Arch. Mech., 48, 5, pp. 911–923, Warszawa 1996

Constitutive relations and internal equilibrium condition for fluid-saturated porous solids Linear description

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USING THE NONLINEAR THEORY established in the paper [5], the constitutive relations for small deformations of the fluid-saturated porous solid are derived. It is assumed that the elastic properties of porous skeleton are non-isotropic while the skeleton pore structure is isotropic. Fluid filling pores is assumed to be barotropic. Such approach made it possible to construct the consistent linear description of elastic behaviour of porous medium in which all material constants are precisely defined and represent mechanical properties of individual constituents. It is shown that the pure elastic properties of fluid-filled anisotropic skeleton are characterized by 36 material constants and reduces to 7 constants for the isotropic case, and to 4 constants when the skeleton is isotropic and its material is incompressible. In each considered case, the only one material constant characterizes mechanical properties of the pore fluid whereas the remaining constants characterize elastic properties of porous skeleton.

1. Introduction

The purpose of this paper is to formulate the linear constitutive theory for fluid-saturated porous elastic solid using as a starting point the results of nonlinear theory established in [5], where the special attention was paid to the consequences of the constituent immiscibility in such a medium.

The elastic properties of porous skeleton are assumed to be anisotropic while its pore structure is isotropic and is described by two scalar parameters: volume porosity f_v and the structural permeability λ (or, equivalently, by parameter $\kappa = \lambda / f_v$), [10].

The comprehensive constitutive macro-description of mechanical behaviour of fluid-saturated porous solids during a deformation process – also within the linear theory – should include all characteristic features resulting from the fact of immiscibility of physical constituents. Therefore, in the case of a solid-fluid elastic system, apart from the constitutive relations for the skeleton stresses and the pore fluid pressure, the additional relations for pore structure parameters and effective skeleton mass density changes must be established. The formulation of such relations should provide clear physically motivated interpretation of interactions between the porous skeleton and pore fluid and to give precisely defined material constants with clear physical interpretation.

In the commonly used linear theory of fluid-saturated porous solids developed by BIOT [1–3], the problem of changes of pore structure parameters and skeleton mass density do not appear. The Biot constitutive relations derived from the internal energy function postulated for the whole aggregate does not provide simple interpretation of mechanical couplings between constituents, and the corresponding material constants characterizing these couplings are complex [3, 7, 8]. The above difficulties are unfortunately not overcome in works in which the linear constitutive relations are obtained from their nonlinear form formulated with the use of the principle of equipresence (see e.g. [4, 6, 9, 12]).

In our analysis of the porous solid deformation process the notions of the external (bulk) deformation defined by the right Cauchy-Green deformation tensor **C** (the infinitesimal strain tensor **E** in the linear case) and of the internal deformation measured by the change of the effective skeleton mass density ϱ^s (or, equivalently, volume porosity f_v) are used. Such approach enables one to obtain the linear constitutive description of elastic solid-fluid composition in which the mechanical coupling between the deformable skeleton and pore fluid appearing in constitutive relations, and the corresponding material constants are well defined and have clear physical meaning.

In the paper, it is shown that the elastic properties of fluid-filled anisotropic skeleton with isotropic pore structure are characterized by 36 material constants and reduces to 7 constants for the isotropic case and to 4 constants when the skeleton is isotropic and its material is incompressible. It should be pointed out that in each considered case, only one material constant characterizes the mechanical properties of the pore fluid whereas the remaining constants characterize elastic properties of the porous skeleton.

2. Initial set of constitutive relations for an elastic porous skeleton filled with barotropic fluid

The starting point for our considerations is the macroscopic nonlinear constitutive description of an elastic porous skeleton filled with barotropic fluid, formulated in the former paper [5]. It is assumed that the skeleton pore structure is isotropic and characterized by two scalar parameters: the volume porosity f_v and structural permeability λ (or, equivalently, parameter $\kappa = \lambda / f_v$). From different forms of the constitutive relations derived for the elastic porous skeleton in this discussion we use that one in which the independent variables are the effective mass density ϱ^s and the right Cauchy-Green deformation tensor

$$\mathbf{C} = \mathbf{F}^T \mathbf{F},$$

where **F** is the porous solid deformation gradient and the superscript T stands for transposition of the tensor.

In such a case the complete set of constitutive equations comprises:

• the constitutive stress-strain relation for the porous skeleton

$$\mathbf{T}^{*s} = -p^f \mathbf{I} + 2\varrho^s \mathbf{F} \frac{\partial \tilde{\varepsilon}^s}{\partial \mathbf{C}} \mathbf{F}^T; \tag{2.1}$$

• the internal, mechanical equilibrium condition for the porous solid-fluid aggregate

$$\frac{p^f}{(\varrho^s)^2} = \frac{\partial \tilde{e}^s}{\partial \varrho^s}; \tag{2.2}$$

• the equation for the κ-parameter variation

$$\kappa = \tilde{\kappa}(\mathbf{C}, \varrho^s); \tag{2.3}$$

• the constitutive relation for the barotropic fluid

$$\frac{p^f}{(\varrho^f)^2} = \frac{\partial \tilde{e}^f}{\partial \varrho^f}. \tag{2.4}$$

In the above equations, the constitutive relations

$$e^s = \tilde{e}^s(\mathbf{C}, \varrho^s), \qquad e^f = \tilde{e}^f(\varrho^f)$$

represent the internal energies of the porous skeleton and fluid, respectively, and $\mathbf{T}^{*s}$ is the effective Cauchy stress tensor related to the partial stress tensor $\mathbf{T}^s$ by expression

$$\mathbf{T}^s = (1 - f_v)\mathbf{T}^{*s}.$$

The quantities p^f and ϱ^f stand for the fluid pore pressure and its mass density, respectively.

The derivative in (2.1) is defined by the identity, [11]

$$\frac{\partial \tilde{e}^s}{\partial \mathbf{C}} \cdot \mathbf{D} = \frac{\partial}{\partial h}\tilde{e}^s(\mathbf{C} + h\mathbf{D}, \varrho^s)\Big|_{h=0}, \tag{2.5}$$

where $\mathbf{D}$ is an arbitrary second order symmetric tensor.

Equations (2.1)–(2.4) have been derived from the internal energy balance equation of porous solid-fluid aggregate which was required to be identically satisfied by the independent internal energy functions postulated for the physical constituents and an arbitrary nondissipative mechanical process. Such approach takes into account the fact of immiscibility of the physical components that provides preservation of their individual physical properties during a deformation process.

The constitutive functions in Eqs. (2.1)–(2.3) are related to the elastic properties of the porous skeleton and do not depend explicitly on the volume porosity f_v. It reduces the number of quantities appearing in these equations simplifying their forms. Therefore the internal equilibrium condition (2.2), that relates the quantities p^f, $\mathbf{C}$ and ϱ^s, may be considered as the equation describing variations of the skeleton mass density ϱ^s during a deformation process. Consequently, variations of the volume porosity parameter f_v are defined by the continuity equation for the skeleton

$$(1 - f_v)\varrho^s \det(\mathbf{F}) = (1 - f_v^0)\varrho_0^s , \tag{2.6}$$

where quantities f_v^0 and ϱ_0^s are the values of f_v and ϱ^s, respectively, in the reference configuration.

All the three quantities: ϱ^s, f_v and κ can not be controlled directly by the boundary conditions and in this sense they play the role of internal parameters.

3. Linear constitutive relations for elastic fluid-filled porous medium

We are interested in the linear constitutive description of elastic porous solid filled with barotropic fluid undergoing small deformations. We consider deformations around the equilibrium state of the medium that is assumed to be its reference configuration. The linear constitutive relations are derived by linearization of the general nonlinear equations (2.1) – (2.4).

Since the fluid does not have the natural stress-free states, both physical constituents (fluid and porous solid) are in some initial stress state (in any arbitrary reference configuration). Assuming that the medium in the reference configuration is homogeneous, its initial state will be characterized by the following set of quantities:

$$\mathbf{T}_0^{*s}, \quad \varrho_0^s, \quad f_v^0, \quad \kappa_0, \quad p_0^f, \quad \varrho_0^f,$$

the values of which, due to (2.1) – (2.4), are related to each other by

$$\mathbf{T}_0^{*s} = -p_0^f \mathbf{I} + 2\varrho_0^s \frac{\partial \widetilde{e}^{\,s}}{\partial \mathbf{C}}\bigg|^0, \tag{3.1}$$

$$\frac{p_0^f}{(\varrho_0^s)^2} = \frac{\partial \widetilde{e}^{\,s}}{\partial \varrho^s}\bigg|^0, \tag{3.2}$$

$$\kappa = \widetilde{\kappa}|^0, \tag{3.3}$$

$$\frac{p_0^f}{(\varrho_0^f)^2} = \frac{\partial \widetilde{e}^{\,f}}{\partial \varrho^f}\bigg|^0, \tag{3.4}$$

where

$$\alpha|^0 = \alpha(\mathbf{C}_0, \varrho_0^s), \qquad \frac{\partial \widetilde{e}^{\,f}}{\partial \varrho^f}\bigg|^0 = \frac{\partial \widetilde{e}^{\,f}}{\partial \varrho^f}(\varrho_0^f)$$

for

$$\alpha = \frac{\partial \widetilde{e}^{\,s}}{\partial \mathbf{C}}, \quad \frac{\partial \widetilde{e}^{\,s}}{\partial \varrho^s}, \quad \widetilde{\kappa}$$

and

$$\mathbf{C}_0 = \mathbf{F}_0^T \mathbf{F}_0 = \mathbf{I}.$$

For further discussion we introduce the solid displacement gradient $\mathbf{H}$

$$\mathbf{H} = \mathbf{F} - \mathbf{I} \tag{3.5}$$

and the Lagrange strain tensor $\mathbf{E}$

$$2\mathbf{E} = \mathbf{C} - \mathbf{I}$$

that are linked by the geometrical relation

$$2\mathbf{E} = \mathbf{H} + \mathbf{H} + \mathbf{H}^T\mathbf{H}. \tag{3.6}$$

Then, at small values of the displacement gradient $\mathbf{H}$, from (3.6) we obtain

$$\mathbf{E} \simeq (\mathbf{H} + \mathbf{H})/2 = \widetilde{\mathbf{E}}$$

and the right Cauchy-Green deformation tensor $\mathbf{C}$ can be expressed as follows

$$\mathbf{C} \simeq \mathbf{I} + 2\widetilde{\mathbf{E}}, \tag{3.7}$$

where $\widetilde{\mathbf{E}}$ is the infinitesimal strain tensor of the skeleton. The quantity $2\widetilde{\mathbf{E}}$ is the linear increment of the deformation tensor $\mathbf{C}$.

To obtain linear constitutive relations from (2.1)–(2.4) we introduce the incremental form of quantities $\mathbf{T}^{*s}$, p^f, ϱ^s, ϱ^f, and κ

$$\begin{aligned} \mathbf{T}^{*s} &= \mathbf{T}_0^{*s} + \Delta\mathbf{T}^{*s}, \qquad & p^f &= p_0^f + \Delta p^f, \\ \varrho^s &= \varrho_0^s + \Delta\varrho^s, \qquad & \varrho^f &= \varrho_0^f + \Delta\varrho^f, \\ & & \kappa = \kappa_0 + \Delta\kappa. & \end{aligned} \tag{3.8}$$

Then, using expressions (3.5), (3.7) and (3.8) in the constitutive relation (2.1), after expansion of the internal energy function we can write the effective stresses in the skeleton as follows

$$\begin{aligned} \mathbf{T}_0^{*s} + \Delta\mathbf{T}^{*s} &= -(p_0^f + \Delta p^f)\mathbf{I} \\ &+ 2(\varrho_0^s + \Delta\varrho^s)(\mathbf{H} + \mathbf{I}) \left[\frac{\partial \widetilde{e}^{\,s}}{\partial \mathbf{C}}\bigg|^0 + 2\frac{\partial^2 \widetilde{e}^{\,s}}{\partial \mathbf{C}^2}\bigg|^0 \cdot \widetilde{\mathbf{E}} + \frac{\partial^2 \widetilde{e}^{\,s}}{\partial \varrho^s \partial \mathbf{C}}\bigg|^0 \Delta\varrho^s + \ldots \right] (\mathbf{H} + \mathbf{I})^T. \end{aligned} \tag{3.9}$$

The above relation, when the condition (3.1) is taken into account and all the nonlinear terms are neglected, assumes the form

$$\begin{aligned} \Delta\mathbf{T}^{*s} + \Delta p^f\mathbf{I} &= \mathbb{C}^* \cdot \widetilde{\mathbf{E}} + \mathbb{K}^* \frac{\Delta\varrho^s}{\varrho_0^s} \\ &+ (\mathbf{T}_0^{*s} + p_0^f\mathbf{I})\frac{\Delta\varrho^s}{\varrho_0^s} + 2p_0^f\widetilde{\mathbf{E}} + \mathbf{H}\,\mathbf{T}_0^{*s} + \mathbf{T}_0^{*s}\mathbf{H}^T, \end{aligned} \tag{3.10}$$

where quantities

$$\mathbb{C}^* = 4\varrho_0^s \frac{\partial^2 \widetilde{e}^s}{\partial \mathbf{C}^2}\bigg|^0, \qquad \mathbb{K}^* = 2(\varrho_0^s)^2 \frac{\partial^2 \widetilde{e}^s}{\partial \varrho^s \partial \mathbf{C}}\bigg|^0 \tag{3.11}$$

are the effective material constants of the porous skeleton which have tensorial character; $\mathbb{C}^*$ is the fourth order tensor and $\mathbb{K}^*$ is the second order tensor.

In a similar way we can obtain the linear form of Eqs. (2.2) and (2.3). They are

$$\Delta p^f = \mathbb{K}^* \cdot \widetilde{\mathbf{E}} + (K_c^* + 2p_0^f)\frac{\Delta \varrho^s}{\varrho_0^s}, \tag{3.12}$$

$$\Delta \kappa = \nu_c^* \frac{\Delta \varrho^s}{\varrho_0^s} + \mathbb{P}^* \cdot \widetilde{\mathbf{E}}, \tag{3.13}$$

where

$$K_c^* = (\varrho_0^s)^2 \frac{\partial^2 \widetilde{e}^s}{\partial (\varrho^s)^2}\bigg|^0, \tag{3.14}$$

$$\nu_c^* = \varrho_0^s \frac{\partial \widetilde{\kappa}}{\partial \varrho^s}, \qquad \mathbb{P}^* = \frac{\partial \widetilde{\kappa}}{\partial \mathbf{C}}. \tag{3.15}$$

In derivation of (3.12) the commutative law of differentiation

$$\mathbb{K}^* = 2(\varrho_0^s)^2 \frac{\partial^2 \widetilde{e}^s}{\partial \varrho^s \partial \mathbf{C}}\bigg|^0 = 2(\varrho_0^s)^2 \frac{\partial^2 \widetilde{e}^s}{\partial \mathbf{C} \partial \varrho^s}\bigg|^0 \tag{3.16}$$

was used.

Equations (3.10), (3.12) and (3.13) are the set of linear constitutive relations for fluid-saturated porous solid of an anisotropic elastic properties and the isotropic pore structure. From the definitions (3.11), (3.14) and (3.15) it is seen that the material constants $\mathbb{C}^*$, $\mathbb{K}^*$, K_c^*, ν_c^* and $\mathbb{P}^*$ characterize the mechanical properties of porous skeleton only and depend on the chosen initial state of the porous solid. The fourth order tensor $\mathbb{C}^*$ in (3.10) is the tensor of elastic constants for the porous skeleton undergoing small external deformations at constant effective skeleton mass density ϱ^s. The material constant K_c^* in (3.12) represents the volumetric modulus of elasticity of the skeleton material corresponding to the pure internal deformation caused by the change of the pore pressure p^f at constant deformation tensor $\mathbf{C}$ ($\widetilde{\mathbf{E}} = 0$, external deformation does not exist).

The second order tensor $\mathbb{K}^*$, as it is seen from the definition (3.16) and relations (3.10) and (3.12), is the tensor characterizing the coupling between two independent kinds of deformations measured by tensor $\mathbf{C}$ and the increment of ϱ^s. Due to the symmetry of tensor $\mathbf{C}$, the tensor $\mathbb{K}^*$ is also symmetric.

Combination of the tensor $\mathbb{K}^*$ with the constant K_c^* in the form

$$\mathbb{V}_p = \mathbb{K}^*/(K_c^* + 2p_0^f) \tag{3.17}$$

characterizes the volume changes of the skeleton material caused by external deformation of the porous solid at constant pore pressure ($p^f = p_0^f$). In such a case from (3.12) we have

$$\frac{\Delta\varrho^s}{\varrho_0^s} = -\mathbb{V}_p \cdot \widetilde{\mathrm{E}}. \tag{3.18}$$

On the other hand, in the case when the external deformation does not exist ($\widetilde{\mathrm{E}} = 0$), from equations (3.10) and (3.12) we obtain the relation

$$\Delta\mathbf{T}^{*s} = -\Delta p^f \mathbf{I} + \mathbb{V}_E \Delta p^f, \tag{3.19}$$

where the tensor $\mathbb{V}_E$ is expressed by tensor $\mathbb{K}^*$and quantities K_c^*, $\mathbf{T}_0^{*s}$ in the following way

$$\mathbb{V}_E = \mathbb{V}_p + (\mathbf{T}_0^{*s} + p_0^f\mathbf{I})/(K_c^* + 2p_0^f). \tag{3.20}$$

It characterizes the change of the skeleton stresses resulting from the internal solid deformation caused by the change of the fluid pore pressure.

Coefficients appearing in equation (3.13) describe the changes of the pore structure parameter κ during the deformation process. The scalar coefficient ν_c^* defined by $(3.15)_1$ characterizes the change of κ as a result of the change of the effective skeleton density at the constant deformation tensor $\mathbf{C}$, whereas the coefficient $\mathbb{P}^*$ is the symmetric second order tensor characterizing the changes of κ caused by the external skeleton deformation defined by $\widetilde{\mathrm{E}}$ at constant effective density ϱ^s.

The changes of the second pore parameter, i.e. the volume porosity, are characterized by the skeleton mass continuity equation (2.6). Its linear form is

$$\frac{\Delta f_v}{1 - f_v^0} = \frac{\Delta\varrho^s}{\varrho_0^s} + \operatorname{tr}(\widetilde{\mathrm{E}}). \tag{3.21}$$

To complete the linear constitutive description of an elastic porous solid-fluid composition it is necessery to linearize the constitutive relation (2.4) for fluid. We have

$$\Delta p^f = a_0^2 \Delta\varrho^f, \tag{3.22}$$

where

$$a_0 = \left(2\varrho_0^f \frac{d\widetilde{e}^f}{d\varrho^f}\bigg|^0 + (\varrho_0^f)^2 \frac{d^2\widetilde{e}^f}{d(\varrho^f)^2}\bigg|^0 \right)^{1/2} \tag{3.23}$$

is the velocity of the wave-front propagation in a bulk fluid.

Constitutive stress-strain relations of the porous solid (3.10), the internal equilibrium equation (3.12), the constitutive relation for the barotropic fluid (3.22) and the equation of changes of the pore parameter κ (3.13) form the complete set of the linear constitutive equations for the elastic fluid-saturated porous solid of an anisotropic mechanical properties and isotropic pore structure.

These equations contain six material constants. Three of them are scalar coefficients (K_c^*, a_0, ν_c^*) and three other are tensorial coefficients of the fourth order ($\mathbb{C}^*$) and second order ($\mathbb{K}^*, \mathbb{P}^*$).

It should be mentiond that in the above description the velocity a_0 or, equivalently, the fluid volume compressibility K^f

$$K^f = \varrho_0 a_0^2 \tag{3.24}$$

is the only material constant characterizing the fluid properties while the remaining parameters characterize the skeleton properties.

Regarding the symmetry of tensorial coefficients we have, in general, $3 + 21 + 6 + 6 = 36$ scalar quantities that have to be determined experimentally. Moreover, in solving any mathematical problem it is necessary to know the quantities $\mathbf{T}_0^{*s}$, ϱ_0^s, κ, f_v^0 and p_0^f characterizing the state of saturated solid in its reference configuration.

4. Linear constitutive relations. Special cases

The obtained constitutive equations of an anisotropic fluid-saturated porous solid are a good basis for derivation of constitutive relations for elastic porous solids with high symmetry of mechanical properties or reduced physical properties. In this section we consider elastic behaviour of the fluid-porous solid composition with skeleton of isotropic mechanical properties, the case when the skeleton material is incompressible and the case when the porous medium is unsaturated. Constitutive relations for porous materials of such reduced properties have simple form and are important in practical applications.

4.1. Porous medium with isotropic skeleton

The constitutive relations (3.10), (3.12), (3.13) and (3.22) will describe the isotropic properties of a porous body if their form is invariant under any orthogonal transformation of the dependent and independent variables

$$\left\{\Delta p^f,\ \ \Delta\varrho^f,\ \ \Delta\varrho^s,\ \ \Delta\kappa,\ \ \Delta\mathbf{T}^{*s},\ \ \widetilde{\mathbf{E}},\ \ \mathbf{H}\right\}.$$

Taking the orthogonal transformations of these variables, i.e.

$$\begin{gathered}'\Delta p^f = \Delta p^f, \qquad '\Delta\varrho^f = \Delta\varrho^f, \qquad '\Delta\varrho^s = \Delta\varrho^s, \qquad '\Delta\kappa = \Delta\kappa,\\ '\Delta\mathbf{T}^{*s} = \mathbf{Q}\,\Delta\mathbf{T}^{*s}\mathbf{Q}^T, \qquad '\widetilde{\mathbf{E}} = \mathbf{Q}\widetilde{\mathbf{E}}\,\mathbf{Q}^T, \qquad '\mathbf{H} = \mathbf{Q}\,\mathbf{H}\,\mathbf{Q}^T,\end{gathered} \tag{4.1}$$

the constitutive relations (3.10), (3.12) and (3.13) assume the form

$$\mathbf{Q}\,\Delta\mathbf{T}^{*s}\mathbf{Q}^T + \Delta p^f\mathbf{I} = \mathbb{C}^*\cdot(\mathbf{Q}\,\widetilde{\mathbf{E}}\,\mathbf{Q}^T) + \mathbb{K}^*\frac{\Delta\varrho^s}{\varrho_0^s} + (\mathbf{T}_0^{*s} + p_0^f\mathbf{I})\frac{\Delta\varrho^s}{\varrho_0^s} + 2p_0^f(\mathbf{Q}\,\widetilde{\mathbf{E}}\,\mathbf{Q}^T) + (\mathbf{Q}\,\mathbf{H}\,\mathbf{Q}^T)\mathbf{T}_0^{*s} + \mathbf{T}_0^{*s}(\mathbf{Q}\,\mathbf{H}\,Q^T)^T, \tag{4.2}$$

$$p^f = \mathbb{K}^*\cdot(\mathbf{Q}\,\widetilde{\mathbf{E}}\,\mathbf{Q}^T) + K_c^*\frac{\Delta\varrho^s}{\varrho_0^s} + 2p_0^f\frac{\Delta\varrho^s}{\varrho_0^s}, \tag{4.3}$$

$$\Delta\kappa = \nu_c^*\frac{\Delta\varrho^s}{\varrho_0^s} + \mathbb{P}^*\cdot(\mathbf{Q}\,\widetilde{\mathbf{E}}\,\mathbf{Q}^T), \tag{4.4}$$

where $\mathbf{Q}$ ($\mathbf{Q}\,\mathbf{Q}^T = \mathbf{I}$) is the orthogonal tensor.

Equations (4.2) – (4.4) will be identical with the corresponding equations (3.10), (3.12) and (3.13) for arbitrary values of variables Δp^f, $\Delta\varrho^s$, $\Delta\kappa$, $\Delta\mathbf{T}^{*s}$, $\widetilde{\mathbf{E}}$, $\mathbf{H}$ and any orthogonal tensor $\mathbf{Q}$ if the following conditions are satisfied

$$\begin{aligned} \mathbf{Q}*\mathbb{C}^* &= \mathbb{C}^*, \\ \mathbf{Q}\,\mathbb{K}^*\,\mathbf{Q}^T &= \mathbb{K}^*, \qquad \mathbf{Q}\,\mathbb{P}^*\mathbf{Q}^T = \mathbb{P}^*, \\ \mathbf{Q}\,\mathbf{T}_0^{*s}\mathbf{Q}^T &= \mathbf{T}_0^{*s}, \end{aligned} \tag{4.5}$$

where $\mathbf{Q}*$ is a linear operator defined by the equation

$$\mathbf{Q}*(\mathbf{v}_1\otimes\mathbf{v}_2\otimes\mathbf{v}_3\otimes\mathbf{v}_4) = \mathbf{Q}\,\mathbf{v}_1\otimes\mathbf{Q}\,\mathbf{v}_2\otimes\mathbf{Q}\,\mathbf{v}_3\otimes\mathbf{Q}\,\mathbf{v}_4$$

and $\otimes$ denotes the tensorial product of vectors.

It follows from (4.5) that the isotropy conditions for the constitutive relations are equivalent to the requirement of isotropy of tensorial material constants $\mathbb{C}^*$, $\mathbb{K}^*$ and $\mathbb{P}^*$ and, additionally, the isotropy of the skeleton stress state $\mathbf{T}_0^{*s}$ in the reference configuration.

The isotropy conditions (4.5) reduce the quantities $\mathbb{C}^*$, $\mathbb{K}^*$ and $\mathbb{P}^*$ and $\mathbf{T}_0^{*s}$ to the following form

$$\begin{aligned} \mathbb{C}^* &= \lambda_\varrho^*\mathbf{I}\otimes\mathbf{I} + 2\mu_\varrho^*\mathbb{J}, \\ \mathbb{K}^* &= K^*\mathbf{I}, \qquad \mathbb{P}^* = \nu_\varrho^*\mathbf{I}, \\ \mathbf{T}_0^{*s} &= -p_0^s\mathbf{I}, \end{aligned} \tag{4.6}$$

where $\mathbb{J}$ is the fourth order unit tensor defined as the identity operator for the second order tensors $\mathbf{A}$ ($\mathbb{J}\cdot\mathbf{A} = \mathbf{A}$). The quantities λ_ϱ^* and μ_ϱ^* are the effective Lamé constants of porous skeleton measured at the constant effective mass density of the skeleton material, and p_0^s is the initial stress in the skeleton.

Using (4.6) in Eqs. (3.10), (3.12) and (3.13) we obtain

$$\Delta \mathbf{T}^{*s} + \Delta p^f \mathbf{I} = 2(\mu_\varrho^* + p_0^f - p_0^s)\widetilde{\mathbf{E}} + \left(\lambda_\varrho^* \operatorname{tr}(\widetilde{\mathbf{E}}) + (K^* + p_0^f - p_0^s)\frac{\Delta \varrho^s}{\varrho_0^s}\right)\mathbf{I}, \tag{4.7}$$

$$\Delta p^f = K^* \operatorname{tr}(\widetilde{\mathbf{E}}) + (K_c^* + 2p_0^s)\frac{\Delta \varrho^s}{\varrho_0^s}, \tag{4.8}$$

$$\Delta \kappa = \nu_\varrho^* \operatorname{tr}(\widetilde{\mathbf{E}}) + \nu_c^* \frac{\Delta \varrho^s}{\varrho_0^s}. \tag{4.9}$$

In the case when the initial stress in the porous skeleton is equal to the intial pore fluid pressure

$$p_0^s = p_0^f,$$

Eq. (4.7) takes the reduced form

$$\Delta \mathbf{T}^{*s} + \Delta p^f \mathbf{I} = 2\mu_\varrho^* \widetilde{\mathbf{E}} + \left(\lambda_\varrho^* \operatorname{tr}(\widetilde{\mathbf{E}}) + K^* \frac{\Delta \varrho^s}{\varrho_0^s}\right)\mathbf{I}. \tag{4.10}$$

Equations (3.22), (4.7) (or (4.10)), (4.8) and (4.9) form the complete set of the linear constitutive relations for fully isotropic porous solid filled with fluid.

Seven material constants

$$\mu_\varrho^*, \quad \lambda_\varrho^*, \quad K^*, \quad K_c^*, \quad \nu_\varrho^*, \quad \nu_c^*, \quad K^f$$

are involved in the description, where the first six constants characterize elastic properties of the porous skeleton and one constant describes the mechanical fuid property.

Methods of determination of these material constants will be discussed n a seperate paper.

4.2. Saturated porous medium with incompressible matrix material

In the analysis of deformation processes of fluid-saturated porous media there are many physical situations in which the skeleton material can be considered as incompressible. The incompressibility condition takes the form

$$\varrho^s = \varrho_0^s \tag{4.11}$$

and is the kinematic constraint that confines the skeleton motion during its deformation.

In such a case the macroscopic volume deformations of porous skeleton arise at the cost of the change of pore volume. This is evidently seen in the skeleton continuity equation (3.21) that has the form

$$\frac{\Delta f_v}{1 - f_v^0} = \operatorname{tr}(\widetilde{\mathbf{E}}). \tag{4.12}$$

The incompressibility condition (4.11) is, at the same time, a particular case of the equation defining changes of the effective skeleton mass density and, as it was shown in [5], it replaces the internal equilibrium condition (in our case, Eqs. (3.12) and (4.8)).

The skeleton material incompressibility has no influence on the form of constitutive relation (3.22) for the fluid pressure, however, it substantially simplifies the form of two other relations (3.10) and (3.13) reducing the number of material constants. We have

$$\Delta \mathbf{T}^{*s} + \Delta p^f \mathbf{I} = (\mathbb{C}^* + 2p_0^f \mathbb{J}) \cdot \widetilde{\mathbf{E}} + \mathbf{H}\,\mathbf{T}_0^{*s} + \mathbf{T}_0^{*s}\mathbf{H}^T, \tag{4.13}$$

$$\Delta\kappa = \nu_\varrho^* \operatorname{tr}(\widetilde{\mathbf{E}}). \tag{4.14}$$

In this case the increment of fluid pressure Δp^f is the part of the skeleton stresses that during the skeleton deformation does the work over the pore fluid but does not change the energetic state of the skeleton due to its material incompressibility. Equations (4.13) and (4.14) for the fully isotropic porous solid, according to the analysis done in Sec. 4.1 assume the form

$$\Delta \mathbf{T}^{*s} + \Delta p^f \mathbf{I} = 2(\mu_\varrho^* + p_0^f - p_0^s)\widetilde{\mathbf{E}} + \lambda_\varrho^* \operatorname{tr}(\widetilde{\mathbf{E}})\mathbf{I}, \tag{4.15}$$

$$\Delta\kappa = \nu_\varrho^* \operatorname{tr}(\widetilde{\mathbf{E}}). \tag{4.16}$$

The above equations form, together with (3.22), the set of three constitutive relations defining the mechanical behaviour of fluid-saturated, isotropic porous solid with incompressible skeleton material. Such porous medium is characterized by four material constants:

$$\mu_\varrho^*,\quad \lambda_\varrho^*,\quad \nu_\varrho^*,\quad K^f.$$

The first three constants describe mechanical properties of porous skeleton and the last one describes the pore fluid.

4.3. Non-saturated porous solid

To obtain the constitutive relations describing the elastic behaviour of an anisotropic porous solid not saturated with fluid, one can assume in the equations (3.10) and (3.12) that the pore fluid pressure p^f is equal to zero ($p^f = 0$). Therefore, these equations get the form

$$\Delta \mathbf{T}^{*s} = \mathbb{C}^* \cdot \widetilde{\mathbf{E}} + (\mathbb{K}^* + \mathbf{T}_0^{*s})\frac{\Delta\varrho^s}{\varrho_0^s} + \mathbf{H}\,\mathbf{T}_0^{*s} + \mathbf{T}_0^{*s}\mathbf{H}^T, \tag{4.17}$$

$$0 = \mathbb{K}^* \cdot \widetilde{\mathbf{E}} + K_c^* \frac{\Delta\varrho^s}{\varrho_0^s}, \tag{4.18}$$

while the equation (3.13) is not changed.

Equation (4.17) will be simplified, if the skeleton reference configuration is its natural configuration, i.e. $\mathbf{T}_0^{*s} = 0$.

We have

$$\Delta \mathbf{T}^{*s} = \mathbb{C}^* \cdot \widetilde{\mathbf{E}} + \mathbb{K}^* \frac{\Delta \varrho^s}{\varrho_0^s}. \tag{4.19}$$

From the internal equilibrium equation (4.18) it follows that for fluid-free porous skeleton, the density change of the skeleton material is uniquely defined by the porous solid strain tensor $\widetilde{\mathbf{E}}$. Therefore the constitutive relation (4.19) can be written in the form

$$\mathbf{T}^{*s} = \mathbb{C}_z^* \cdot \widetilde{\mathbf{E}} \tag{4.20}$$

which is analogous to that of non-porous elastic solid.

Tensor

$$\mathbb{C}_z^* = \mathbb{C}^* - (\mathbb{K}^* \otimes \mathbb{K}^*)/K_c^* \tag{4.21}$$

is the equivalent elasticity tensor of the effective elastic constants of a porous skeleton.

In the isotropic case relation (4.20) is

$$\mathbf{T}^{*s} = 2\mu_z^* \widetilde{\mathbf{E}} + \lambda_z^* \operatorname{tr}(\widetilde{\mathbf{E}})\mathbf{I}, \tag{4.22}$$

where

$$\mu_z^* = \mu_\varrho^*, \qquad \lambda_z^* = \lambda_\varrho^* - (\nu_\varrho^*)^2/K_c^*.$$

If, additionally, the incompressibility of the skeleton material is assumed, the stress in the skeleton can be written as

$$\mathbf{T}^{*s} = 2\mu_\varrho^* \widetilde{\mathbf{E}} + \lambda_\varrho^* \operatorname{tr}(\widetilde{\mathbf{E}})\mathbf{I}. \tag{4.23}$$

The material coefficients appearing in relations (4.20), (4.22) and (4.23) play an analogous role as those in the classical linear elasticity of solids, and their measurement can be done in the classical way. These material constants completely assure the determination of stress and strain state in the porous skeleton. For description of the change of the skeleton mass density ϱ^s, or the change of the pore structure parameter κ it is necessary to evaluate additional coefficients appearing in Eqs. (4.18) and (3.13) or in their reduced forms (4.9) and (4.8). Measurement of these coefficients requires some new methods to be proposed.

5. Final remarks

The complete set of constitutive relations for a fluid-saturated porous solid with anisotropic properties of elastic skeleton and isotropic pore structure characterized by two parameters have been formulated in the paper. It comprises: the constitutive relations for the effective skeleton stresses and the pore fluid pressure, the internal mechanical equilibrium condition and the equation of changes of the pore structure parameter κ. These relations are supplemented with the skeleton continuity equation which describe the changes of porosity f_v.

Considerations have been based on the nonlinear constitutive relations of such medium obtained in the paper [5], where the consequences of the constituent immiscibility for these relations have been analysed.

Such approach made it possible to construct the consistent linear description of elastic behaviour of porous skeleton filled with barotropic fluid in which all material constants are precisely defined and have clear physical meaning. Also the character of couplings appearing in the constitutive relations and their interpretation are simpler.

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Received March 14, 1996.