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Constitutive relations and internal equilibrium condition for fluid-saturated porous solids Nonlinear theory

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NONLINEAR CONSTITUTIVE relations for the fluid-saturated porous elastic solid with isotropic pore structure undergoing pure mechanical large deformations are developed. The fluid-solid composition is considered as the immiscible mixture consisting of physically identifiable constituents preserving its own individual, physical properties during deformation process. Considerations are based on the balance equation for the internal energy of the whole composition which is required to be satisfied identically by the internal energy constitutive functions postulated for particular components independently. Constitutive relations for partial stresses of particular constituents are obtained, and the internal equilibrium condition for the whole composition is established. These relations for the media with incompressible matrix material and for nonsaturated porous skeleton are discussed.

1. Introduction

MACRO-CONTINUUM constitutive modelling of fluid-saturated porous solids has been a subject of wide discussion through the last decades. Nonlinear models of such materials are based mostly upon the fundamental notions of the Classical Mixture Theory, [5, 23], and its reformulated form – the Theory of Interacting Continua, [9, 10]. Classical mixtures are considered to be composed of miscible constituents (miscible mixture) and within that theory, a fluid-filled porous medium is treated as the superposition of two continua (solid and fluid) characterized by two independent velocity fields. In such approach, the microstructure of solid-fluid composition is not taken into account in formulation of the balance equations and constitutive relations. At the same time, constitutive theories of classical mixtures quickly become complex and unwieldy, even for the simplest constitutive assumptions, when they are based on the principle of equipresence [23], which assumes that each constitutive quantity of a particular component depends on a set of independent variables for the whole solid-fluid composition (see e.g. [1, 2, 7, 11, 12]).

It is evident, however, that such materials as saturated sands, soils, porous rocks, sintered metals, sponges etc. consist of physically identifiable solid matrix and a fluid filling its pores that retain their material integrity, and thus their individual physical properties, during a deformation process. Therefore, porous materials filled with fluid, contrary to the classical mixtures, have internal geometrical structure reflecting the fact of immiscibility of constituents and characteristics of this structure play important role in both transport phenomena and constitutive modelling of such materials. This also proves that in the local sense, each constituent will obey the constitutive relations for that constituent alone.

Among many works developing the macro-continuum constitutive modelling of solid-fluid mixture there are papers that regard the immiscibility effect by incorporating in the description the parameter of volume porosity characterizing the volume fractions of the constituents (see for example [4, 6, 8, 13, 19–22]). Most of these papers have in common the fact that they apply the principle of equipresence in formulation of constitutive relations for partial quantities of the individual components of the solid-fluid mixture ([6, 8, 19–22]), or for the energy constitutive functions concerning the whole aggregate ([4, 13]). Such approach does not prove to be self-consistent in treating the immiscibility effect as the immanent feature of the porous solid-fluid composition.

The extensive literature concerning the different descriptions of immiscible and structured mixtures can be found in the review paper [3].

The purpose of this study is to develop, within the macro-continuum description, the nonlinear constitutive relations for fluid-saturated porous solids undergoing pure mechanical deformations where the main consequences of the immiscibility, i.e. the skeleton pore structure characteristics and mutual independence of mechanical properties of individual constituents are taken into account. The components are assumed to be elastic and the pore structure has isotropic and homogeneous properties in the macroscopic (averaged) sense.

Considerations are based on the balance equation for the internal energy of the whole composition, which is required to be satisfied identically by the internal energy constitutive functions postulated for particular components, the functional forms of which reflect their individual features.

This enables one to obtain two nonlinear constitutive relations for stresses (one for porous skeleton and the other for pore fluid), and the relation for interface interaction force. Moreover, the additional relation is derived which is the condition of internal equilibrium for the solid-fluid composition. It relates the pore fluid pressure with independent variables describing the deformation state of porous skeleton.

In the paper three particular cases of constitutive relations are also considered. They concern the porous solids with isotropic mechanical properties, with incompressible material of the skeleton and the case when porous solid is not saturated with fluid.

2. Balance equations for mass, linear momentum and internal energy

In our considerations we use the macroscopic continuum description of fluid-saturated porous solid, the pore structure of which is isotropic and characterized by two parameters: the volume porosity f_v and the structural permeability λ (or equivalently $\kappa = \lambda / f_v < 1$). The quantity f_v represents the fluid volume fraction and λ is the measure of inhomogeneity of the fluid micro-velocity field in its flow relative to the skeleton, caused by the pore structure [14–16]. The characteristic feature of this theory is that the description of kinematics and dynamics of the porous solid-fluid mixture is referred to the so-called virtual components while the description of constitutive properties of the mixture is formulated for its physical constituents.

The physical constituents are: the fluid ($|^f$) and the porous skeleton ($|^s$), which are chemicaly inert and their mass is conserved. Therefore the appropriate continuity equations have the classical form used within the mixture theory [5]

$$\frac{\partial \overline{\varrho}^s}{\partial t} + \operatorname{div}(\overline{\varrho}^s \mathbf{v}^s) = 0,$$
$$\frac{\partial \overline{\varrho}^f}{\partial t} + \operatorname{div}(\overline{\varrho}^f \mathbf{v}^f) = 0,$$

where $\overline{\varrho}^s$ and $\overline{\varrho}^f$ are the partial densities of porous solid and fluid, respectively, and $\mathbf{v}^s$ and $\mathbf{v}^f$ stand for the mass average velocities of the constituents.

The virtual constituents are formed by the porous skeleton and fluid associated with it – the first virtual constituent ($|^1$) moving at the skeleton velocity $\overset{1}{\mathbf{v}}$, and the free fluid – the second virtual constituent ($|^2$) moving at its own velocity $\overset{2}{\mathbf{v}}$. These velocities are related to velocities of the physical constituents as follows, [14, 15]:

$$\overset{1}{\mathbf{v}} = \mathbf{v}^s,$$
$$\overset{2}{\mathbf{v}} = \mathbf{v}^s + \frac{1}{\kappa}(\mathbf{v}^f - \mathbf{v}^s).$$

The virtual constituents in the macroscopic description result from the requirement that the whole linear momentum and kinetic energy of the particular constituents considered within the Elementary Volume Element and described by the quantities defined at the micro-level (pore, grain level) should be fully represented by the macroscopic (averaged) quantities at the macro-level.

Since during a deformation process the amount of associated fluid can change, the virtual constituents form systems interchanging their masses and the coresponding continuity equations have the following form, [15],

$$\begin{aligned} &\frac{\partial \overset{1}{\varrho}}{\partial t} + \operatorname{div}(\overset{1}{\varrho}\overset{1}{\mathbf{v}}) = g,\\ &\frac{\partial \overset{2}{\varrho}}{\partial t} + \operatorname{div}(\overset{2}{\varrho}\overset{2}{\mathbf{v}}) = -g. \end{aligned} \tag{2.1}$$

The function g is the mass exchange intensity between the free and associated fluid and is given explicitly by the expression

$$g = \overline{\varrho}^{s} \frac{\overset{1}{D}}{Dt} \left[(1-\kappa) \frac{\overline{\varrho}^{f}}{\overline{\varrho}^{s}} \right].$$

The motion equations of the virtual constituents are, [14–16],

$$\begin{aligned} \overset{1}{\varrho} \frac{\overset{1}{D} \overset{1}{\mathbf{v}}}{dt} &= \operatorname{div} \overset{1}{\mathbf{T}} + \overset{1}{\varrho} \mathbf{b} + \overset{1}{\pi} + \frac{1}{2} g (\overset{2}{\mathbf{v}} - \overset{1}{\mathbf{v}}), \\ \overset{2}{\varrho} \frac{\overset{2}{D} \overset{2}{\mathbf{v}}}{dt} &= \operatorname{div} \overset{2}{\mathbf{T}} + \overset{2}{\varrho} \mathbf{b} + \overset{2}{\pi} + \frac{1}{2} g (\overset{2}{\mathbf{v}} - \overset{1}{\mathbf{v}}); \end{aligned} \tag{2.2}$$

$$\frac{D^{\alpha}(\,)}{Dt} = \frac{\partial(\,)}{\partial t} + \overset{\alpha}{\mathbf{v}} \cdot \operatorname{grad}(\,), \qquad \alpha = 1,2,$$

where $\overset{1}{\pi}$ and $\overset{2}{\pi}$ ($\overset{1}{\pi} = \pi = -\overset{2}{\pi}$) stand for the internal interaction forces between constituents and $\mathbf{b}$ is the external body force per unit mass. The last terms on the RHS of Eqs. (2.2) represent the coupling between virtual constituents caused by the linear momentum exchange accompanying their mass exchange.

The partial densities $\overset{1}{\varrho}$, $\overset{2}{\varrho}$ and partial Cauchy stresses $\overset{1}{\mathbf{T}}$, $\overset{2}{\mathbf{T}}$ of the virtual constituents are related to the partial densities $\overline{\varrho}^{s}$, $\overline{\varrho}^{f}$ and partial Cauchy stresses $\mathbf{T}^{s}$, $\mathbf{T}^{f}$ of the physical constituents through the following equations, [14]

$$\overset{1}{\varrho} = \overline{\varrho}^{s} + (1-\kappa)\overline{\varrho}^{f}, \qquad \overset{2}{\varrho} = \kappa \overline{\varrho}^{f},, \tag{2.3}$$

$$\overset{1}{\mathbf{T}} = \mathbf{T}^{s} + (1-\kappa)\mathbf{T}^{f}, \qquad \overset{2}{\mathbf{T}} = \kappa \mathbf{T}^{f}, \tag{2.4}$$

where

$$\overline{\varrho}^{f} = f_v \varrho^{f}, \qquad \overline{\varrho}^{s} = (1-f_v)\varrho^{s}$$

and ϱ^{f}, ϱ^{s} stand for the effective density of fluid and porous skeleton, respectively. Stress tensors $\mathbf{T}^{s}$ and $\mathbf{T}^{f}$ are assumed to be symmetric, so that stresses $\overset{1}{\mathbf{T}}$ and $\overset{2}{\mathbf{T}}$ are also symmetric.

The local form of the internal energy balance equation can be formulated both for the individual components or for the whole solid-fluid composition. In our case we use the second one that allows us to avoid the specification of the terms describing the interchange of energy between constituents.

Accounting for the immiscibility of the physical constituents, the internal energy of the porous solid-fluid composition is considered as the sum of the internal energies of these constituents. When thermal efects are disregarded, its form is as follows, [16, 17],

$$(2.5) \qquad \bar{\varrho}^s \frac{\overset{1}{D} e^s}{Dt} + (1-\kappa)\bar{\varrho}^f \frac{\overset{1}{D} e^f}{Dt} + \kappa\bar{\varrho}^f \frac{\overset{2}{D} e^f}{Dt} = \boldsymbol{\pi} \cdot (\overset{2}{\mathbf{v}} - \overset{1}{\mathbf{v}}) + \operatorname{tr}(\overset{1}{\mathbf{T}}{}^T \overset{1}{\mathbf{L}}) + \operatorname{tr}(\overset{2}{\mathbf{T}}{}^T \overset{2}{\mathbf{L}}),$$

where e^f and e^s are the internal energies per unit mass for the fluid and the solid skeleton, respectively, and the tensors

$$\overset{1}{\mathbf{L}} = \operatorname{grad} \overset{1}{\mathbf{v}}, \qquad \overset{2}{\mathbf{L}} = \operatorname{grad} \overset{2}{\mathbf{v}}$$

are the velocity gradients of the virtual constituents. The subscript T stands for the transposition of the tensor.

The particular terms of the LHS of (2.5) describe the rate of changes of the internal energy in the matrix material, associated fluid and free fluid, respectively, which are balanced by the rate of work of volume and surface forces represented by the RHS terms of (2.5). Equation (2.5) will be used in further part of this work to derive the necessary constitutive relations.

3. Constitutive relations for the elastic porous solid filled with barotropic fluid. The internal equilibrium of the system

In this section we formulate constitutive relations for porous solid filled with fluid undergoing large elastic deformations. It is assumed that both, porous skeleton and fluid filling pores have elastic properties, and mutual solid-fluid interaction on the interface is that of mechanical type only. We disregard the viscous effects of the fluid confining our considerations to the pure elastic interactions.

Under the above assumptions, the fluid-saturated porous solid forms the non-dissipative system of two immiscible constituents, each of which preserves its own physical properties during a deformation process. The mechanical behaviour of such system is entirely described by the mass and linear momentum balance equations of virtual constituents (2.1) and (2.2), respectively, and appropriate constitutive relations which have to be formulated for the physical components. At the same time, the balance equation (2.5) for the internal energy of the system must be identically satisfied by constitutive relations for an arbitrary mechanical process.

We apply the internal energy balance equation (2.5) to obtain nonlinear constitutive equations for the elastic porous solid filled with fluid. Their forms will be derived from Eq. (2.5) which has to be identically satisfied by the postulated functions for the internal energies of individual physical constituents of the solid-fluid system.

Such method of derivation of the constitutive relations is analogous to the clasical approach used for the hyperelastic medium.

3.1. Constitutive postulates for the fluid and porous skeleton internal energies

The essence of the immiscibility is the fact that the physical constituents of porous solid-fluid mixture remain separated during a deformation process and then, in the local sense, each constituent shall obey the constitutive relations for that constituent alone. Therefore it is reasonable to define the internal energy for each physical constituent independently by the field quantities describing its own state of deformation.

In the case of the elastic (barotropic) fluid filling pores of the skeleton, its local state is defined by the effective fluid mass density ϱ^f. Thus, the constitutive function for the fluid internal energy can be written as follows:

$$e^f = \hat{e}^f(\varrho^f). \tag{3.1}$$

The local deformation state of the elastic porous skeleton filled with fluid, contrary to a non-porous material, is characterized by two kinds of independent variables describing, say, the internal and external skeleton deformations. The internal deformation of the skeleton is connected with a change of its geometrical pore structure and is measured by variations of the pore structure parameters: f_v and λ (or equivalently κ). Both the pore structure parameters will be used in the description as the internal state variables.

The external skeleton deformation (bulk deformation of a porous sample) is defined by the deformation gradient

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial}{\partial \mathbf{X}} \chi_k(\mathbf{X}, t), \tag{3.2}$$

where

$$\mathbf{x} = \chi_k(\mathbf{X}, t) \equiv \chi(k^{-1}(\mathbf{X}), t)$$

is the deformation function of the porous body B which relates the position $\mathbf{X}$ of the skeleton particle (macroscopic particle) $X \in B$ in the reference configuration

$$\mathbf{X} = \mathbf{k}(X)$$

to its position $\mathbf{x}$ in the current configuration

$$\mathbf{x} = \chi(X, t).$$

The derivative in (3.2) is defined by the identity, [18],

$$\frac{\partial \chi_k}{\partial \mathbf{X}} \mathbf{D} \equiv \frac{\partial}{\partial h} \chi_k(\mathbf{X} + h\mathbf{D}, t)|_{h=0},$$

where $\mathbf{D}$ is an arbitrary vector quantity.

The deformation process in which the pore structure parameters of porous skeleton changes while the deformation gradient $\mathbf{F}$ is constant and equal to the identity tensor is called the pure internal deformation, whereas the case when the pore parameters are constant during the deformation measured by the gradient $\mathbf{F}$ is called the pure external deformation.

Taking the above into account, the constitutive relation for the internal energy of porous elastic solid can be proposed in the following form

$$e^s = e^s_k(\mathbf{F}, f_v, \kappa). \tag{3.3}$$

Because of dependence of the deformation gradient $\mathbf{F}$ on the choice of reference configuration, the function e^s_k must also depend on the reference configuration to ensure the value of internal energy e^s to be insensitive to changes of this configuration.

It is commonly accepted that each constitutive relation should satisfy the principle of material objectivity, that is to be independent of the choice of reference frame.

The relation (3.1) as the scalar-valued scalar function satisfies this principle automatically whereas the objectivity condition for the relation (3.3) takes the form of the following identity, [18],

$$e^s_k(\mathbf{QF}, f_v, \kappa) \equiv e^s_k(\mathbf{F}, f_v, \kappa), \tag{3.4}$$

that has to be satisfied for every orthogonal transformation $\mathbf{Q}$ ($\mathbf{Q}^T = \mathbf{Q}^{-1}$) and for arbitrary values of independent variables $\mathbf{F}$, f_v and κ.

The condition (3.4) when applied to (3.3) yields

$$e^s = \widehat{e}^s_k(\mathbf{C}, f_v, \kappa), \tag{3.5}$$

where

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \tag{3.6}$$

is the right Cauchy–Green deformation tensor.

Accounting for the fact that the independent variables $\mathbf{C}$, f_v and κ are objective quantities, the scalar-valued function (3.5) satisfies the objectivity principle for arbitrary form of $\widehat{e}^s_k$.

The representation (3.5) is the general (nonlinear) constitutive relation for the internal energy of porous solid of anisotropic elastic properties and the isotropic pore structure.

The relation (3.5) is not the only form that represents the internal energy of porous skeleton. We can derive two other, equivalent forms replacing the current volume porosity f_v with the porosity f^L_v (say Lagrangean porosity) defined by

$$f^L_v = f_v J, \tag{3.7}$$

where

$$J = \det(\mathbf{F}) = [\det(\mathbf{C})]^{1/2}, \tag{3.8}$$

or with the skeleton density ϱ^s related to the porosity f_v by the skeleton continuity equation

$$\varrho^s(1 - f_v)J = \varrho_0^s(1 - f_v^0) = \overline{\varrho}_0^s\,. \tag{3.9}$$

In (3.9) ϱ_0^s and f_v^0 stand for the values of ϱ^s and f_v, respectively, in the reference configuration.

The porosity f_v^L is the ratio of the pore volume contained in the Elementary Volume Element (macro-particle) of the porous body in the current configuration to the total volume of the same Volume Element in the reference configuration. Variation of this quantity, contrary to variation of f_v, is the local absolute measure of the change of a pore volume during a deformation process.

Similarly, the skeleton mass density variation may be considered as the local absolute measure of the change of a skeleton volume. In such a case the density ϱ^s, similarly to the porosity f_v, plays the role of internal state variable.

These are good points in application of f_v^L and ϱ^s to the skeleton internal energy formulation. We obtain

$$e^s = \widehat{e}_k^{sf}(\mathbf{C}, f_v^L, \kappa), \tag{3.10}$$

$$e^s = \widehat{e}_k^{s\varrho}(\mathbf{C}, \varrho^s, \kappa), \tag{3.11}$$

where the constitutive functions $\widehat{e}_k^{sf}$ and $\widehat{e}_k^{s\varrho}$ are defined by the following identities

$$\widehat{e}_k^{sf}(\mathbf{C}, f_v^L, \kappa) \equiv \widehat{e}_k^{s}(\mathbf{C}, f_v^L/J, \kappa), \tag{3.12}$$

$$\widehat{e}_k^{s\varrho}(\mathbf{C}, \varrho^s, \kappa) \equiv \widehat{e}_k^{s}(\mathbf{C}, 1 - \overline{\varrho}_0^s/\varrho^s J, \kappa). \tag{3.13}$$

The constitutive functions for the fluid (3.1), and for the porous skeleton (3.5) or its alternative forms (3.10) and (3.11) entirely describe the energetic state of elastic porous solid filled with fluid and undergoing finite deformations.

3.2. Constitutive relations for stresses. The condition of mechanical internal equilibrium

To establish constitutive stress-strain relations for each constituent of the fluid-porous solid immiscible mixture, and relations describing mutual solid-fluid interaction, we apply the approach characteristic for the hyperelastic medium. We introduce the relations (3.1) and (3.5) to the energy balance equation (2.5) for the whole porous solid-fluid mixture which has to be identically satisfied for an arbitrary mechanical process. Using, moreover, the continuity equation $(2.1)_2$, Eq. (2.5) can be written as follows

$$\begin{aligned}(3.14)\qquad &\operatorname{tr}\left\{\left[2\overline{\varrho}^{s}\mathbf{F}\frac{\partial \widehat{e}_{k}^{s}}{\partial \mathbf{C}}\mathbf{F}^{T}-\frac{d\widehat{e}^{f}}{d\varrho^{f}}\left(\varrho^{f}\right)^{2}f_{v}(1-\kappa)\mathbf{I}-\overset{1}{\mathbf{T}}\right]\overset{1}{\mathbf{L}}\right\}\\ &-\operatorname{tr}\left\{\left[\frac{d\widehat{e}^{f}}{d\varrho^{f}}\left(\varrho^{f}\right)^{2}\kappa f_{v}\mathbf{I}+\overset{2}{\mathbf{T}}\right]\overset{2}{\mathbf{L}}\right\}+\left[\overline{\varrho}^{s}\frac{\partial \widehat{e}_{k}^{s}}{\partial f_{v}}-\frac{d\widehat{e}^{f}}{d\varrho^{f}}\left(\varrho^{f}\right)^{2}\right]\frac{\overset{1}{D}f_{v}}{Dt}\\ &-\left[\frac{d\widehat{e}^{f}}{d\varrho^{f}}\left(\varrho^{f}\right)^{2}\operatorname{grad}(\lambda)+\boldsymbol{\pi}\right]\cdot(\overset{1}{\mathbf{v}}-\overset{2}{\mathbf{v}})+\frac{\partial \widehat{e}_{k}^{s}}{\partial \kappa}\frac{\overset{1}{D}\kappa}{Dt}=0.\end{aligned}$$

Equation (3.14) is the linear function of the independent quantities

$$\overset{1}{\mathbf{L}},\quad \overset{2}{\mathbf{L}},\quad (\overset{1}{\mathbf{v}}-\overset{2}{\mathbf{v}}),\quad \frac{\overset{1}{D}f_{v}}{Dt},\quad \frac{\overset{1}{D}\kappa}{Dt},$$

for an arbitrary mechanical process in the body. Since these quantities can assume arbitrary values, equation (3.14) will be identically satisfied if the corresponding coefficients are equal to zero. Defining the quantity

$$(3.15)\qquad p^{f}=\left(\varrho^{f}\right)^{2}\frac{d\widehat{e}^{f}}{d\varrho^{f}}$$

which is considered as the effective pore pressure, from (3.14) we have

$$(3.16)\qquad \overset{1}{\mathbf{T}}+(1-\kappa)f_{v}p^{f}\mathbf{I}=2\overline{\varrho}^{s}\mathbf{F}\frac{\partial \widehat{e}_{k}^{s}}{\partial \mathbf{C}}\mathbf{F}^{T},$$

$$(3.17)\qquad \overset{2}{\mathbf{T}}=-\kappa f_{v}p^{f}\mathbf{I},$$

$$(3.18)\qquad p^{f}=\overline{\varrho}^{s}\frac{\partial \widehat{e}_{k}^{s}}{\partial f_{v}},$$

$$(3.19)\qquad \frac{\partial \widehat{e}_{k}^{s}}{\partial \kappa}=0,$$

$$(3.20)\qquad \boldsymbol{\pi}=-p^{f}\operatorname{grad}(\lambda).$$

Condition (3.20) describes the force exerted on the solid skeleton by fluid filling its pores. From (3.20) it is seen that, despite the lack of fluid viscosity effects in the considerations, the solid-fluid interface interaction force does exist due to the nonhomogeneity of the skeleton pore structure.

The expression (3.19) is the necessary condition for minimum of the skeleton internal energy function at constant strain tensor $\mathbf{C}$ and volume porosity f_v. If, additionally, the sufficient condition is satisfied, i.e.

$$\frac{\partial^{2}\widehat{e}_{k}^{s}}{\partial \kappa^{2}}>0,$$

then Eq. (3.19) indicates that the saturated porous body undergoes a deformation process in such a way that the parameter κ takes values for which the skeleton internal energy has a minimum, (Fig. 1). Therefore, the condition (3.19) may be treated as the implicit equation of variation of the κ-parameter during a deformation process.

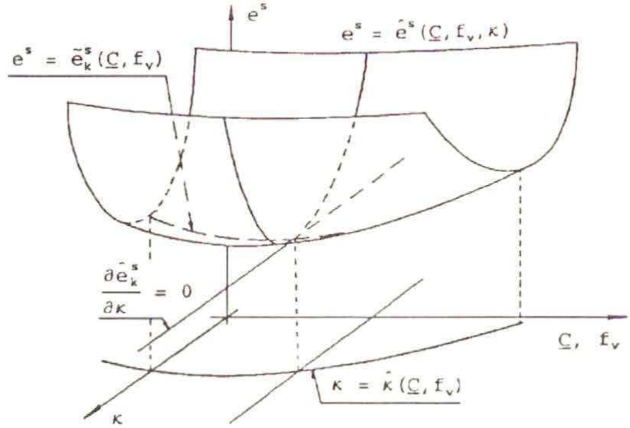

FIG. 1. Illustration of the changes of the pore structure parameter κ during the deformation process.

Assuming that (3.19) is the smooth function of κ, it can be rewritten (at least locally) in the explicit form

$$\kappa = \widehat{\kappa}(\mathbf{C}, f_v), \tag{3.21}$$

that has to satisfy the condition

$$\frac{\partial \widehat{e}^{\,s}_k}{\partial \kappa}(\mathbf{C}, f_v, \kappa)\Big|_{\kappa=\widehat{\kappa}(\mathbf{C}, f_v)} \equiv 0.$$

Equation (3.21) offers the possibility of exclusion of the κ-parameter from the set of independent variables defining the skeleton internal energy function. In such a case, the constitutive functions (3.5), (3.10) and (3.11) take the form

$$\widetilde{e}^{\,s}_k(\mathbf{C}, f_v) \equiv \widehat{e}^{\,s}_k(\mathbf{C}, f_v, \widehat{\kappa}(\mathbf{C}, f_v)), \tag{3.22}$$

$$\widetilde{e}^{\,sf}_k(\mathbf{C}, f^L_v) \equiv \widehat{e}^{\,sf}_k(\mathbf{C}, f^L_v, \widehat{\kappa}^{\,L}(\mathbf{C}, f^L_v)), \tag{3.23}$$

$$\widetilde{e}^{\,s\varrho}_k(\mathbf{C}, \varrho^s) \equiv \widehat{e}^{\,s\varrho}_k(\mathbf{C}, \varrho^s, \widehat{\kappa}^{\,\varrho}(\mathbf{C}, \varrho^s)), \tag{3.24}$$

where

$$\widehat{\kappa}^{\,L}(\mathbf{C}, f^L_v) \equiv \widehat{\kappa}(\mathbf{C}, f^L_v/J), \tag{3.25}$$

$$\widehat{\kappa}^{\,\varrho}(\mathbf{C}, \varrho^s) \equiv \widehat{\kappa}(\mathbf{C}, 1 - \overline{\varrho}^{\,s}_0/\varrho^s J). \tag{3.26}$$

From the condition (3.19), due to identities (3.12) and (3.13), we obtain

$$\frac{\partial \widehat{e}_k^{sf}}{\partial \kappa} = \frac{\partial \widehat{e}_k^{s\varrho}}{\partial \kappa} = 0,$$

and consequently (3.12) and (3.13) reduce to the following identities, respectively

$$\widetilde{e}_k^{sf}(\mathbf{C}, f_v^L) \equiv \widetilde{e}_k^{s}(\mathbf{C}, f_v^L/J), \tag{3.27}$$

$$\widetilde{e}_k^{s\varrho}(\mathbf{C}, \varrho^s) \equiv \widetilde{e}_k^{s}(\mathbf{C}, 1 - \overline{\varrho}_0^s/\varrho^s J). \tag{3.28}$$

From the above consideration it is seen that the condition (3.19) should be treated not only as the condition of independence of the skeleton internal energy of the parameter κ, but also as the equation of changes of this parameter. Such interpretation of the condition (3.19) is supported by the analysis given in Sec. 4.3 where the skeleton mass density changes of non-saturated porous material is obtained as a particular case of the internal equilibrium condition (for $p^f = 0$) in the same form as the condition (3.19).

Now, taking into account relations (3.22), (3.19) and (2.4), Eqs. (3.16)–(3.18) can be written in the following form

$$\mathbf{T}^s = 2\overline{\varrho}^s \mathbf{F} \frac{\partial \widetilde{e}_k^s}{\partial \mathbf{C}} \mathbf{F}^T, \tag{3.29}$$

$$\mathbf{T}^f = -f_v p^f \mathbf{I}, \tag{3.30}$$

$$p^f = \overline{\varrho}^s \frac{\partial \widetilde{e}_k^s}{\partial f_v}. \tag{3.31}$$

Equations (3.29) and (3.30) (together with (3.15)) are the constitutive relations for the partial stresses of an elastic porous skeleton and of a barotropic fluid filling its pores, respectively. Equation (3.31) relates the pore fluid pressure p^f with the deformation tensor $\mathbf{C}$ and the volume porosity f_v; quantities which define the state of deformation of the porous skeleton. It is the condition for internal mechanical equilibrium between porous skeleton and fluid filling its pores. As will be shown in Sec. 4.2, this equation does not appear in the description in the case when the skeleton material is incompressible.

The condition (3.31) and the skeleton continuity equation (3.9) define (optionally) changes of the two internal parameters: the volume porosity f_v and the skeleton mass density ϱ^s.

From the above considerations it is seen that the constitutive functions

$$\frac{\partial \widetilde{e}_k^s}{\partial \mathbf{C}}; \qquad \frac{\partial \widetilde{e}_k^s}{\partial f_v}$$

in relations (3.29) and (3.31) are defined by the mechanical properties of porous skeleton and do not depend on the properties of the fluid filling its pores. In such constitutive formulation the mechanical coupling between fluid and porous skeleton appears only in the internal equilibrium condition (3.31) where the fluid pore pressure p^f is present.

It is worth to note that relation similar to (3.31) was considered by KENYON in his paper [13] on the equilibrium theory of the solid-fluid mixture. He introduced constitutive postulate relating the volume porosity f_v to the fluid bulk density $\overline{\varrho}^f$ and $J = \det(\mathbf{F})$, without any physical motivation. Such relation can be considered as a particular case of the equation (3.31) that has the resonable physical interpretation.

The form of the constitutive equation (3.29) and the internal equilibrium condition (3.31) will change if the Lagrangean porosity f_v^L or the skeleton mass density ϱ^s, instead of the current volume porosity f_v, is used in the expression for skeleton internal energy. In the first case, after differentiation of (3.27) with respect to $\mathbf{C}$ and f_v^L, we obtain

$$\frac{\partial \tilde{e}_k^{sf}}{\partial \mathbf{C}} = \frac{\partial \tilde{e}_k^s}{\partial \mathbf{C}} - \frac{1}{2} f_v \frac{\partial \tilde{e}_k^s}{\partial f_v} \mathbf{C}^{-T},$$
$$\frac{\partial \tilde{e}_k^{sf}}{\partial f_v^L} = \frac{1}{J} \frac{\partial \tilde{e}_k^s}{\partial f_v}.$$

The above relations, when applied to (3.29) and (3.31) yield

$$\mathbf{T}^s - f_v p^f \mathbf{I} = 2\overline{\varrho}^s \mathbf{F} \frac{\partial \tilde{e}_k^{sf}}{\partial \mathbf{C}} \mathbf{F}^T, \tag{3.32}$$

$$p^f = \overline{\varrho}_0^s \frac{\partial \tilde{e}_k^{sf}}{\partial f_v^L}. \tag{3.33}$$

Taking Eq. (3.30) into account, we can conclude that Eq. (3.32) is the constitutive relation for the total stress

$$\mathbf{T} = \mathbf{T}^s + \mathbf{T}^f$$

in the solid-fluid composition.

Equations (3.32) and (3.33) coincide with the equations derived in another way by BIOT [4].

In the second case, after differentiation of (3.28) with respect to $\mathbf{C}$ and ϱ^s we have

$$\frac{\partial \tilde{e}_k^{s\varrho}}{\partial \mathbf{C}} = \frac{\partial \tilde{e}_k^s}{\partial \mathbf{C}} - \frac{1}{2}(1 - f_v) \frac{\partial \tilde{e}_k^s}{\partial f_v} \mathbf{C}^{-T},$$
$$\frac{\partial \tilde{e}_k^{s\varrho}}{\partial \varrho^s} = \frac{\overline{\varrho}_0^s}{(\varrho^s)^2 J} \frac{\partial \tilde{e}_k^s}{\partial f_v}.$$

These relations, when applied to (3.29) and (3.31), give

$$\mathbf{T}^s = -(1 - f_v)p^f\mathbf{I} + 2\overline{\varrho}^s\mathbf{F}\frac{\partial \tilde{e}_k^{s\varrho}}{\partial \mathbf{C}}\mathbf{F}^T, \tag{3.34}$$

$$\frac{p^f}{(\varrho^s)^2} = \frac{\partial \tilde{e}_k^{s\varrho}}{\partial \varrho^s}. \tag{3.35}$$

From (3.34) it is seen that the stress in the skeleton is composed of two parts; the first part

$$-(1 - f_v)p^f\mathbf{I}$$

is due to the presence of the pore fluid in the skeleton pores, and the second part

$$2\overline{\varrho}^s\mathbf{F}\frac{\partial \tilde{e}_k^{s\varrho}}{\partial \mathbf{C}}\mathbf{F}^T$$

is due to the deformation of porous solid.

Introducing the effective stress tensor in the skeleton by the definition

$$\mathbf{T}^{*s} = \mathbf{T}^s/(1 - f_v), \tag{3.36}$$

from (3.34) we obtain the constitutive equation for the effective stresses in the following form

$$\mathbf{T}^{*s} = -p^f\mathbf{I} + 2\varrho^s\mathbf{F}\frac{\partial \tilde{e}_k^{s\varrho}}{\partial \mathbf{C}}\mathbf{F}^T. \tag{3.37}$$

It should be noted that the constitutive relation (3.37) as well as the internal equilibrium condition (3.35) and the constitutive equation (3.15) for the effective pore fluid pressure, do not depend explicitly on the volume porosity f_v. It limits the number of the quantities appearing in these equations thus simplifying their forms.

In such a case the condition (3.35) can be considered as the equation describing variations of the skeleton mass density ϱ^s (particularly in the case of non-saturated pores ($p^f = 0$); see Sec. 4.3). Then the skeleton continuity equation (3.12) plays the role of equation for the volume porosity changes.

The constitutive relations (3.15) and (3.37), the condition (3.35) and Eq. (3.21) form a complete set of the constitutive equations for the fluid-saturated porous solid of elastic mechanical properties (in general anisotropic) and of the isotropic, initially homogeneous pore structure. This set of equations is supplemented by the relation (3.20) describing the solid-fluid interface interaction force.

4. Constitutive relations. Special cases

In this section the nonlinear constitutive relations for the practically important fluid-saturated porous media of simplified mechanical properties are analysed. We consider three particular cases that concern porous solids: of isotropic mechanical properties, of an incompressible skeleton material and the case when porous solid is not saturated with fluid.

4.1. Saturated porous solid with isotropic properties of skeleton

The constitutive relations (3.29), (3.31) and (3.21) will describe elastic properties of the isotropic porous skeleton filled with fluid if all of them are isotropic relations. This can be achieved by imposing the isotropy condition on the relation (3.4) for the skeleton internal energy.

This condition takes the following form, [18],

$$\widehat{e}_k^s(\mathbf{Q}\,\mathbf{C}\,\mathbf{Q}^T, f_v, \kappa) \equiv \widehat{e}_k^s(\mathbf{C}, f_v, \kappa), \tag{4.1}$$

which has to be fulfilled for all orthogonal transformations $\mathbf{Q}$ ($\mathbf{Q}^T = \mathbf{Q}^{-1}$) and all values of independent variables $\mathbf{C}$, f_v, κ. Such requirement imposed on the fluid internal energy (3.1) is satisfied identically.

The condition (4.1) shows that the skeleton internal energy is an invariant of the deformation tensor $\mathbf{C}$ and thus, it can be considered as a function of the invariants of $\mathbf{C}$.

In such case Eq. (3.5) becomes

$$e^s = \widehat{e}_k^s(\mathbf{C}, f_v, \kappa) \equiv \overline{e}_k^s(I_1^C, I_2^C, I_3^C, f_v, \kappa), \tag{4.2}$$

where

$$I_1^C = \operatorname{tr}(\mathbf{C}), \qquad I_2^C = \frac{1}{2}(\operatorname{tr}^2(\mathbf{C}) - \operatorname{tr}(\mathbf{C}^2)), \qquad I_3^C = \det(\mathbf{C}),$$

are the principal invariants of the tensor $\mathbf{C}$.

Relation (4.2) is a general form of the constitutive equation for the internal energy of the skeleton with isotropic mechanical properties.

Isotropy of (4.2), due to the relation (3.19), results in isotropy of Eq. (3.21) and consequently, due to the identity (3.22), leads to the isotropy of the stress-strain relation (3.29) and of the internal equilibrium condition (3.31).

We obtain

$$\kappa = \overline{\kappa}(I_1^B, I_2^B, I_3^B, f_v), \tag{4.3}$$

$$\mathbf{T}^s = 2\overline{\varrho}^s\left\{I_3^B E_3^B \mathbf{I} + (E_1^B + I_1^B E_2^B)\mathbf{B} - E_2^B \mathbf{B}^2\right\}, \tag{4.4}$$

$$p^f = \overline{\varrho}^s \frac{\partial \overline{\overline{e}}_k^s}{\partial f_v}, \tag{4.5}$$

where

$$\overline{\overline{e}}_k^s(I_1^B, I_2^B, I_3^B, f_v) \equiv \overline{e}_k^s(I_1^B, I_2^B, I_3^B, f_v, \overline{\kappa}(I_1^B, I_2^B, I_3^B, f_v)) \tag{4.6}$$

and the right Cauchy-Green deformation tensor $\mathbf{C}$ is replaced with the left deformation tensor

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T,$$

the invariants of which are identical

$$I_\alpha^B = I_\alpha^C, \qquad \alpha = 1,2,3.$$

The quantity E_α^B stands for

$$E_\alpha^B = \frac{\partial \overline{\overline{e}}{}_k^s}{\partial I_\alpha^B}, \qquad \alpha = 1,2,3.$$

Relations (3.15), (4.3)–(4.5) form the set of the constitutive equations describing the mechanical behaviour of the isotropic, elastic porous solid-fluid composition.

Using the relations (3.7) and (3.9) we can derive two other, equivalent sets of constitutive relations in which, instead of the volume porosity f_v, the mass density ϱ^s or Lagrangean porosity f_v^L are used as idependent constitutive variable.

4.2. Saturated porous medium with incompressible skeleton material

Incompressibility of the porous skeleton material is defined by the condition

$$\varrho^s = \varrho_0^s. \tag{4.7}$$

It is a kinematic constraint confining the skeleton motion during its deformation. This condition, at the same time, is the special (trivial) case of the equation describing changes of the skeleton mass density and replaces in this role the condition of internal equilibrium for the solid-fluid composition.

Taking (4.7) into account, the skeleton continuity equation (3.9) reduces to the relation

$$f_v = 1 - (1 - f_v^0)/J \tag{4.8}$$

that uniquely defines the volume porosity changes by means of the skeleton deformation gradient $\mathbf{F}$.

The assumption (4.7) and relation (4.8) eliminate the density ϱ^s and porosity f_v or f_v^L from the set of independent variables describing the internal energy of the skeleton. We have

$$e^s = \overset{*}{e}{}_k^s(\mathbf{C}, \kappa). \tag{4.9}$$

Now, requiring the balance equation of the internal energy (2.5) to be identically satisfied by relations (3.1) and (4.9), one can find the constitutive equations for the interface interaction force and the effective fluid stresses identical with the relations (3.20) and (3.15), respectively. At the same time, the function of the

κ-parameter variation and the constitutive relation for the skeleton effective stress take forms similar to (3.19) and (3.37), respectively, i.e.

$$\frac{\partial \overset{*}{e}{}^{s}_{k}}{\partial \kappa} = 0, \tag{4.10}$$

$$\mathbf{T}^{*s} = -p^{f}\mathbf{I} + 2\varrho_0^s \mathbf{F}\frac{\partial \tilde{\tilde{e}}{}^{s}_{k}}{\partial \mathbf{C}}\mathbf{F}^{T}, \tag{4.11}$$

where

$$\tilde{\tilde{e}}{}^{s}_{k}(\mathbf{C}) = \overset{*}{e}{}^{s}_{k}(\mathbf{C}, \tilde{\tilde{\kappa}}(\mathbf{C}))$$

and

$$\kappa = \tilde{\tilde{\kappa}}(\mathbf{C})$$

is the explicit form of the relation (4.10) defined by the identity

$$\frac{\partial \overset{*}{e}{}^{s}_{k}}{\partial \kappa}(\mathbf{C}, \kappa)\Big|_{\kappa = \tilde{\tilde{\kappa}}(\mathbf{C})} \equiv 0.$$

The term in (4.11), related to the fluid pressure p^f represents the stresses in the skeleton caused by the presence of the fluid in pores. These stresses contribute to the pore fluid energy during the skeleton deformation. However, due to incompressibility of the skeleton material, they do not influence the skeleton internal energy.

It should be pointed out that the set of constitutive relations mentioned above does not contain the condition of internal, mechanical equilibrium for the considered solid-fluid composition. This results from the fact that the skeleton mass density has been excluded from the set of independent variables.

4.3. The non-saturated porous solid

Constitutive description of non-saturated, elastic porous solids can be obtained from the constitutive relation (3.37) and the internal equilibrium condition (3.35) through the assumption that the effective fluid pressure is equal to zero ($p^f = 0$). In such case, we have

$$\mathbf{T}^{*s} = 2\varrho^s \mathbf{F}\frac{\partial \tilde{e}{}^{s\varrho}_{k}}{\partial \mathbf{C}}\mathbf{F}^{t}, \tag{4.12}$$

$$\frac{\partial \tilde{e}{}^{s\varrho}_{k}}{\partial \varrho^s} = 0. \tag{4.13}$$

The form of Eq. (4.13) is similar to that of (3.19). Therefore, we conclude that during a deformation process of elastic, non-saturated porous solid, the skeleton mass density ϱ^s takes values for which the skeleton internal energy has a minimum. This additionally justifies our interpretation of (3.31) as the internal equilibrium condition between the pore fluid and skeleton.

If $\tilde{e}_k^{s\varrho}$ is a smooth function, equation (4.13) can be rewritten in the form

$$\varrho^s = \tilde{\varrho}^s(\mathbf{C}) \tag{4.14}$$

that explicitly describes the skeleton mass density changes. Then the identity

$$\frac{\partial \hat{e}_k^{s\varrho}}{\partial \varrho^s}(\mathbf{C}, \varrho^s)\Big|_{\varrho_s = \tilde{\varrho}^s(\mathbf{C})} \equiv 0$$

is satisfied.

Introducing (4.14) into the skeleton the mass continuity equation (3.9) we obtain the equation for the changes of volume porosity

$$f_v = 1 - \overline{\varrho}_0^s / J\tilde{\varrho}^s(\mathbf{C}). \tag{4.15}$$

Moreover, the equation (4.14) eliminates the skeleton mass density from the set of independent variables describing the internal energy of the skeleton. We have

$$e^s = \tilde{e}_k^s(\mathbf{C}) \equiv \tilde{e}_k^{s\varrho}(\mathbf{C}, \tilde{\varrho}^s(\mathbf{C})). \tag{4.16}$$

Thus, the constitutive relation (4.12), when (4.13) is taken into account, reduces to the form

$$\mathbf{T}^{*s} = 2\varrho^s \mathbf{F} \frac{\partial \tilde{e}_k^s}{\partial \mathbf{C}} \mathbf{F}^T, \tag{4.17}$$

similar to that for a non-porous solid.

5. Final remarks

Nonlinear constitutive relations for the fluid-saturated porous solid immiscible mixture undergoing pure mechanical large deformations have been developed.

Considerations have been based on the balance equation for the internal energy of the whole composition which was required to be satisfied identically by the internal energy constitutive functions postulated independently for individual components.

General constitutive relations for partial stresses in an anisotropic, elastic skeleton and barotropic fluid have been formulated and the internal equilibrium condition for the composition has been established. This condition relates the pore fluid pressure to independent variables describing the state of porous skeleton and it does not appear in the constitutive description when the skeleton material is incompressible. Also the constitutive relations for the medium with simplified physical properties have been discussed.

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