Arch. Mech., 48, 5, pp. 933–946, Warszawa 1996

The stationary transverse Euler and Stokes gas flows through a cylindrical region with large variations of density and viscosity coefficient

Z. PŁOCHOCKI and B. KAŹMIERCZAK (WARSZAWA)

THE FLOW of a gas in space, which encounters a cylindrical region, where the density of the gas (and its viscosity coefficient) changes abruptly, is considered both in the Euler and the Stokes approximations. The flow is homogeneous at infinity. Density and viscosity coefficients of the gas are assumed to be constants, which are different outside and inside the cylinder. The analytical solutions of the problem are found in both cases. These solutions may be useful for building the models of flow in flames or laser-sustained (or generated) plasmas.

1. Introduction

MODEL EXAMPLES of a stationary gas flow through a region with large variation of density (and viscosity coefficient) may be useful for constructing the simple hydraulic models of gas flow in systems with large heat perturbation, as for example – in flames or laser-generated or sustained plasmas. The idea of such models depends on the assumption, that the constant density of a gas inside the region is small as compared to (also constant) density outside the region. Such a density distribution is thought to be generated by a suitable temperature field, therefore in fact the viscosity coefficient of the gas should also be assumed to vary in a similar way.

The first such a hydraulic model was proposed in [1] for a spherical region in the Euler approximation. Numerical solution of the Navier–Stokes equations for such a flow was presented in [2]. The Stokes approximation of a gas flow through a spherical region was analyzed in [3]. The stationary transverse gas flow through a cylindrical region both in the Euler and the Stokes approximations is examined in the present paper.

2. General assumptions

Let us consider a stationary and homogeneous at infinity, transverse gas flow through a cylinder of radius R. The z-axis of the Cartesian coordinate system is the symmetry axis of the cylinder. At infinity the gas flows along the x-axis toward the cylinder. The flow is assumed to be plane in the sense, that the z-coordinate of the velocity is identically equal to zero. The gas density and the shear viscosity coefficient are assumed in the form:

$$
\begin{aligned}
\overline{\varrho} &:= \frac{\varrho}{\varrho_\infty} = \varepsilon_\varrho + (1-\varepsilon_\varrho)H(\overline{r}-1),\\
\overline{\eta} &:= \frac{\eta}{\eta_\infty} = \frac{1}{\varepsilon_\eta} - \frac{1-\varepsilon_\eta}{\varepsilon_\eta}H(\overline{r}-1),
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
\varepsilon_\varrho &:= \frac{\varrho_{\rm int}}{\varrho_\infty},\\
\varepsilon_\eta &:= \frac{\eta_\infty}{\eta_{\rm int}},
\end{aligned}
\tag{2.2}
$$

where ϱ_∞, $\varrho_{\rm int}$, η_∞, $\eta_{\rm int}$ stand for constant density and shear viscosity coefficient outside and inside the cylinder, respectively, $H(x-x_0)$ stands for the Heaviside function, $\overline{r} = r/R$ is dimensionless r-coordinate, and cylindrical coordinate system r, φ, z is used. Let us note that because the assumed distributions of ϱ and η may be thought to be generated by a suitable temperature field T, therefore the quantities ε_ϱ and ε_η are interrelated. In the case of an ideal gas ($\varrho \propto 1/T$, $\eta \propto \sqrt{T}$) this relationship has the form:

$$
\varepsilon_\eta = \sqrt{\varepsilon_\varrho}\,.
\tag{2.3}
$$

The solution of the governing equations, which describe the velocity and pressure fields, will be looked for separately outside and inside the cylinder, and next these external and internal solutions will be matched using the continuity conditions for the mass and momentum flux densities at the surface of the cylinder.

3. The Euler approximation

3.1. Formulation of the problem

According to the assumptions adopted, the governing equations in the cylindrical coordinate system both outside ($\overline{r} > 1$) and inside the cylinder ($\overline{r} < 1$) can be written in the following dimensionless form:

$$
\begin{aligned}
\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}(\overline{r}\,\overline{v}_r) + \frac{1}{\overline{r}}\frac{\partial}{\partial\varphi}\overline{v}_\varphi &= 0,\\
\overline{v}_r\frac{\partial\overline{v}_r}{\partial\overline{r}} + \frac{\overline{v}_\varphi}{\overline{r}}\frac{\partial\overline{v}_r}{\partial\varphi} - \frac{\overline{v}_\varphi^2}{\overline{r}} + \frac{1}{2\overline{\varrho}}\frac{\partial\overline{\Delta p}}{\partial\overline{r}} &= 0,\\
\overline{v}_r\frac{\partial\overline{v}_\varphi}{\partial\overline{r}} + \frac{\overline{v}_\varphi}{\overline{r}}\frac{\partial\overline{v}_\varphi}{\partial\varphi} + \frac{\overline{v}_r\overline{v}_\varphi}{\overline{r}} + \frac{1}{2\overline{\varrho}}\frac{1}{\overline{r}}\frac{\partial\overline{\Delta p}}{\partial\varphi} &= 0,
\end{aligned}
\tag{3.1}
$$

where

$$
\overline{v}_\alpha := \frac{v_\alpha}{v_\infty}, \qquad \alpha = r,\varphi, \qquad \overline{\Delta p} := 2\frac{p-p_\infty}{\varrho_\infty v_\infty^2}\,,
$$

where, in turn, v_∞ and p_∞ stand for the velocity modulus and pressure at $\overline{r} = \infty$, respectively.

The boundary and matching conditions are:

$$
\begin{aligned}
&\text{at} \quad \overline{r} = \infty : \begin{cases} \overline{v}_r = \cos\varphi, \\ \overline{v}_\varphi = -\sin\varphi, \\ \overline{\Delta p} = 0; \end{cases} \\
&\text{at} \quad \overline{r} = 0 : |\overline{v}_r|,\ |\overline{v}_\varphi|,\ |\overline{\Delta p}| < \infty; \\
&\text{at} \quad \overline{r} = 1 : \begin{cases} [\![\, \overline{\varrho}\,\overline{v}_r \,]\!] = 0, \\ [\![\, \tfrac{1}{2}\overline{\Delta p} + \overline{\varrho}\,\overline{v}_r^2 \,]\!] = 0, \\ [\![\, \overline{\varrho}\,\overline{v}_r \overline{v}_\varphi \,]\!] = 0; \end{cases}
\end{aligned}
\tag{3.2}
$$

where

$$[\![\, \psi \,]\!] := \psi(\overline{r} = 1 + 0) - \psi(\overline{r} = 1 - 0) =: \psi^{\text{ext}}(\overline{r} = 1) - \psi^{\text{int}}(\overline{r} = 1).$$

Because all the considerations will run in terms of the dimensionless variables introduced only, therefore from now on, all the bars will be ignored.

3.2. The solution

The velocity field is looked for in the form:

$$
\begin{aligned}
v_r &= f(r)\cos\varphi, \\
v_\varphi &= -g(r)\sin\varphi.
\end{aligned}
\tag{3.3}
$$

Substituting Eqs. (3.3) into Eq. $(3.1)_1$ one may obtain the following relationship between the functions f and g:

$$g = (rf)' = rf' + f, \tag{3.4}$$

where prime denotes the derivative with respect to r. Substituting Eqs. (3.3) into Eqs. $(3.1)_{2,3}$ and using Eq. (3.4) one may obtain:

$$
\begin{aligned}
\frac{1}{2\varrho}\frac{\partial \Delta p}{\partial r} &= r(f')^2 + ff' - \left\{ r(f')^2 + 2ff' \right\}\cos^2\varphi, \\
\frac{1}{2\varrho}\frac{\partial \Delta p}{\partial \varphi} &= \left\{ r^2 ff'' + rff' - r^2(f')^2 \right\}\sin\varphi\cos\varphi.
\end{aligned}
\tag{3.5}
$$

Integrating Eq. $(3.5)_2$ one may obtain:

$$
\begin{aligned}
\frac{1}{2\varrho}\Delta p &= \phi(r) - \frac{1}{2}\chi(r)\cos^2\varphi, \\
\chi(r) &:= r^2 ff'' + rff' - r^2(f')^2.
\end{aligned}
\tag{3.6}
$$

Comparison of Eqs. (3.6) and $(3.5)_1$ gives, after some algebra:

$$\phi' = \frac{1}{2}(\chi - f^2)', \qquad \left\{\frac{f''}{f} + \frac{3}{r}\frac{f'}{f}\right\}' = 0. \tag{3.7}$$

From Eq. $(3.7)_2$ one has immediately:

$$f'' + \frac{3}{r} f' = \beta f, \tag{3.8}$$

where β stands for an integration constant. If $\beta = 0$, then Eq. (3.8) gives

$$f_0 = C_1 + \frac{C_2}{r^2}, \tag{3.9}$$

where C_1 and C_2 stand for integration constants.

If $\beta = -c^2 < 0$, then after substituting:

$$f_- = \frac{\psi_-(\zeta)}{r}, \qquad \zeta = c\,r,$$

Eq. (3.8) is transformed to the Bessel equation of the first order, therefore:

$$f_- = \frac{1}{r}\left\{C_3 J_1(c\,r) + C_4 Y_1(c\,r)\right\}, \tag{3.10}$$

where C_3 and C_4 stand for integration constants, J_1 – for the Bessel function of the first kind and the first order, and Y_1 – for the Bessel function of the second kind (the Neumann or the Weber function) and of the first order.

If $\beta = \widehat{c}^2 > 0$, then in the same way one may obtain:

$$f_+ = \frac{1}{r}\left\{C_5 I_1(\widehat{c}\,r) + C_6 K_1(\widehat{c}\,r)\right\}, \tag{3.11}$$

where C_5 and C_6 are integration constants, I_1 is the modified Bessel function of the first kind and the first order, and K_1 is the modified Beessel function of the first kind (the MacDonald function) and the first order.

The boundary condition at infinity can be fulfilled only by the function given by Eq. (3.9) with $C_1 = 1$. The boundary condition at $r = 0$ can be satisfied only by the functions given by Eqs. (3.10) and (3.11) with $C_4 = 0 = C_6$. The matching conditions at $r = 1$ can be satisfied only by the pair: f_0 as an external solution (outside the cylinder) and f_- as an internal one (inside the cylinder). Thus, we obtain:

$$\begin{aligned} f^{\text{ext}} &= 1 - 2\frac{a}{r^2}, \qquad && r > 1, \\ f^{\text{int}} &= b\frac{J_1(c\,r)}{r}, \qquad && r < 1, \end{aligned} \tag{3.12}$$

where the superscripts ext and int refer to the external and to the internal region of the cylinder, respectively, and a, b, c stand for constants (which have to be determined from the matching conditions at $r = 1$).

Thus, the velocity and pressure fields outside and inside the cylinder, which satisfy the boundary conditions, may be written in the form:

$$
\begin{aligned}
r > 1: \qquad v_r^{\text{ext}} &= \left(1 - \frac{2a}{r^2}\right)\cos\varphi,\\
v_\varphi^{\text{ext}} &= -\left(1 + \frac{2a}{r^2}\right)\sin\varphi,\\
\Delta p^{\text{ext}} &= -\frac{4a}{r^2}\left(1 + \frac{a}{r^2}\right) + \frac{8a}{r^2}\cos^2\varphi,\\
r < 1: \qquad v_r^{\text{int}} &= b\frac{J_1}{r}\cos\varphi,\\
v_\varphi^{\text{int}} &= -b\left(cJ_0 - \frac{J_1}{r}\right)\sin\varphi,\\
\Delta p^{\text{int}} &= d - \varepsilon_\varrho b^2\left\{\left(cJ_0 - \frac{J_1}{r}\right)^2 + J_1^2c^2\right\}\\
&\quad + \varepsilon_\varrho b^2\left\{(J_0^2 + J_1^2)c^2 - 2\frac{J_0J_1}{r}c\right\}\cos^2\varphi,
\end{aligned}
\tag{3.13}
$$

where the abbreviation $J_n = J_n(c\,r)$, $n = 0, 1$ was used.

The constants: a, b, c, d have to be determined from the matching conditions at the cylinder surface (Eqs. $(3.2)_{5-7}$). In fact, using these conditions one may obtain, after some algebra, the following set for these constants:

$$
\begin{aligned}
a &= \frac{1}{2M}\{ch_0 - h_1 - \varepsilon_\varrho h_1\},\\
b &= \frac{2}{M},\\
&(h_0^2 + h_1^2)\varepsilon_\varrho c^2 = M^2,\\
d &= -4a(1 + a) + \varepsilon_\varrho b^2\{(h_0c - h_1)^2 + c^2h_1^2\}\\
&= 2 - 4a(1 + a) - \varepsilon_\varrho b^2h_1^2,
\end{aligned}
\tag{3.14}
$$

where

$$M = ch_0 - h_1 + \varepsilon_\varrho h_1;$$

the second formula for d may be obtained, after some algebra, from the first one using properties of Eqs. $(3.14)_{1-3}$; and, for distinguishing, the abbreviation $h_n = J_n(c)$, $n = 0, 1$ was used.

The scheme of calculations is as follows. First, the third equation is solved with respect to $\varepsilon_\varrho(c)$, and next the inverse function $c(\varepsilon_\varrho)$ is numerically calculated.

Then from the first and the second equations the quantities $a(\varepsilon_\varrho)$ and $b(\varepsilon_\varrho)$ are obtained. Finally, from the fourth equation the quantity $d(\varepsilon_\varrho)$ is calculated. In this way all the constants considered are obtained (in numerical way) as the functions of ε_ϱ:

$$
\begin{aligned}
a &= a(\varepsilon_\varrho) \cong a_0 - \alpha_1\sqrt{\varepsilon_\varrho}, && a_0 = 0.5, && \alpha_1 = 0.4773,\\
b &= b(\varepsilon_\varrho) \cong b_0 + \alpha_2\frac{1}{\sqrt{\varepsilon_\varrho}}, && b_0 = 0.3276, && \alpha_2 = 1.6405,\\
c &= c(\varepsilon_\varrho) \cong c_0 - \alpha_3\sqrt{\varepsilon_\varrho}, && c_0 = 1.8412, && \alpha_3 = 1.6141,\\
d &= d(\varepsilon_\varrho) \cong d_0 - \alpha_4\sqrt{\varepsilon_\varrho}, && d_0 = 0.0888, && \alpha_4 = 0.3639,
\end{aligned}
\tag{3.15}
$$

where the approximate relationships represent the asymptotical behaviour of these constants as $\varepsilon_\varrho \to 0$. Substituting the constants calculated into Eqs. (3.13) we obtain the final solution of the problem examined.

The asymptotical behaviour of the flow functions outside and inside the cylinder at small ε_ϱ is, according to the structure of the solution, completely determined by the asymptotical behaviour of the functions f^{ext} and f^{int}, which are given by the formulae:

$$
\begin{aligned}
f^{\text{ext}} &\cong 1 - \frac{1}{r^2}\left(1 - 2\alpha_1\sqrt{\varepsilon_\varrho}\right),\\
f^{\text{int}} &\cong -\alpha_5 J_0(c_0 r) + \frac{J_1(c_0 r)}{r}\left(\alpha_6 + \frac{\alpha_7}{\sqrt{\varepsilon_\varrho}}\right),\\
\alpha_5 &= 2.6480, \qquad \alpha_6 = 1.7658, \qquad \alpha_7 = 1.6405,
\end{aligned}
$$

where α_1 is given by Eq. $(3.15)_{1/3}$, and c_0 – by Eq. $(3.15)_{3/2}$.

3.3. Results

From the results given in the previous subsection one may obtain all the information about the flow examined. Examples of two types of such an information will be present.

The information of the first type concerns the flow fields at a given ε_ϱ. The example value $\varepsilon_\varrho = 2.5 \times 10^{-2}$ is assumed. Thus, the lower half of Fig. 1. presents the streamlines picture. Figure 2 presents the dimensionless x-coordinate of velocity:

$$v_x = v_r \cos\varphi - v_\varphi \sin\varphi$$

at the flow symmetry plane ($\varphi = 0, \pi$, respectively) as a function of dimensionless x-coordinate (as referred to the cylinder radius). Figure 3 presents the dependence of the dimensionless pressure difference Δp on the dimensionless x-coordinate at the flow symmetry plane.

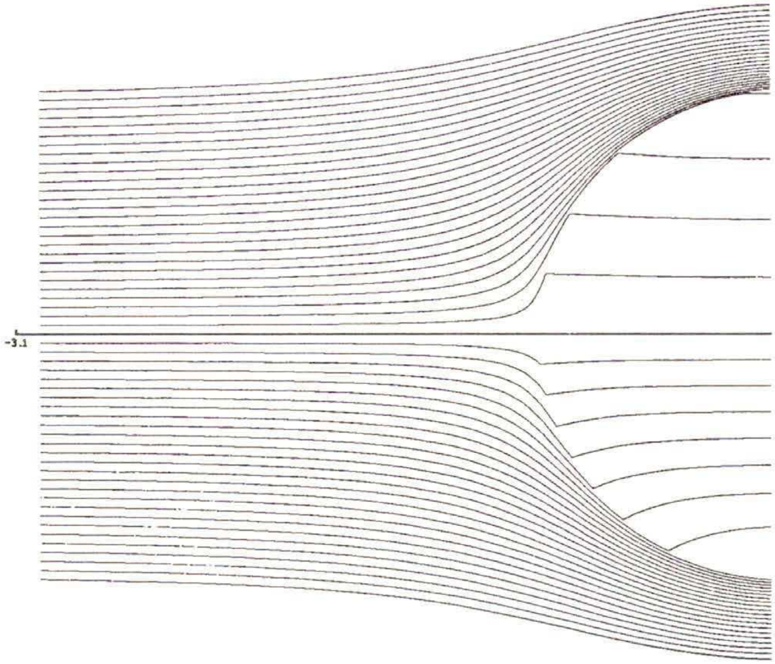

FIG. 1. Streamline pictures for the flow through the sphere in the Euler (the lower half) and Stokes (the upper half) approximations under the assumptions: $\varepsilon_\eta = \sqrt{\varepsilon_\varrho}$, $\varepsilon_\varrho = 2.5 \times 10^{-2}$.

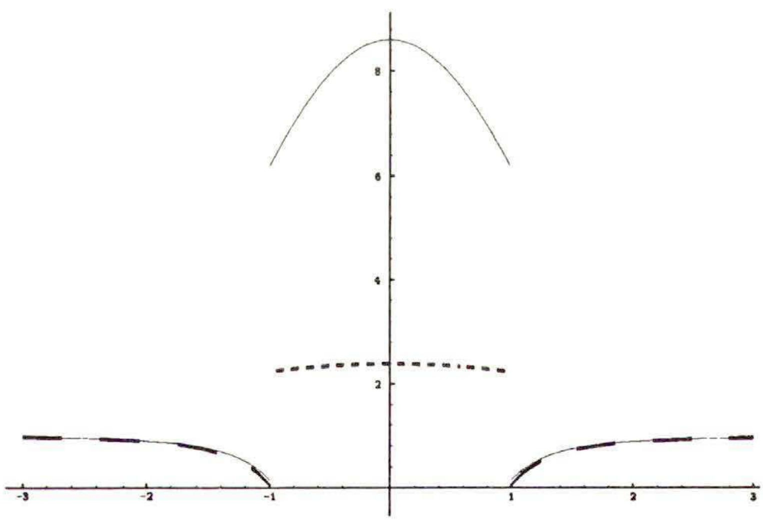

FIG. 2. Dimensionless velocity (as referred to v_∞) at the flow symmetry axis as a function of the dimensionless z-coordinate (as referred to R) under the same assumptions about ε_η and ε_ϱ as in the case of Fig. 1, in the Euler (solid line) and Stokes (dashed line) approximations.

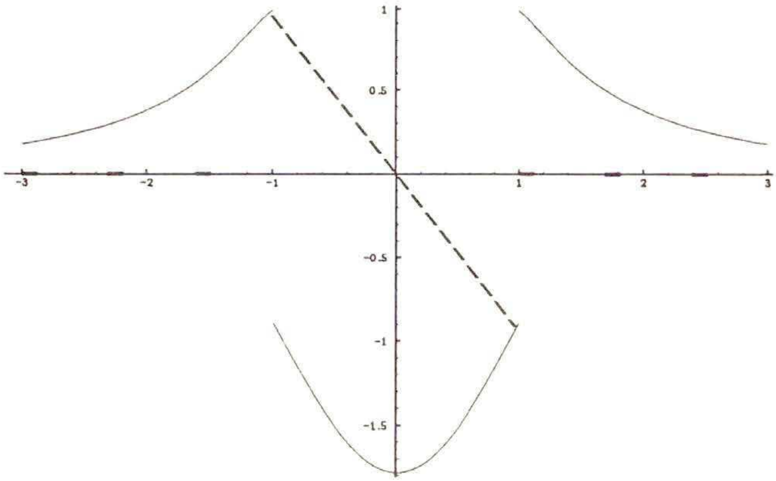

FIG. 3. Scaled relative pressure at the flow symmetry axis for $\varepsilon_\eta = \sqrt{\varepsilon_\varrho}$, $\varepsilon_\varrho = 2.5 \times 10^{-2}$;
solid line – the Euler approximation: $2(p - p_\infty)/(\varrho_\infty v_\infty^2)$,
dashed line – the Stokes approximation: $2(p - p_\infty)/(\varrho_\infty v_\infty^2)(\mathrm{Re})/(20)$.

The information of the second type concerns the characteristics of the flow considered as functions of ε_ϱ, as for example: velocity and pressure at the symmetry plane at the center and the boundary of the cylinder (Fig. 4 a, Fig. 5 a) [1]:

$$
\begin{aligned}
&v_x^{\mathrm{ext}}(1) = 1 - 2a && \cong 2\alpha_1\sqrt{\varepsilon_\varrho}, && \\
&v_x^{\mathrm{int}}(1) = bJ_1(c) && \cong \alpha_8 + \frac{\alpha_9}{\sqrt{\varepsilon_\varrho}}, && \alpha_8 = 0.1906,\quad \alpha_9 = 0.9546, \\
&v_x^{\mathrm{int}}(0) = \tfrac{1}{2}bc && \cong -\alpha_{10} + \frac{\alpha_{11}}{\sqrt{\varepsilon_\varrho}}, && \alpha_{10} = 1.0224,\quad \alpha_{11} = 1.5102, \\
&[\![v_x]\!] = 1 - 2a - bJ_1(c) && \cong -v_x^{\mathrm{int}}(1), && \\
&\Delta p^{\mathrm{ext}}(1) = 4a(1-a) && \cong 1 - \alpha_{12}\varepsilon_\varrho, && \alpha_{12} = 0.9112, \\
&\Delta p^{\mathrm{int}}(1) = d - \varepsilon_\varrho b^2 J_1^2(c) && \cong -\alpha_{13} - \alpha_{14}\sqrt{\varepsilon_\varrho}, && \alpha_{13} = 0.8223,\quad \alpha_{14} = 0.7278, \\
&\Delta p^{\mathrm{int}}(0) = d - \frac{1}{4}\varepsilon_\varrho b^2 c^2 && \cong -\alpha_{15} + \alpha_{16}\sqrt{\varepsilon_\varrho}, && \alpha_{15} = 2.1920,\quad \alpha_{16} = 2.7242, \\
&[\![\Delta p]\!] = 4a(1-a) - d + \varepsilon_\varrho b^2 J_1^2(c) && \cong \alpha_{17} + \alpha_{14}\sqrt{\varepsilon_\varrho}, && \alpha_{17} = 1.8223,
\end{aligned}
\tag{3.16}
$$

[1] Note, that the part of the gas flux flowing through the cylinder (per unit of its length) as referred to the flux incoming from infinity is given by $v_x^{\mathrm{ext}}(1)$ (if follows from an immediate calculation and application of Eqs. (3.4), $(3.2)_5$ and $(3.16)_1$).

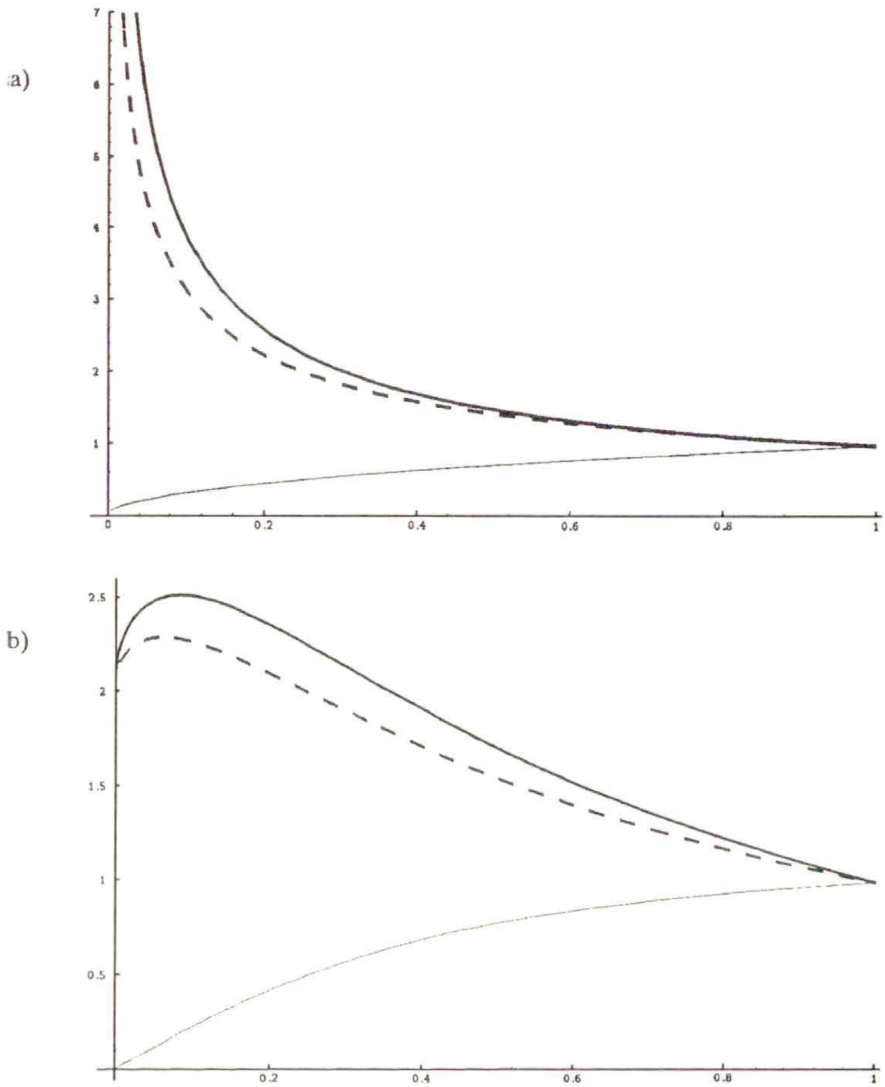

FIG. 4. Dependence of $\overline{v}_z^{\rm ext}(1)$ (solid line), $\overline{v}_z^{\rm int}(1)$ (dashed line) and $\overline{v}_z^{\rm int}(0)$ (bold line) on ε_ϱ for the flow through the cylinder in the Euler (a) and Stokes (b) approximation under the assumption: $\varepsilon_\eta = \sqrt{\varepsilon_\varrho}$.

where the first column represents the exact formulae, the second one – the asymptotic formulae for small ε_ϱ; α_1 is given by Eq. $(3.15)_{1/3}$;

$$\psi(1) := \psi(\varphi = \pi, r = 1),$$

$$\psi(0) := \psi(\varphi = \pi, r = 0),$$

and $[\![\psi]\!]$ is defined by the equation following Eqs. (3.2).

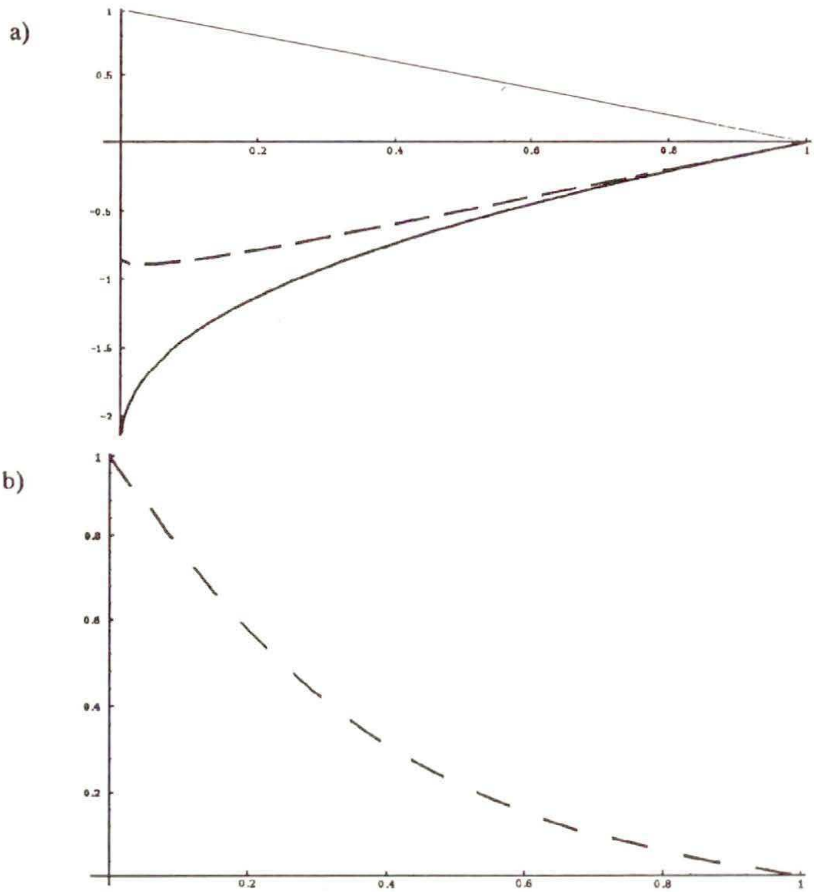

FIG. 5. Dependence of $\overline{\Delta p}^{\,\mathrm{ext}}(1)$ (solid line), $\overline{\Delta p}^{\,\mathrm{int}}(1)$ (dashed line) and $\overline{\Delta p}^{\,\mathrm{int}}(0)$ (bold line) on ε_ϱ for the flow through the cylinder in the Euler approximation (a) and $\frac{Re}{20}\overline{\Delta p}^{\,\mathrm{int}}(1)$ in the Stokes approximation (b) under the same assumptions as in the case of Fig. 3.

4. The Stokes approximation

4.1. The problem

The governing equations in this case may be written in the form (in terms of the same dimensionless variables as previously):

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}(r\,v_r)+\frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi}=0,\\
&\frac{1}{2}\,\mathrm{Re}\,\frac{1}{\eta}\,\frac{\partial \Delta p}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \varphi^2}-\frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}-\frac{v_r}{r^2},\\
&\frac{1}{2}\,\mathrm{Re}\,\frac{1}{\eta}\,\frac{1}{r}\,\frac{\partial \Delta p}{\partial \varphi}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\varphi}{r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}-\frac{v_\varphi}{r^2},
\end{aligned}
\tag{4.1}
$$

where the Reynolds number

$$\mathrm{Re} = \frac{\varrho_\infty v_\infty R}{\eta_\infty}$$

plays the role of the scale factor only, and η stands for the dimensionless shear viscosity coefficient (dimension coefficient as referred to η_∞).

The boundary conditions at $r = \infty$ and $r = 0$ are the same, as in the Euler approximation (Egs. $(3.2)_{1-4}$), the matching conditions are:

$$\text{(4.2)} \qquad \text{at } r = 1: \qquad \begin{aligned} &[\![\varrho v_r]\!] = 0, \\ &\left[\!\!\left[\frac{1}{2}\,\mathrm{Re}\,\Delta p - 2\eta \frac{\partial v_r}{\partial r} \right]\!\!\right], \\ &\left[\!\!\left[\eta \left(\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \varphi} \right) \right]\!\!\right]. \end{aligned}$$

4.2. The solution

Applying the same procedure as in the previous case, one may find the general solution of the problem outside and inside the cylinder, which satisfies the boundary conditions, namely:

$$\text{(4.3)} \qquad \begin{aligned} r > 1: \qquad v_r^{\mathrm{ext}} &= \left(1 - \frac{2\widetilde{a}}{r^2}\right)\cos\varphi, \\ v_\varphi^{\mathrm{ext}} &= -\left(1 + \frac{2\widetilde{a}}{r^2}\right)\sin\varphi, \\ \mathrm{Re}\,\Delta p^{\mathrm{ext}} &= 0, \\ r < 1: \qquad v_r^{\mathrm{int}} &= (\widetilde{b} + \widetilde{c}r^2)\cos\varphi, \\ v_\varphi^{\mathrm{int}} &= -(\widetilde{b} + 3\widetilde{c}r^2)\sin\varphi, \\ \mathrm{Re}\,\Delta p^{\mathrm{int}} &= \widetilde{d} + \frac{16\widetilde{c}}{\varepsilon_\eta} r\cos\varphi. \end{aligned}$$

The constants $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$ have to be determined from the matching conditions at the cylider surface. In fact, substituting Eqs. (4.3) into Eqs. (4.2) we obtain the following set of equations for the constants considered:

$$\text{(4.4)} \qquad \begin{aligned} 1 - 2\widetilde{a} &= \varepsilon_\varrho(\widetilde{b} + \widetilde{c}), \\ 2\widetilde{a}\varepsilon_\eta &= -c, \\ \widetilde{d} &= 0. \end{aligned}$$

It is seen that we have two equations for three constants: $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$.

Thus, in order to obtain a unique solution we should adopt an additional condition, and the continuity condition for the tangent component of velocity at the cylinder surface is assumed (²):

$$[\![v_\varphi]\!] = 0, \tag{4.5}$$

which leads to the following additional equation:

$$1 + 2\tilde{a} = \tilde{b} + 3\tilde{c}. \tag{4.6}$$

Now, solving Eqs. (4.4) and Eq. (4.6) we obtain:

$$\begin{aligned} \tilde{a} &= \frac{1}{2}\frac{1-\varepsilon_\varrho}{1+\varepsilon_\varrho(1+2\varepsilon_\eta)} \cong \frac{1}{2} - \varepsilon_\varrho, \\ \tilde{b} &= \frac{2+3\varepsilon_\eta - \varepsilon_\varrho\varepsilon_\eta}{1+\varepsilon_\varrho(1+2\varepsilon_\eta)} \cong 2 + 3\sqrt{\varepsilon_\varrho}, \\ \tilde{c} &= -\frac{(1-\varepsilon_\varrho)\varepsilon_\eta}{1+\varepsilon_\varrho(1+2\varepsilon_\eta)} \cong -\sqrt{\varepsilon_\varrho}, \\ \tilde{d} &= 0, \end{aligned} \tag{4.7}$$

where the first equation in a given line represents the exact relationship, and the second one – the asymptotical expression as $\varepsilon_\varrho \to 0$ (under the assumption $\varepsilon_\eta = \sqrt{\varepsilon_\varrho}$).

Thus, Eqs. (4.3) with Eqs. (4.7) represent the solution of the problem as expressed by Eqs. (4.1), Eqs. $(3.2)_{1-4}$ and Eqs. (4.2), which is unique in the class of functions specified by Eqs. (3.3) (and under the assumption expressed by Eq. (4.5)).

4.3. Results

Similarly to the case of the Euler approximation, two types of information, which is contained in the formulae given in the previous subsection, will be presented.

The information of the first type concerns the flow fields at a given ε_ϱ. The example value $\varepsilon_\varrho = 2.5 \times 10^{-2}$ is adopted, and ε_η as given by Eq. (2.3) is assumed. Thus, the upper half of Fig. 1 presents the streamlines picture. Figure 2 presents the dimensionless x-coordinate of velocity (see the formula given at the beginning of Subsec. 3.3.) at the flow symmetry plane ($\varphi = \pi, 0$, respectively). Figure 3 presents the dependence of the dimensionless pressure difference on the dimensionless x-coordinate at the symmetry plane.

The information of the second type concerns, as previously, the characteristics of the flow considered as functions of ε_ϱ (under the same assumption about ε_η

(²) For comments on this assumption – see [3].

as above), namely – velocity and pressure at the symmetry plane at the center and at the boundary of the cylinder, in the same convention as in the case of the Euler approximation (Eqs. (3.16)) (Fig. 4 b, Fig. 5 b) ([3]):

$$
\begin{aligned}
& v_x^{\text{ext}}(1) && = 1 - 2\tilde{a} && \cong 2\varepsilon_\varrho, \\
& v_x^{\text{int}}(1) && = \tilde{b} + \tilde{c} && \cong 2 + 2\sqrt{\varepsilon_\varrho}, \\
& v_x^{\text{int}}(0) && = \tilde{b} && \cong 2 + 3\sqrt{\varepsilon_\varrho}, \\
& [\![v_x]\!] && = 1 - 2\tilde{a} - \tilde{b} - \tilde{c} && \cong -v_x^{\text{int}}(1), \\
& \operatorname{Re} \Delta p^{\text{ext}}(1) && = 0, && \\
& \operatorname{Re} \Delta p^{\text{int}}(1) && = -16\frac{\tilde{c}}{\varepsilon_\eta} && \cong 16 - 32\varepsilon_\varrho, \\
& \operatorname{Re} \Delta p^{\text{int}}(0) && = 0, && \\
& \operatorname{Re} [\![\Delta p]\!] && = -\operatorname{Re} \Delta p^{\text{int}}(1). &&
\end{aligned}
\tag{4.8}
$$

5. Conclusions

Comparing the results obtained for the cylindrical case (in particular – the asymptotic relationships) with those for the spherical case (see [1] and [3]) one may conclude, that:

1. The velocity and the pressure fields and their dependence on ε_ϱ in the Euler approximation are very similar in both flow geometries; there occur only relatively small quantitative differences; the influence of low density region on the flow fields is, in general, greater in the case of cylinder as compared to that in the case of a sphere;

2. The same concerns the flow through the cylinder as compared to that through the sphere in the Stokes approximation;

3. The similarities and differences between the flow through the cylinder in the Euler and in the Stokes approximations are, generally, the same as in the case of flow through the sphere (see discussion in [3]).

Acknowledgments

The authors are deeply grateful to Prof. Z. PERADZYŃSKI for valuable comments.

([3]) The part of the gas flux flowing through the cylinder (per unit of its length) as referred to the flux incoming from infinity is given by $v_x^{\text{ext}}(1)$.

References

1. K.G. GUS'KOV, YU.P. RAYZER and S.T. SURZHIKOV, *On an observed speed of the optical discharge slow propagation* [in Russian], Kvantovaya Elektronika, **17**, 7, 937–942, 1990.
2. Z. PERADZYŃSKI and E. ZAWISTOWSKA, *Flow in a region with large density variations*, Arch. Mech., **46**, 6, 881–891, 1984.
3. Z. PŁOCHOCKI, B. KAŹMIERCZAK and Z. PERADZYŃSKI, *The stationary Stokes flow through a spherical region with large variations of density and viscosity coefficient*, Arch. Mech., **48**, 4, 765–778, 1996.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received March 22, 1996.