R}(x,t)$ by use of an approximate expression for $\mathsf{R}(k;t)$. By virtue of the dispersion relation

$$\omega^2 \simeq \omega_0^2(1+6p^2/\omega_0^2+60p^4/\omega_0^4+\ldots), \qquad p\to 0$$

and following the derivation of Eq. (4.1), we have

$$(4.8)\qquad \mathsf{R}(k;t) \simeq \omega_0 e^{-s_L(k)t}\sin(\omega_0^2t^2+6p^2t^2+60p^4t^4/\omega_0^2t^2)^{1/2} \qquad \text{for}\quad p\to 0,$$

where $s_L(k)$ is a damping coefficient.

In view of the symmetry (4.7) we may expect the following approximate formula

$$(4.9)\qquad \mathsf{R}(x,t) \simeq -(\omega_0 a/\pi^{1/2}t)e^{-\beta(x,t)}\sin(\omega_0 t(1+6X^2+\ldots)^{1/2}) \qquad \text{for}\quad t\to\infty,$$

where $X = xa/\omega_0t^2$ and $\beta(x,t)$ is a damping rate. Analytical expression for $\beta(x,t)$ is not known.

Analyzing Figs. 6 and 7, we note that for $X = 0$ and fixed $T_n = 30\pi$, $5)\pi$, 100π, 200π and 300π, the amplitudes behave according to the asymptotic relaton

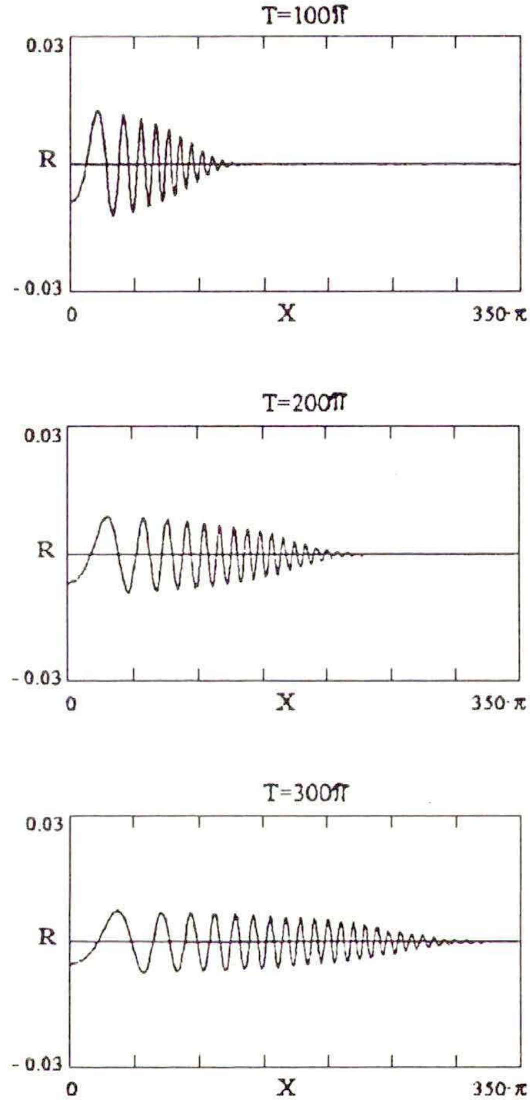

FIG. 6. Maxwellian plasma resolvent kernels $\mathsf{R}(X,T)$ vs X for $T = 100\pi$, 200π, and 300π.

(4.2), that is $\mathsf{R}_n \simeq CT_n^{-1/2}\sin(T_n + \pi/4)$ where C is a constant. We conclude that for $T_n \geq 30\pi$ and X small enough, the Maxwellian resolvent behaves qualitatively in accordance with the formula (4.9).

The characteristic feature of the curves in Figs. 6–8 is a distribution of resolvent zeros (nodes) for fixed time T and $x > 0$ according to (4.9). First of all we can not find such values of T that $\mathsf{R}(x,T)$ is zero for all x, $x \in (-\infty,\infty)$, as in the case of diffusive transitions of oscillations (see Eq. (3.14) and (3.15)), where $\mathsf{R}(x,\omega_0 t = m\pi) = 0$ for $x \in (-\infty,\infty)$.

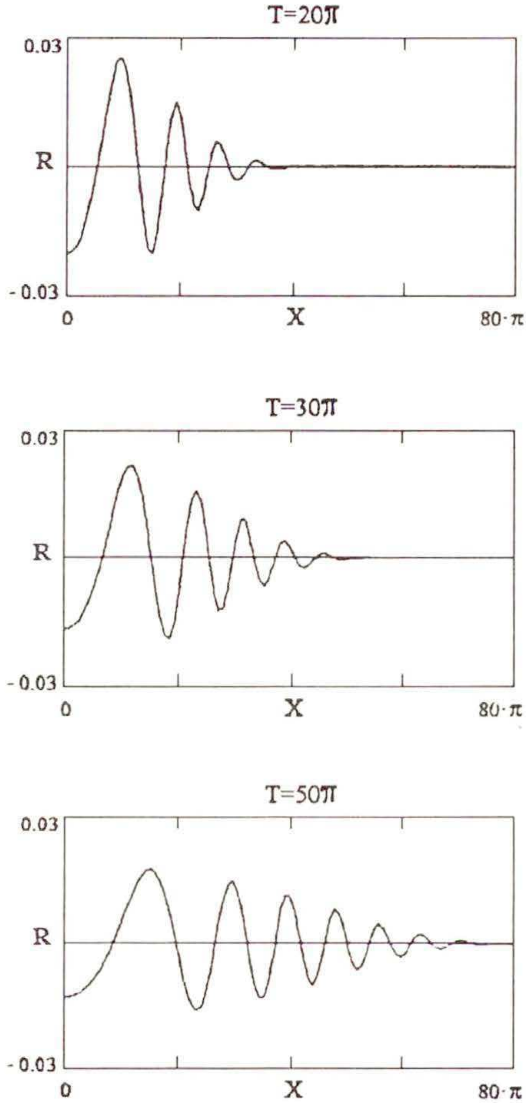

FIG. 7. Maxwellian plasma resolvent kernels $\mathsf{R}(X,T)$ vs X for $T = 20\pi,\ 30\pi$, and 50π.

Also, a wave front is not marked contrary to the case of square equilibrium, see Fig. 9. The rate of spatial damping of the signal versus X is high for shorter times, i.e. $T_n = \pi$, 6π, $9\ 1/6\pi$ and 20π, Figs. 7, 8. For $T_n = 50\pi$, 100π, 200π and 300π the rate of damping is nearly linear.

In the case of a dispersive wave, Eq. (3.10), the wave front propagates with velocity α but zeros are subject to dispersion and travel with the following velocities: $v_m = dx/dt = \alpha(1 - \kappa_m^2/\omega_0^2 t^2)^{-1/2}$, where $\mathsf{J}_0(\kappa_m) = 0$ and for $(\kappa_m/\omega_0 t)^2 > 1$, $\mathsf{R}(x,t) = 0$. In the case of Maxwellian plasmas, $v_m \simeq (m\pi/a)(6(1-\omega_0^2 t^2/m^2\pi^2))^{-1/2}$

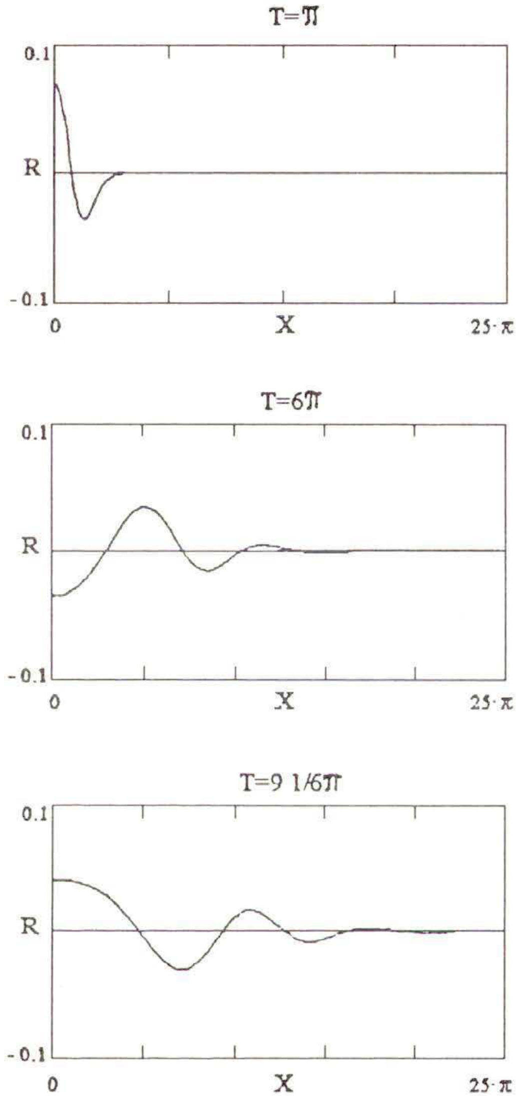

FIG. 8. Maxwellian plasma resolvent kernels $R(X, T)$ vs X for $T = \pi,\ 6\pi$, and $9\ 1/6\pi$.

according to Eq. (4.9), which is an asymptotic relation for $\omega_0 t/m\pi \gg 1$, hence v_m are purely imaginary.

By use of Eq. (4.9), we may write

$$\omega_0 t(1 + 6x^2a^2/\omega_0^2t^4)^{1/2} \simeq \omega_0 t + 3x^2a^2/\omega_0 t^3.$$

If $6x^2a^2/\omega_0^2t^4 \ll 1$ and denoting $X^2 = 6x^2a^2/t^2$, $T = \omega_0 t$, the equation

$$\sin(T + (1/2)X^2/T) = 0$$

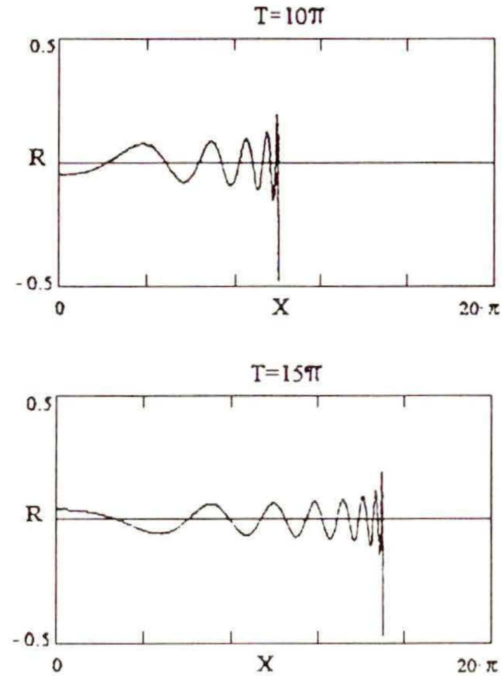

FIG. 9. Resolvent kernel for "square" velocity equilibrium; $\mathsf{R}(X,T) = -0.5\mathsf{J}_0((T^2 - X)^{1/2})$ vs X for $T = 10\pi$ and 15π.

must be satisfied. According to computer calculations, at least in the range $T_n = 100\pi \div 300\pi$, we have

$$X^2_{m,n} \simeq 2T_n d_m\,, \qquad d_m \simeq \pi + 3m\pi, \qquad m = 0, 1, 2...$$

hence, we may write

$$\begin{aligned} &\frac{X^2_{m,n+1} - X^2_{m,n}}{T_{n+1} - T_n} \simeq 2d_m\,, \\ &X^2_{m,n}/X^2_{m,n+1} \simeq T_n/T_{n+1}\,. \end{aligned} \tag{4.10}$$

We conclude, that the m-th zero (node) of the resolvent is moving along the X-axis in accordance with the law of diffusive transition. We note that the m-th zero is related to the m-th diffusive constant, $d_m = \pi + 3m\pi$.

Finally, we emphasize the fundamental difference between a diffusive transition of oscillations and wave propagation, both being based on our convolution equations, which uniquely transform the kernel $\mathsf{K}(x,t)$ into the respective resolvent $\mathsf{R}(x,t)$.

If we assume the solution in the form

$$\mathsf{R}(x,t) = -\omega_0\varrho(x,t)sin(\omega_0 t),$$

where

$$\int_{-\infty}^{\infty} \varrho(x,t)\,dx = 1,$$

then Eq. (3.2) takes the form

$$(4.11)\qquad \mathsf{F}_0(x,t) - t\varrho(x,t) = \omega_0 \int_0^t dt_1 (t-t_1)\sin(\omega_0 t_1) \times \left[\int_{-\infty}^{\infty} \varrho(x_1,t_1) \frac{\mathsf{F}_0(x-x_1,t-t_1)}{t-t_1}\,dx_1 - \varrho(x,t) \right],$$

where $\mathsf{K}(x,t) = -\omega_0^2 \mathsf{F}_0(x,t)$,

$$\sin(\omega_0 t) = \omega_0 t - \omega_0^2 \int_0^t (t-t_1)\sin(\omega_0 t_1)\,dt_1 .$$

If $\varrho(x,t) = (1/t)\mathsf{F}_0(x,t)$, then the resolvent equation implies the following Chapman - Kolmogoroff equation

$$(4.12)\qquad \int_{-\infty}^{\infty} \varrho(x-x_1,t-t_1)\varrho(x_1,t_1)\,dx_1 = \varrho(x,t)$$

and

$$(4.13)\qquad \int_{-\infty}^{\infty} x^2 \varrho(x,t)\,dx = 2Dt.$$

The equation (4.12) possesses a unique solution (see Eq. (3.15)). When the integral (4.13) does not exist (e.g. unlimited energy), then Eq. (4.12) can possess different solutions, (see Eq. (3.13) and (3.14)). The wave propagation can be derived by reduction of Eq. (3.2) to a wave equation, (see Eq. (3.11)). The case of Maxwellian equilibrium cannot be reduced neither to Chapman - Kolmogoroff equation or the wave equation.

However, as numerical calculations indicate, there is a set of values $(x_{m,n}; t_n)$ for which the resolvent $\mathsf{R}(x,t)$ comes to nodes and they travel along the x-axis according to the diffusive law, see Eq. (4.10). Moreover, on the basis of the Eq. (3.2), an approximate dispersion relation can be derived and an approximate wave equation can be regained, (see Eq. (3.11)).

5. Discussion and conclusions

In this article we have studied space-time responses of linearized Vlasov plasmas on the basis of multiple integral convolution equations. An initial-value problem for Vlasov - Poisson/Ampere equations can be reduced to the integral equation and the solution to the problem is expressed in terms of a forcing function $G(x,t)$ and its convolution with a resolvent kernel $\mathsf{R}(x,t)$ (see Eq. (3.1)). The forcing function is responsible for the initial disturbance and the resolvent is responsible for an equilibrium velocity distribution, see Eq. (3.2). Resolvent kernel equations (3.2) are eligible for computer calculations.

We have presented three types of exact analytical solutions of the space-time resolvent equations. The solutions can be classified following the space-tine behaviour. The first one is a dispersive wave solution (Riemann function) in the case of the simplified electron plasma equilibrium, called "square equilibrium" Then the resolvent equation (3.2) can be reduced to dispersion wave equation and the Bohm - Gross dispersion relation is satisfied. The second one is calculated for the Lorentz equilibrium of electron plasmas. We call this type of space-time behaviour "diffusive transition of oscillations" since the space-time amplitude of oscilations satisfies the Chapman - Kolmogoroff equation and there is no wave speed and no dispersion relation. On the ground of the two types of resolvent kernels, the solution to an initial-value problem of Vlasov - Poisson/Ampere equations can be determined if the respective forcing function is known. The last type of the exact solution of Eq. (3.2) is also a diffusive transition of oscillations with the amplitude being a Gaussian function (3.15). This example is not exactly applicable to linearized plasma equations since it has not been derived from any equilibrium, but it turns out that the resolvent approximates the Maxwellian plasma behaviour for fixed x and long time t according to (3.15) and due to the computer calculated results, Figs. 3 and 4. By use of the resolvent equation (3.2) one can easily prove the time-reversibility as well as the space-reflexivity for a given plasma kernel.

The main results of this paper concern the Maxwellian plasmas having the properties which can be summarized as follows. The nature of the plasma response is a compound of a diffusive transition, being essentially a plasma oscillation mode with the ω_0 - plasma frequency and the Gaussian type of amplitude profiles, and a damped dispersive wave mode. Differentiation of these two properties is not an easy task and we have not a ready conclusion but it seens that the Maxwellian plasma response exhibits mainly diffusive transition in space for fixed values of time in a long time range, and damped wave behaviour for fixed values of x with respect to time t. We note that the plasma response appears immediately in the whole space of x, and the zeros (nodes) travel according to (4.10) at least for long times. The step-density disturbance of electron Maxwellian plasmas appears to be the electric current forcing function, which is propotional to Maxwellian plasma kernel, hence the resolvent kernel is a plasma response to the step-density disturbance. It is noteworthy that the solitary plasma waves can be excited experimentally by strong step-density disturbances in ion-electron plasmas.

By inspecting the series representing the resolvent and its Fourier transform, Eqs. (4.4) and (4.6), we found the symmetry property with respect to Fourier transforms. It can be used for constructing approximate formulae of $\mathsf{R}(x,t)$ if the approximate expressions of their Fourier transforms are known, and vice versa.

Acknowledgment

The authors are grateful to Prof. H. ZORSKI for critical reading of the manuscript.

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Received April 22, 1996.