On existence theorems of periodic traveling wave solution to the generalized forced Kadomtsev–Petviashvili equation

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THIS PAPER is concerned with periodic traveling wave solutions of the generalized forced Kadomtsev-Petviashvili equation in the form $(u_t + [f(u)]_x + \alpha u_{xxx})_x + \beta u_{yy} = h_0$. The basic approach to this problem is to establish an equivalence relationship between a periodic boundary value problem and nonlinear integral equations with symmetric kernels by using the Green's function method. The integral representations generate compact operators in a Banach space of real-valued continuous periodic functions with a given period $2T$. Schauder's fixed point theorem is then used to prove the existence of nonconstant periodic traveling wave solutions.

1. Introduction

THE KORTEWEG – DE VRIES EQUATION (KdV equation for short) is a nonlinear evolution equation governing long one-dimensional, small amplitude, surface gravity waves propagating in a shallow channel of water [1]. It has many applications in the study of other physical problems, such as plasma waves, lattice waves, and waves in elastic rods, etc. A two-dimensional generalization of the KdV equation is the Kadomtsev – Petviashvili equation (referred to as KP equation henceforth), which was obtained in 1970 in the study of plasma [2]. The evolution described by the KP equation is weakly nonlinear, weakly dispersive, and weakly two-dimensional, with all the three effects being of the same order. The KP equation has also been proposed as a model for the surface waves and internal waves in channels of varying depth and width [3].

Twenty years ago, in an impressive survey on the KdV equation, MIURA listed seven open problems of the KdV equation [2]. The seventh open problem concerns the forced KdV equation. At that time the physical basis for the forced KdV equation was not clear. PATOINE and WARN were the first two who used the forced KdV equation as a physical model equation in 1982 [5]. However, it was not until 1984 that AKYLAS first systematically derived the forced KdV equation from the model of long nonlinear water wave forced by a moving pressure [6]. After that, WU [7] and SHEN [8] also derived the forced KdV equation in the study of long water waves in a two-dimensional channel forced by a bottom topography and/or an external pressure applied on the free surface. In a recent paper [9], SHEN derived the one-dimensional stationary forced KdV equation of the form $\lambda u_t + \alpha u u_x + \beta u_{xxx} = h_x$ for the long nonlinear water waves flowing over long bumps, and proved the existence of positive solitary wave solutions to the stationary forced KdV equation with the boundary value conditions $u(\pm\infty) = u'(\pm\infty) = 0$.

In this paper the author considers the generalized forced KP equation of the form

$$(1.1) \qquad (u_t + [f(u)]_x + \alpha u_{xxx})_x + \beta u_{yy} = h_0,$$

where α and β are positive numbers, and f is a C^2 function of its argument. When $f(u) = u^2/2$, $\alpha = 1$, and $\beta = 3$, Eq. (1.1) reduces to the two-dimensional forced KP equation of the form

$$(1.2) \qquad (u_t + uu_x + u_{xxx})_x + 3u_{yy} = h_0,$$

which is a two-dimensional generalization of the equation obtained by Akylas, Wu, and Shen. The author will prove an existence theorem of nonconstant periodic traveling wave solution to the generalized forced KP equation following the idea of LIU and PAO [10].

The author applies the Green's function method to derive nonlinear integral equations which are equivalent to the generalized forced KP equation with periodic boundary conditions. Imposing suitable conditions, the author establishes the existence of solutions to the integral equations, and hence proves the existence of periodic traveling wave solutions to Eq. (1.1). Furthermore, we note that the nonconstant periodic traveling wave solutions are infinitely differentiable.

The content of the paper is arranged as follows. In Sec. 2, the author converts the generalized forced KP equation into nonlinear integral equations using the Green's function method. Section 3 contains the proof of the existence theorem for these integral equations.

2. Formulation of the problem

We start from the generalized forced KP equation

$$(2.1) \qquad (u_t + [f(u)]_x + \alpha u_{xxx})_x + \beta u_{yy} = h_0,$$

where f is a C^2 function of its argument and h_0 is a nonconstant function of x, y and t. We are interested in the periodic traveling wave solutions of the form $U(z) = u(x, y, t)$, where $z = ax + by - \omega t$ with a, b, and ω being real constants. Without any loss of generality we assume $a > 0$. Consider the case that $h_0(x, y, t) = a^2 h(z)$ is a $2T$-periodic continuous function of z, where T is a preassigned positive number. Substitution of the $U(z)$ into Eq. (2.1) leads then to the fourth-order nonlinear ordinary differential equation

$$(2.2) \qquad U^{(4)}(z) = \frac{C}{\alpha a^2} U''(z) - \frac{1}{\alpha a^2} \Big[f(U(z))(U'(z))^2$$

$$+ f'(U(z))U''(z) \Big] + \frac{1}{\alpha a^2} h(z),$$

where $C = (\omega a - \beta b^2)/a^2$. We impose the following periodic boundary conditions

$$(2.3) \qquad U^{(n)}(0) = U^{(n)}(2T), \qquad n = 0, 1, 2, 3.$$

In addition, in order to rule out non-zero constant solutions, another condition is introduced

$$(2.4) \qquad \int_0^{2T} U(z)\, dz = 0.$$

Thus, any solution of the boundary value problem consisting of Eqs. (2.2)–(2.4) can be extended to a $2T$-periodic traveling wave solution to Eq. (2.1).

Integrating both sides of Eq. (2.2) with respect to z twice and using Eqs. (2.3), (2.4), we obtain

$$(2.5) \qquad U''(z) - \frac{C}{\alpha a^2} U(z) = E - \frac{1}{\alpha a^2} [f(U(z)) - H(z)],$$

$$(2.6) \qquad U^{(n)}(0) = U^{(n)}(2T), \qquad n = 0, 1,$$

where

$$E = \frac{1}{2T} \cdot \frac{1}{\alpha a^2} \int_0^{2T} [f(U(z)) - H(z)]\, dz,$$

and $H(z)$ is a $2T$-periodic function of z such that $H''(z) = h(z)$. Conversely, integrating both sides of Eq. (2.5) from 0 to $2T$ and using Eqs. (2.6) we are led to Eq. (2.4), and direct differentiations of Eq. (2.5) will give us Eqs. (2.2), (2.3). Therefore, we have proved the following theorem by noting from Eq. (2.5) that $U \in C^2[0, 2T]$ implies $U \in C^4[0, 2T]$ since f is a C^2 function of its argument.

THEOREM 1. *Suppose that* $C \neq 0$; *a function* $U(z)$ *is a solution of the boundary value problem* Eqs. (2.2)–(2.4) *if and only if it is a solution of the boundary value problem* Eqs. (2.5) *and* (2.6).

From now on we consider only the two cases: 1. $C > 0$, and 2. $C < 0$ but $-C/(\alpha a^2) \neq (k\pi/T)^2$ with k being any integer.

Denote the function $f(U(z)) - H(z)$ on the right-hand side of Eq. (2.5) by $F(U(z))$. Treating the right-hand side of Eq. (2.5) as a forcing term and using the Green's function method [11], the boundary value problem Eqs. (2.5), (2.6) can be converted to an integral equation

$$(2.7) \qquad U(z) = \frac{1}{\alpha a^2} \int_0^{2T} K_i(z, s) F(U(s))\, ds,$$

where the kernels K_i, $i = 1, 2$, are defined as follows:

1. When $C > 0$, let $\lambda_1 = \sqrt{C/(\alpha a^2)}$; then

$$(2.8) \qquad K_1(z, s) = \frac{\cosh \lambda_1(T - |z - s|)}{2\lambda_1 \sinh \lambda_1 T} - \frac{1}{2\lambda_1^2 T}, \qquad \forall z, s \in [0, 2T].$$

2. When $C < 0$ but $-C/(\alpha a^2) \neq (k\pi/T)^2$ with k being any integer, let $\lambda_2 = \sqrt{-C/(\alpha a^2)}$; then

$$(2.9) \qquad K_2(z, s) = \frac{\cos \lambda_2(T - |z - s|)}{2\lambda_2 \sin \lambda_2 T} - \frac{1}{2\lambda_2^2 T}, \qquad \forall z, s \in [0, 2T].$$

LEMMA 1. The kernels K_1 and K_2 have the following properties:

$$K_i(0, s) = K_i(2T, s), \qquad \forall s \in [0, 2T], \qquad i = 1, 2,$$
$$K_i(z, 2T - s) = K_i(2T - z, s), \qquad \forall s \in [0, 2T], \qquad i = 1, 2.$$

P r o o f. Straightforward computations follow from the definitions of the kernels K_i, $i = 1, 2$, given in Eqs. (2.8), (2.9). $\square$

THEOREM 2. *A function* $U(z)$ *is a solution of the boundary value problem* Eqs. (2.5), (2.6) *if and only if it is a solution of the integral equation* (2.7).

P r o o f. The "if" part can be proved by direct differentiations of Eq. (2.7) and the "only if" part is based on the Green's function method by treating the right-hand side of Eq. (2.5) as a nonhomogeneous term. $\square$

3. Existence theorem

To show the existence of $2T$-periodic traveling wave solutions to Eq. (2.1) it is suffucient to show that solutions to the Eq. (2.7) exist.

To this end we define C_{2T} as a collection of real-valued continuous functions, $v(z)$, on $[0, 2T]$ such that $v(0) = v(2T)$. Equip C_{2T} with the sup norm $\|\cdot\|$ as $\|v\| = \sup_{0 \leq z \leq 2T} |v(z)|$, for each $v \in C_{2T}$. Then $(C_{2T}, \|\cdot\|)$ is a Banach space.

We now define operators $\mathcal{A}_i$, $i = 1, 2$, on C_{2T} as

$$(3.1) \qquad \mathcal{A}_i v(z) = \frac{1}{\alpha a^2} \int_0^{2T} K_i(z, s) F(v(s)) \, ds, \qquad \forall v \in C_{2T},$$

where the kernels K_i, $i = 1, 2$, are given in Eqs. (2.8), (2.9). We shall demonstrate that there exist functions v in C_{2T} such that $v = \mathcal{A}_i v$, $i = 1, 2$, and hence, prove that there exist solutions to Eq. (2.7).

Let

$$(3.2) \qquad Q_i \geq \max_{0 \leq z \leq 2T} \int_0^{2T} |K_i(z,s)| \, ds, \qquad i = 1, 2,$$

$$(3.3) \qquad \tau_1 = 1, \qquad \tau_2 = |\sin \lambda_2 T|.$$

A consequence of Lemma 1 can now be stated.

LEMMA 2. Let v be an element of C_{2T}. If $v(z) = v(2T - z)$ for $z \in [0, 2T]$, then $\mathcal{A}_i v(z) = \mathcal{A}_i v(2T - z)$, $i = 1, 2$.

We now define $B(0, r)$ to be a closed ball in C_{2T} and let $M = \sup[\| F(v)\| : v \in B(0, r)]$. We then have the following existence theorem.

THEOREM 3. $\mathcal{A}_i$, $i = 1, 2$, is a compact operator from C_{2T} into C_{2T}. In particular, if $Q_i M / (\alpha a^2) \leq r$, $i = 1, 2$, then $\mathcal{A}_i$ maps $B(0, r)$ into itself. Hence, the integral equation (2.7) has at least one solution in $B(0, r)$.

P r o o f. First we show $\mathcal{A}_i : C_{2T} \to C_{2T}$, $i = 1, 2$. Since it is obvious from Lemma 1 that $\mathcal{A}_i v(0) = \mathcal{A}_i v(2T)$ for each $v \in C_{2T}$, $i = 1, 2$, it suffices to show that $\mathcal{A}_i v$, $i = 1, 2$, is continuous on $[0, 2T]$.

Let v be an arbitrary function in C_{2T}; we have then

$$(3.4) \qquad \frac{d\mathcal{A}_1 v(z)}{dz} = \frac{-1}{2\alpha a^2 \sinh \lambda_1 T} \int_0^z \sinh \lambda_1 (T - z + s) F(v(s)) \, ds$$

$$+ \frac{1}{2\alpha a^2 \sinh \lambda_1 T} \int_z^{2T} \sinh \lambda_1 (T + z - s) F(v(s)) \, ds,$$

$$(3.5) \qquad \frac{d\mathcal{A}_2 v(z)}{dz} = \frac{1}{2\alpha a^2 \sin \lambda_2 T} \int_0^z \sin \lambda_2 (T - z + s) F(v(s)) \, ds$$

$$+ \frac{-1}{2\alpha a^2 \sin \lambda_2 T} \int_z^{2T} \sin \lambda_2 (T + z - s) F(v(s)) \, ds.$$

The existence of $d\mathcal{A}_1 v / dz$ and $d\mathcal{A}_2 v / dz$ implies that both $\mathcal{A}_1 v$ and $\mathcal{A}_2 v$ are continuous on $[0, 2T]$, and hence, $\mathcal{A}_i : C_{2T} \to C_{2T}$, $i = 1, 2$.

Let **S** be any bounded subset of C_{2T}, i.e., there exists an $L_0 > 0$ such that $\|v\| < L_0$ for all $v \in$ **S**. Then there must be an $M_0 > 0$ such that

$$\| F(v)\| = \sup_{0 \leq z \leq 2T} |F(v(z))| \leq \sup_{-L_0 \leq w \leq L_0} |F(w)| \leq M_0, \qquad \forall v \in \mathbf{S}.$$

Thus from Eqs. (3.1), (3.4), (3.5) we shall have

$$\|\mathcal{A}_i v\| \leq \frac{1}{\alpha a^2} Q_i M_0, \qquad \forall v \in \mathbf{S}, \quad i = 1, 2,$$

$$\|d\mathcal{A}_i v / dz\| \leq \frac{T}{\alpha a^2 \tau_i} M_0, \qquad \forall v \in \mathbf{S}, \quad i = 1, 2.$$

Therefore, $\mathcal{A}_i \mathbf{S}$, $i = 1, 2$, is uniformly bounded and equi-continuous, and by the Ascoli–Arzela Theorem both $\mathcal{A}_1$ and $\mathcal{A}_2$ are compact.

To show that $\mathcal{A}_i$, $i = 1, 2$, has a fixed point in $B(0, r)$ when $Q_i M / (\alpha a^2) \leq r$, $i = 1, 2$, we write

$$|\mathcal{A}_i v(z)| = \frac{1}{\alpha a^2} \left| \int_0^{2T} K_i(z, s) F(v(s)) \, ds \right|$$

$$\leq \frac{1}{\alpha a^2} \int_0^{2T} |K_i(z, s)| \, |F(v(s))| ds$$

$$\leq \frac{Q_i M}{\alpha a^2} \leq r, \qquad \forall v \in B(0, r).$$

This implies that $\|\mathcal{A}_i v\| \leq r$ for all $v \in B(0, r)$, $i = 1, 2$, and hence, $\mathcal{A}_i$, $i = 1, 2$, maps $B(0, r)$ into itself. Therefore, by the Schauder's fixed point theorem we proved that $\mathcal{A}_i$ has a fixed point in $B(0, r)$ for each $i = 1, 2$. And hence, Eq. (2.7) has a solution for each case of $C > 0$ and $C < 0$ with $-C/(\alpha a^2) \neq (k\pi/T)^2$. □

It is worth noting that as long as $\int_0^{2T} K_i(z, s) H(s) \, ds \neq 0$, $i = 1, 2$, by Theorem 3, there exists a nonconstant function $v(z)$ on $[0, 2T]$ such that $v = \mathcal{A}_i v$, $i = 1, 2$, which implies that $v(z)$ is infinitely differentiable on $[0, 2T]$ since $\mathcal{A}_i v$ is differentiable on $[0, 2T]$. The extension of the $v(z)$ to a $2T$-periodic function $V(z)$ provides an infinitely differentiable $2T$-periodic traveling wave solution to the generalized forced KP equation.

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