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BRIEF NOTES

A note on the hyperelastic constitutive equation for rotated Biot stress

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THE FORWARD-ROTATED BIOT STRESS and the right stretch strain are defined, and the virtual work of the rotated stress is found. It is shown that it involves a corotational variation of the Green–McInnis–Naghdi type. For the strain energy assumed in terms of principal invariants of the right stretching tensor, a constitutive equation and a constitutive (4th rank) operator for the Biot stress is derived. Subsequently, they are subjected to the rotate-forward operation, and it is demonstrated how their structure is carried over to the rotated measures.

1. Introduction

THE CO-ROTATIONAL FORMULATIONS are applied to many problems of mechanics, ranging from finite strain plasticity to large rotation shells, mostly due to relative simplicity of manipulating on orthogonal rotation tensors.

In finite strain plasticity, see e.g. DIENES [3] and JOHNSON, BAMMANN [4], the so-called rotated description is based on a back-rotated Kirchhoff stress $\boldsymbol{\Sigma} = \mathbf{Q}^T \boldsymbol{\tau} \mathbf{Q}$ and a back-rotated spatial rate of deformation $\mathbf{D} = \mathbf{Q}^T \mathbf{d}\, \mathbf{Q}$, where $\mathbf{d} = \mathrm{sym}\,(\dot{\mathbf{F}}\, \mathbf{F}^{-1})$. The rotated measures are exploited to define a constitutive equation, which later is converted to $\overset{\circ}{\boldsymbol{\tau}}$ and $\mathbf{d}$, where $\overset{\circ}{\boldsymbol{\tau}}$ is the Green – McInnis – Naghdi objective stress rate.

It was noticed by several authors, e.g. see the introduction to CRISFIELD [2], that nonlinearities resulting from large rotations of beams or shells can be eliminated if corotational local frames are introduced. Among recent works using the corotational frames, we would like to mention contributions of RANKIN, BROGAN [6], SIMO [7], SIMO, VU-QUOC [9], and CRISFIELD [2]. In RANKIN, BROGAN [6] a general framework to handle large rotations has been constructed, in which already existing linear finite elements can be embedded. In [7] and [9] a finite strain/rotation beam model for dynamics has been consistently derived from three-dimensional equations. In [2] an issue of symmetry of the tangent operator for the finite rotation beam has been undertaken. In all these papers separation of frame rotations simplified the equations.

In the present note we extend the concept of the corotational frame used for beams and shells and introduce a forward-rotated description: the rotated stress and strain measures, and the corotational variation. We address in detail an issue of a hyperelastic constitutive equation and a constitutive operator for the rotated measures as derived from the constitutive relations for the Biot stress. The forward-rotated description, as a general concept, can be found convenient in problems involving independent rotation fields, not only in beam or shell theories but also in three-dimensional elasticity formulated as e.g. in SIMO, FOX, HUGHES [8].

Notation

Small letters – vectors, capital letters – 2nd rank tensors, capital letters with a superscribed digit 4 – 4th rank tensors, dots $\cdot$ – scalar products, colons : – contractions of a 4th and a 2nd rank tensors yielding a 2nd rank tensor, $\otimes$ – tensorial products.

2. Rotated stress and strain

In this section the rotated strain and stress measures are introduced and a corresponding form of the virtual work of stress is given.

The Cauchy (true) stress, $\mathbf{T}$, can be expressed in terms of other stress measures as follows, see e.g. OGDEN [5],

$$\mathbf{T} = J^{-1}\boldsymbol{\tau} = J^{-1}\mathbf{P}\,\mathbf{F}^T = J^{-1}\mathbf{F}\,\mathbf{S}\,\mathbf{F}^T, \tag{2.1}$$

where $\boldsymbol{\tau}$ is the Kirchhoff stress, $\mathbf{P}$ is the 1st Piola - Kirchhoff stress, (its transpose is a nominal stress), $\mathbf{S}$ is the 2nd Piola - Kirchhoff stress. Besides, $\mathbf{F}$ denotes the gradient of deformation, and $J = \det \mathbf{F}$.

Let us introduce a symmetric Biot stress tensor, $\mathbf{T}^B \equiv \mathrm{sym}\,(\mathbf{Q}^T\mathbf{P})$. The rotation tensor $\mathbf{Q} \in SO(3)$ is obtained from the polar decomposition of the deformation gradient. The Biot stress $\mathbf{T}^B$ and the right stretch strain $\mathbf{E}$ are work conjugates because the virtual work of stress can be expressed as follows

$$\mathbf{P} \cdot \delta\mathbf{F} = \mathbf{T}^B \cdot \delta\mathbf{E}, \tag{2.2}$$

where $\mathbf{E} = \mathbf{U} - \mathbf{I}$ is the right stretch strain, and $\mathbf{U} \equiv (\mathbf{F}^T\mathbf{F})^{1/2}$ is the right stretching tensor. This tensor appears also in the (right) polar decomposition of the deformation gradient, $\mathbf{F} = \mathbf{Q}\mathbf{U}$. On the basis of Eq. (2.1), the Biot stress is related to other stress measures in the following way

$$\mathbf{T}^B \equiv \mathrm{sym}\,(\mathbf{Q}^T\mathbf{P}) = \mathrm{sym}\,(\mathbf{Q}^T\boldsymbol{\tau}\mathbf{F}^{-T}) = \mathrm{sym}\,(\mathbf{U}\,\mathbf{S}). \tag{2.3}$$

The Biot stress tensor $\mathbf{T}^B$ and the right stretch strain $\mathbf{E}$ can be used to introduce a set of rotated measures defined as follows

$$\mathbf{T}^* \equiv \mathbf{Q}\,\mathbf{T}^B\mathbf{Q}^T, \qquad \mathbf{E}^* \equiv \mathbf{Q}\,\mathbf{E}\,\mathbf{Q}^T, \tag{2.4}$$

for which the virtual work of stress (2.2) yields

$$\mathbf{T}^B \cdot \delta\mathbf{E} = \mathbf{T}^* \cdot \overset{\circ}{\delta}\,\mathbf{E}^*, \tag{2.5}$$

where

$$\overset{\circ}{\delta}\,\mathbf{E}^* \equiv \mathbf{Q}\,\delta\mathbf{E}\,\mathbf{Q}^T = \mathbf{Q}\,\delta(\mathbf{Q}^T\mathbf{E}^*\mathbf{Q})\,\mathbf{Q}^T.$$

The above corotational variation corresponds with the Green - McInnis - Naghdi objective time derivative, and consists of the rotate-back, take a variation and rotate-forward operations. The definition (2.4) yields

$$\mathbf{E}^* = \mathbf{V} - \mathbf{I}, \tag{2.6}$$

where $\mathbf{V} = \mathbf{Q}\,\mathbf{U}\,\mathbf{Q}^T$ is a left stretching tensor defined as $\mathbf{V} \equiv (\mathbf{F}\,\mathbf{F}^T)^{1/2}$. Hence, $\mathbf{E}^*$ is the left stretch strain. The rotated Biot stress is related to other stress measures as follows

$$\mathbf{T}^* = \operatorname{sym}(\mathbf{P}\,\mathbf{Q}^T) = \operatorname{sym}(\boldsymbol{\tau}\,\mathbf{V}^{-1}) = \operatorname{sym}(\mathbf{F}\,\mathbf{S}\,\mathbf{V}^{-1}). \tag{2.7}$$

We can see that $\mathbf{T}^*$ is different than other spatial stress measures, such as Cauchy stress $\mathbf{T}$ or Kirchhoff stress $\boldsymbol{\tau}$.

3. Constitutive equation for rotated measures

In this section a constitutive equation and a constitutive operator for $\mathbf{T}^B$ and $\mathbf{U}$ are introduced for an isotropic hyperelastic (Green) material. Next, the same constitutive equation and the constitutive operator are expressed in terms of the rotated tensors, $\mathbf{T}^*$ and $\mathbf{V}$.

Let us assume the existence of a strain energy function $W(\mathbf{U})$. On arguments discussed e.g. in OGDEN [5], a strain energy given in terms of $\mathbf{U}$ is objective, and provides a response function, which is invariant under an observer transformation. On the basis of the representation theorem for isotropic functions, we can write

$$W(\mathbf{U}) = \tilde{W}(I_1(\mathbf{U}), I_2(\mathbf{U}), I_3(\mathbf{U})), \tag{3.1}$$

where the principal invariants of $\mathbf{U}$ are defined as follows

$$I_1(\mathbf{U}) = \operatorname{tr}\mathbf{U}, \qquad I_2(\mathbf{U}) = \frac{1}{2}\left[(\operatorname{tr}\mathbf{U})^2 - \operatorname{tr}\mathbf{U}^2\right], \qquad I_3(\mathbf{U}) = \det\mathbf{U}. \tag{3.2}$$

A constitutive equation for the Biot stress tensor is defined as

$$\mathbf{T}^B \equiv \frac{\partial W(\mathbf{U})}{\partial \mathbf{U}} = \frac{\partial \tilde{W}(I_1(\mathbf{U}), I_2(\mathbf{U}), I_3(\mathbf{U}))}{\partial \mathbf{U}}. \tag{3.3}$$

From the chain rule of differentiation we obtain

$$\frac{\partial \tilde{W}}{\partial \mathbf{U}} = \frac{\partial \tilde{W}}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{U}} + \frac{\partial \tilde{W}}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{U}} + \frac{\partial \tilde{W}}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{U}}. \tag{3.4}$$

Taking into account that

$$\frac{\partial I_1}{\partial \mathbf{U}} = \mathbf{I}, \qquad \frac{\partial I_2}{\partial \mathbf{U}} = I_1\,\mathbf{I} - \mathbf{U}, \qquad \frac{\partial I_3}{\partial \mathbf{U}} = I_3\,\mathbf{U}^{-1}, \tag{3.5}$$

the constitutive equation can be rewritten as a polynomial of $\mathbf{U}$

$$\mathbf{T}^B = \beta_0\mathbf{I} + \beta_1\mathbf{U} + \beta_2\mathbf{U}^{-1}, \tag{3.6}$$

where β_0, β_1 and β_2 are scalar coefficients depending on the invariants and derivatives of $\tilde{W}$ with respect to the invariants. Note that using the Cayley–Hamilton theorem, the above equation can be converted to a second order polynomial of $\mathbf{U}$. A variation of stress with respect to the strain can be written as

$$\delta\mathbf{T}^B = \frac{\partial \mathbf{T}^B}{\partial \mathbf{U}} : \delta\mathbf{U} = \overset{4}{\mathbf{C}} : \delta\mathbf{U}, \tag{3.7}$$

where the constitutive operator (elasticity tensor) can be defined as a 4-th rank tensor

$$\overset{4}{\mathbf{C}} \equiv \frac{\partial \mathbf{T}^B}{\partial \mathbf{U}} = \frac{\partial^2 W(\mathbf{U})}{\partial \mathbf{U}\partial \mathbf{U}} = \frac{\partial^2 \tilde{W}(I_1(\mathbf{U}), I_2(\mathbf{U}), I_3(\mathbf{U}))}{\partial \mathbf{U}\partial \mathbf{U}}. \tag{3.8}$$

Hence, from the formula for the derivative of the product of a scalar and a second rank tensor we have

$$\begin{aligned}\overset{4}{\mathbf{C}} = \frac{\partial \mathbf{T}^B}{\partial \mathbf{U}} &= \mathbf{I} \otimes \frac{\partial \beta_0}{\partial \mathbf{U}} + \beta_0 \frac{\partial \mathbf{I}}{\partial \mathbf{U}} \\ &\quad + \mathbf{U} \otimes \frac{\partial \beta_1}{\partial \mathbf{U}} + \beta_1 \frac{\partial \mathbf{U}}{\partial \mathbf{U}} + \mathbf{U}^{-1} \otimes \frac{\partial \beta_2}{\partial \mathbf{U}} + \beta_2 \frac{\partial \mathbf{U}^{-1}}{\partial \mathbf{U}},\end{aligned} \tag{3.9}$$

where

$$\frac{\partial \beta_i}{\partial \mathbf{U}} = \frac{\partial \beta_i}{\partial I_k}\frac{\partial I_k}{\partial \mathbf{U}} \qquad \text{for} \quad i = 0,1,2 \quad \text{and} \quad k = 1,2,3 \tag{3.10}$$

due to the chain rule of differentiation. We can say that in Eq. (3.9) the 1st, 3rd and 5th components are expressed in terms of nine tensorial products, provided by all combinations of $\mathbf{I}$, $\mathbf{U}$ and $\mathbf{U}^{-1}$. Furthermore, for the 2nd, 4th and 6th components (and a symmetric $\mathbf{U}$), we have

$$\begin{aligned}&\frac{\partial \mathbf{I}}{\partial \mathbf{U}} = \overset{4}{\mathbf{0}}, \qquad \frac{\partial \mathbf{U}}{\partial \mathbf{U}} = \frac{1}{2}(\overset{4}{\mathbf{I}}_a + \overset{4}{\mathbf{I}}_c), \\ &\frac{\partial \mathbf{U}^{-1}}{\partial \mathbf{U}} = -\frac{1}{2}\left\{\mathbf{U}^{-1}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{U}^{-1}\right\} \otimes \{\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i\},\end{aligned} \tag{3.11}$$

where $\mathbf{e}_i$ are vectors of an orthonormal frame. The 4th rank invariants used here are $\overset{4}{\mathbf{I}}_a = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j$ and $\overset{4}{\mathbf{I}}_c = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i$, and operate on an arbitrary 2nd rank $\mathbf{A}$ as follows: $\overset{4}{\mathbf{I}}_a\mathbf{A} = \mathbf{A}$ and $\overset{4}{\mathbf{I}}_c\mathbf{A} = \mathbf{A}^T$, see [1].

The derivation of $\partial\mathbf{U}^{-1}/\partial\mathbf{U}$, being more complicated, is described below. Consider $\mathbf{I} = \mathbf{U}\,\mathbf{U}^{-1}$ as a tensor-valued function of a tensor argument. As $\mathbf{U}$ is symmetric, it may be replaced by $\frac{1}{2}[\mathbf{U} + \mathbf{U}^T]$, and thus $\mathbf{U}^{-1}$ can also be considered as a function of $\frac{1}{2}[\mathbf{U} + \mathbf{U}^T]$. A directional derivative of $\mathbf{I}$ at $\mathbf{U}$ in direction $\mathbf{A}$ yields

$$\frac{\partial\mathbf{I}}{\partial\mathbf{U}} \,:\, \mathbf{A} \equiv \left[\frac{d}{d\varepsilon}\mathbf{I}(\mathbf{U} + \varepsilon\mathbf{A})\right]_{\varepsilon=0} = \mathbf{0}, \tag{3.12}$$

where $\mathbf{A}$ is an arbitrary 2nd rank tensor. After straightforward calculations, from (3.12) we obtain

$$\frac{\partial\mathbf{U}^{-1}}{\partial\mathbf{U}} \,:\, \mathbf{A} = -\frac{1}{2}\mathbf{U}^{-1}(\mathbf{A} + \mathbf{A}^T)\mathbf{U}^{-1}. \tag{3.13}$$

To introduce a constitutive operator, we have to rewrite the above equation as a contraction of a fourth rank tensor and a second rank tensor $\mathbf{A}$. Introducing the 4th rank invariants we have

$$(\mathbf{A} + \mathbf{A}^T) = (\overset{4}{\mathbf{I}}_a + \overset{4}{\mathbf{I}}_c) \,:\, \mathbf{A} = \mathbf{e}_i \otimes \mathbf{e}_j \left\{[\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i]\cdot\mathbf{A}\right\}, \tag{3.14}$$

where the identity $(\mathbf{T} \otimes \mathbf{S}) \,:\, \mathbf{Q} = \mathbf{T}(\mathbf{S}\cdot\mathbf{Q})$ is used. Note that the product in the parentheses is a scalar. Substituting Eq. (3.14) into Eq. (3.13), and recovering the 4th rank tensor, we obtain

$$\frac{\partial\mathbf{U}^{-1}}{\partial\mathbf{U}} \,:\, \mathbf{A} = \left[-\frac{1}{2}\left\{\mathbf{U}^{-1}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{U}^{-1}\right\} \otimes \{\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i\}\right] \,:\, \mathbf{A}, \tag{3.15}$$

where the 4th rank tensor given by Eq. (3.11) can be easily identified. □

Having derived the constitutive equation (3.6) and the elasticity tensor (3.7) for the Biot stress $\mathbf{T}^B$, we can find the respective equations for the rotated stress $\mathbf{T}^*$. For $\mathbf{T}^B$ given by Eq. (3.6) we obtain

$$\mathbf{T}^* \equiv \mathbf{Q}\,\mathbf{T}^B\,\mathbf{Q}^T = \mathbf{Q}\left(\beta_0\mathbf{I} + \beta_1\mathbf{U} + \beta_2\mathbf{U}^{-1}\right)\mathbf{Q}^T. \tag{3.16}$$

On the basis of identities

$$\mathbf{Q}\,\mathbf{I}\,\mathbf{Q}^T = \mathbf{I}, \qquad \mathbf{Q}\,\mathbf{U}\,\mathbf{Q}^T = \mathbf{V}, \qquad \mathbf{Q}\,\mathbf{U}^{-1}\mathbf{Q}^T = \mathbf{V}^{-1} \tag{3.17}$$

we have

$$\mathbf{T}^* = \beta_0\mathbf{I} + \beta_1\mathbf{V} + \beta_2\mathbf{V}^{-1}, \tag{3.18}$$

which is a polynomial of the left stretching tensor $\mathbf{V}$.

Next, we find the elasticity tensor for the rotated stress $\mathbf{T}^*$,

$$\text{(3.19)} \qquad \overset{\circ}{\delta}\,\mathbf{T}^* \equiv \mathbf{Q}\,\delta\mathbf{T}^B\,\mathbf{Q}^T = \mathbf{Q}\left[\overset{4}{\mathbf{C}} \,:\, \frac{1}{2}\delta\mathbf{U}\right]\mathbf{Q}^T = \mathbf{Q}\left[\frac{\partial\mathbf{T}^B}{\partial\mathbf{U}} \,:\, \delta\mathbf{U}\right]\mathbf{Q}^T,$$

where the expression for $\partial\mathbf{T}^B/\partial\mathbf{U}$ is given by Eq. (3.9). Consider the 1st, 3rd and 5th component of this equation contracted with $\delta\mathbf{U}$. As mentioned earlier, these components contain nine tensorial products, and the contraction can be written as $(\mathbf{A}_i \otimes \mathbf{A}_j) \,:\, \delta\mathbf{U}$, where $\mathbf{A}_i,\ \mathbf{A}_j \in \{\mathbf{I}, \mathbf{U}, \mathbf{U}^{-1}\}$. Furthermore, $(\mathbf{A}_i \otimes \mathbf{A}_j) \,:\, \delta\mathbf{U} = \mathbf{A}_i(\mathbf{A}_j \cdot \delta\mathbf{U})$, where in the parentheses we have a scalar. Hence,

$$\text{(3.20)} \qquad \mathbf{Q}\left[(\mathbf{A}_i \otimes \mathbf{A}_j) \,:\, \delta\mathbf{U}\right]\mathbf{Q}^T = \left[\mathbf{Q}\,\mathbf{A}_i\mathbf{Q}^T\right](\mathbf{A}_j \cdot \delta\mathbf{U}) = \mathbf{B}_i(\mathbf{A}_j \cdot \delta\mathbf{U})$$

where $\mathbf{Q}\,\mathbf{A}_i\mathbf{Q}^T = \mathbf{B}_i$ and $\mathbf{B}_i \in \{\mathbf{I}, \mathbf{V}, \mathbf{V}^{-1}\}$ in accordance with Eq. (3.17). Besides, for the scalar product we have

$$\text{(3.21)} \qquad \begin{aligned} \mathbf{A}_j \cdot \delta\mathbf{U} = \operatorname{tr}(\mathbf{A}_j\delta\mathbf{U}) &= \operatorname{tr}\left(\mathbf{Q}^T(\mathbf{Q}\,\mathbf{A}_j\mathbf{Q}^T)(\mathbf{Q}\delta\mathbf{U}\,\mathbf{Q}^T)\mathbf{Q}\right) \\ &= \operatorname{tr}(\mathbf{B}_j\,\overset{\circ}{\delta}\,\mathbf{V}) = \mathbf{B}_j \cdot \overset{\circ}{\delta}\,\mathbf{V}, \end{aligned}$$

where $\mathbf{Q}\,\mathbf{A}_j\mathbf{Q}^T = \mathbf{B}_j$. Hence

$$\text{(3.22)} \qquad \mathbf{Q}\left[(\mathbf{A}_i \otimes \mathbf{A}_j) \,:\, \delta\mathbf{U}\right]\mathbf{Q}^T = (\mathbf{B}_i \otimes \mathbf{B}_j) \,:\, \overset{\circ}{\delta}\,\mathbf{V}.$$

For the 2nd component of Eq. (3.9) we have $\partial\mathbf{I}/\partial\mathbf{U} = \overset{4}{\mathbf{0}}$ and the respective term does not need to be considered. For the 4th and 6th component we have

$$\text{(3.23)} \qquad \begin{aligned} \frac{\partial\mathbf{U}}{\partial\mathbf{U}} \,:\, \delta\mathbf{U} &= \frac{1}{2}(\overset{4}{\mathbf{I}}_a + \overset{4}{\mathbf{I}}_c) \,:\, \delta\mathbf{U} = \frac{1}{2}(\delta\mathbf{U} + \delta\mathbf{U}^T), \\ \frac{\partial\mathbf{U}^{-1}}{\partial\mathbf{U}} \,:\, \delta\mathbf{U} &= -\frac{1}{2}\mathbf{U}^{-1}(\delta\mathbf{U} + \delta\mathbf{U}^T)\mathbf{U}^{-1}, \end{aligned}$$

where Eq. (3.13) was used to derive the second equation. Applying the rotation operations to both of these equations we obtain

$$\text{(3.24)} \qquad \begin{aligned} \mathbf{Q}\left(\frac{\partial\mathbf{U}}{\partial\mathbf{U}} \,:\, \delta\mathbf{U}\right)\mathbf{Q}^T &= \frac{1}{2}\mathbf{Q}(\delta\mathbf{U} + \delta\mathbf{U}^T)\mathbf{Q}^T = \frac{1}{2}(\overset{\circ}{\delta}\,\mathbf{V} + \overset{\circ}{\delta}\,\mathbf{V}^T), \\ \mathbf{Q}\left(\frac{\partial\mathbf{U}^{-1}}{\partial\mathbf{U}} \,:\, \delta\mathbf{U}\right)\mathbf{Q}^T &= -\frac{1}{2}\mathbf{Q}\,\mathbf{U}^{-1}(\delta\mathbf{U} + \delta\mathbf{U}^T)\mathbf{U}^{-1}\mathbf{Q}^T \\ &= -\frac{1}{2}\mathbf{V}^{-1}(\overset{\circ}{\delta}\,\mathbf{V} + \overset{\circ}{\delta}\,\mathbf{V}^T)\mathbf{V}^{-1}. \end{aligned}$$

Note that as a result of the rotate-forward operation in the above formulas, $\mathbf{U}$ is replaced by $\mathbf{V}$, $\mathbf{U}^{-1}$ by $\mathbf{V}^{-1}$, and $\delta\mathbf{U}$ by $\overset{\circ}{\delta}\mathbf{V}$. Hence, we may introduce an elasticity tensor $\overset{4}{\mathbf{C}}{}^*$ relating $\overset{\circ}{\delta}\mathbf{T}^*$ with $\overset{\circ}{\delta}\mathbf{V}$,

$$\overset{\circ}{\delta}\mathbf{T}^* = \mathbf{Q}\left[\overset{4}{\mathbf{C}} : \delta\mathbf{U}\right]\mathbf{Q}^T \equiv \overset{4}{\mathbf{C}}{}^* : \overset{\circ}{\delta}\mathbf{V} \tag{3.25}$$

of the same structure as $\overset{4}{\mathbf{C}}$.

For an infinitesimal deformation, when $\mathbf{F} \approx \mathbf{I}$, we have

$$\mathbf{U} = \mathbf{V} = \mathbf{I}, \qquad \frac{\partial\mathbf{U}^{-1}}{\partial\mathbf{U}} = -\frac{1}{2}(\overset{4}{\mathbf{I}}_a + \overset{4}{\mathbf{I}}_c), \qquad \frac{\partial\mathbf{V}^{-1}}{\partial\mathbf{V}} = -\frac{1}{2}(\overset{4}{\mathbf{I}}_a + \overset{4}{\mathbf{I}}_c) \tag{3.26}$$

and therefore the linearized elasticity tensors $\overset{4}{\mathbf{C}}$ and $\overset{4}{\mathbf{C}}{}^*$ are identical.

4. Conclusion

We have shown that under the rotate-forward operation, the structure of a general hyper-elastic constitutive equation and the respective constitutive operator for the Biot stress is carried over to the respective relations for the rotated Biot stress, with $\mathbf{U}$ replaced by $\mathbf{V}$, and $\delta\mathbf{U}$ by $\overset{\circ}{\delta}\mathbf{V}$, where the corotational variation is of the Green–McInnis–Naghdi type.

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