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Singularities of aerodynamic transfer functions calculated on the basis of an unsteady lifting surface model in subsonic flow

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A DIRECT METHOD to calculate the Laplace transformed pressure distribution on subsonic lifting surfaces is considered. The kernel function is analytically continued in the entire p-plane (of the non-dimensional Laplace variable), and the discretizing procedure follows the lifting lines (or doublet-lattice) method developed for simple harmonic motion. The aerodynamic influence coefficient matrix is a function of Mach number M and the complex variable p. In the first part of the paper, some analytical properties of this matrix were investigated on the basis of numerical calculations performed for an aspect-ratio-3 rectangular wing. The main conclusion of this paper is that for $M \neq 0$, there exist a large (probably infinite) set of latent roots of the matrix in the left half of the p-plane which (usually) reflect in poles of the transfer functions. For $M \to 1$, all latent roots tend to the origin ($p = 0$). For $M \to 0$, all latent roots move to infinity and probably, for $M = 0$ there are no roots in the finite part of the p-plane. The distribution of latent roots in the p-plane does not depend on the number of aerodynamic elements introduced by the discretization (within the limits of accuracy of the calculation method). The algebraic equations are well-conditioned in the right half of the p-plane and in a strip parallel to the imaginary axis in left half of the p-plane. The width of this strip depends on the Mach number. In the second part of the paper, an approximation to the aerodynamic transfer functions based on the identified singularities and calculated left and right-hand latent vectors of the aerodynamic influence coefficients matrix is developed. It avoids the ill-posed analytical continuation from the imaginary axis in the whole p-plane. The results clarify also some unexpected phenomena observed in Laplace-domain calculations, and described in the literature.

1. Introduction

THE KNOWLEDGE of unsteady aerodynamic forces acting on a flexible aeroplane undergoing small perturbations from a steady equilibrium state of trimmed, rectilinear flight, is essential for stability analyses of the motion of the structure. The prediction of the unsteady aerodynamic loads is complicated by the fact that the unsteady flowfield surrounding the body is not determined solely by the instantaneous state variables of the structure, but it depends also on the past history of the motion of the body. The aerodynamic forces exhibit heredity due to the influence of vorticity shed into the wake at earlier instants of time.

The input data in a lifting surface aerodynamic model is the upwash distribution $w(x, y, t)$ on the wing surface S (Fig. 1). Assuming that all linear coordinates (x, y, z) are nondimensionalized by a reference length b (usual root semichord), and introducing nondimensional time t

$$t = \frac{U \cdot t_{\text{real}}}{b}, \tag{1.1}$$

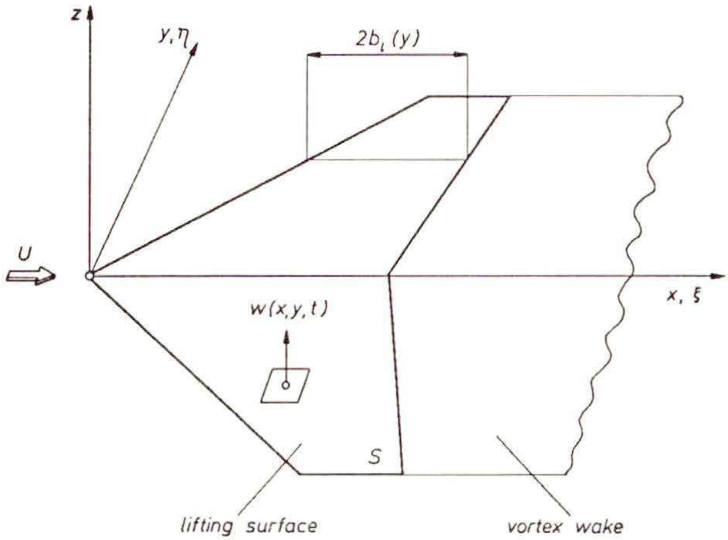

FIG. 1.

where U is the flight velocity, the expression for the upwash distribution can be put in the form

$$\frac{w(x,y,t)}{U} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}, \tag{1.2}$$

where $h(x, y, t)$ denotes the normal (nondimensionalized) displacements of the wing surface.

The lifting surface integral equation relates upwash and lifting pressure coefficient $c_p(x, y, t)$ (i.e. pressure difference $\Delta p(x, y, t)$ between the upper and lower surface, nondimensionalized by the dynamic pressure $\varrho U^2/2$) on the wing. The original form of the lifting surface equation, given in 1940 by KÜSSNER [1] applies to harmonic motion, when

$$w(x, y, t) = \widehat{w}(x, y, ik)e^{ikt} \qquad \text{and} \qquad c_p(x, y, t) = \widehat{c}_p(x, y, ik)e^{ikt}.$$

The lifting surface equation relates in this case the amplitudes of upwash and pressure coefficient

$$\frac{\widehat{w}(x,y,ik)}{U} = \frac{1}{8\pi} \iint\limits_S K(M, x_0, y_0, ik)\widehat{c}_p(\xi, \eta, ik)\, d\xi\, d\eta, \tag{1.3}$$

where $x_0 = x - \xi$, $y_0 = y - \eta$, M stands for the Mach number and

$$k = \frac{\omega b}{U} \tag{1.4}$$

is the nondimensional frequency coefficient (called also reduced frequency).

The kernel of this equation is singular and the solution is sought in the class of functions vanishing on the trailing edge – this is a necessary condition for the uniqueness of the solution, and physically it expresses the Kutta condition.

There were developed many different methods for discretization of the lifting surface equation in the frequency domain. One of the most useful is the doublet-lattice technique of ALBANO and RODDEN [2]. The calculations in this paper were made mainly by the lifting-lines method [3, 4], with algorithms very similar to the doublet-lattice method, but usually with better convergence properties (with respect to the number of introduced aerodynamic elements).

For many years, the unsteady aerodynamic theories and its applications have focused primarily on the frequency domain, since the knowledge of aerodynamic forces at harmonic disturbances is sufficient for the determination of flutter boundaries. The advent of active control technology for flexible aircraft has renewed interest in unsteady aerodynamic forces given in the time and Laplace domains.

The displacements of the structure are usually described by means of a finite set of generalized coordinates $q_1(t)$, $q_2(t)$, $\ldots$, $q_n(t)$ defined on the basis of a set of assumed modes

$$h(x,y,t) = \sum_{k=1}^{n} h_k(x,y) \cdot q_k(t), \tag{1.5}$$

where the functions $h_k(x,y)$ $(k = 1, 2, \ldots, n)$ correspond in the most cases to natural vibration modes of the structure. The upwash distribution on the lifting surface may be expressed in terms of the generalized coordinates and generalized velocities

$$\frac{w(x,y,t)}{U} = \sum_{k=1}^{n} \frac{\partial h_k(x,y)}{\partial x} \cdot q_k(t) + \sum_{k=1}^{n} h_k(x,y) \cdot \dot{q}_k(t). \tag{1.6}$$

The generalized aerodynamic forces (related to the dynamic pressure and b^2) are defined by integrals taken over the surface

$$f_k(t) = \iint_S h_k(x,y) c_p(x,y,t)\, dS \qquad \text{for} \qquad k = 1, 2, \ldots, n. \tag{1.7}$$

The problem consist in determination of the generalized force vector $\{f(t)\}$ (with n elements (1.7)) for a given motion, described by the function $\{q(\tau)\}$ for $-\infty < \tau \leq t$. Independently of the details of the aerodynamic model, the aerodynamic operator which relates $\{q(t)\}$ to $\{f(t)\}$ possesses always some basic properties, such as single-valuedness, linearity, time-invariance and continuity. According to a theorem of SCHWARTZ [6], these four properties can be replaced by the entirely equivalent condition that states that this operator has a (distributional) convolution representation.

$$\{f(t)\} = [A(M,t)] * \{q(t)\}, \tag{1.8}$$

where $[A(M,t)]$ is the unit impulse response matrix function (called also hereditary matrix [5]), the (j,k) element of which is the generalized indicial response in the j-th mode due to the pressure $c_p(x,y,t)$ generated by the motion in the k-th mode with $q_k(t) = \delta(t)$. The elements of this matrix depend also on the Mach number M. The aerodynamic forces can depend only on the history and not on the future of the motion. That means that the aerodynamic system is causal, and therefore

$$[A(M,t)] = 0 \qquad \text{for} \qquad t < 0.$$

Direct calculation of the elements of $[A(M,t)]$ for arbitrary time may be difficult and in practice, these functions are usually determined only by means of the inversion of Fourier or Laplace transform. Taking the Laplace transformation of the convolution (1.8) it follows that

$$\left\{\widehat{f}(p)\right\} = \left[\widehat{A}(M.p)\right]\{\widehat{q}(p)\}, \tag{1.9}$$

where p is the Laplace variable, and the circumflex accents ($\widehat{\ }$) denote transforms

$$\left\{\widehat{f}(p)\right\} = \mathcal{L}\{f(t)\}, \quad \{\widehat{q}(p)\} = \mathcal{L}\{q(t)\} \quad \text{and} \quad \left[\widehat{A}(M,p)\right] = \mathcal{L}[A(M,t)]. \tag{1.10}$$

The aerodynamic transfer functions matrix $[\widehat{A}(M,p)]$ is a Laplace transform of a real distribution and is real whenever p is real. Hence

$$\left[\widehat{A}(M,p)\right]^* = \left[\widehat{A}(M,p^*)\right], \tag{1.11}$$

where the star (*) denotes complex conjugate values.

The convolution (1.8) and the Laplace transformation should be interpreted on the basis of the theory of distributions [6]. The aerodynamic transfer functions grow with increasing $|p|$ like $O(|p|)$ in the case of compressible flow, and like $O(|p|^2)$ in the incompressible case. Additionally, the distributional Laplace transform does not contain explicitly the initial values and this simplifies the analysis.

If the Laplace variable is pure imaginary $p = ik$, then (1.9) determines the steady-state frequency response function, which relates the amplitudes of generalized coordinates to the amplitudes of generalized forces in harmonic motion.

$$\left\{\widehat{f}(ik)\right\} = \left[\widehat{A}(M,ik)\right]\{\widehat{q}(ik)\}, \tag{1.12}$$

where k is the frequency coefficient defined in (1.4), and

$$\left[\widehat{A}(M,ik)\right] = \lim_{p\to ik}\left[\widehat{A}(M,p)\right]$$

is the matrix of harmonic transfer functions.

The elements of the matrix $[\widehat{A}(M,ik)]$ can be calculated numerically for given values M and k on the basis of the lifting surface equation (1.3), and the relations (1.6) and (1.7).

The aerodynamic transfer matrix $[\widehat{A}(M,p)]$ is the final product of aerodynamic calculations and it is usually determined by means of the analytic continuation of the elements of matrix $[\widehat{A}(M,ik)]$ from the imaginary axis into the whole complex plane. Two types of approximation have been used in practice for this purpose.

1. The first approach begins with calculating the values of harmonic transfer functions over a specified range of the frequency coefficients $k = k_1, k_2, \ldots, k_m$. Next, the harmonic transfer functions are approximated by rational function which fit best the calculated values. The last step is the analytic continuation of the resulting rational functions into the whole p-plane.

2. In the second (direct) approach, the kernel function of the lifting surface equation (1.3) is extended from the imaginary axis to the entire complex plane $K(M,x_0,y_0,ik) \to K(M,x_0,y_0,p)$ by means of an exact analytic continuation. The elements of the aerodynamic transfer functions are calculated directly for a given value of the Laplace variable p on the basis of this generalized lifting surface equation.

Both approaches have their own advantages and disadvantages. The elements of $[\widehat{A}(M,p)]$ are holomorphic functions with branch points $p = 0$ and $p = -\infty$ (for $M < 1$) which are neglected in the approximation by rational functions. When the transfer functions are approximated by polynomials or rational functions, it is possible to cast the aeroelastic (and aeroservoelastic) equations of motion in a linear time-invariant state-space form (instead of integro - differential form), although the size of the state vector increases due to the approximation. Currently there are three basic formulations used in approximating the aerodynamic transfer functions by means of rational functions: least-squares [7], modified matrix-Padé [8] and minimum-state [9]. The common disadvantage of these methods is the necessity of numerical realisation of an ill-posed analytic continuation.

The direct analytic continuation of functions which appear in the expression for harmonic aerodynamic forces gave rise in the past to arguments on the validity of the results in the left-hand half-plane of the Laplace variable and was rejected in a series of articles [10, 11, 12, 13]. This problem was later resolved by MILNE [14], EDWARDS [15] and others, but nowadays some doubts arose about the possibility of a practical utilisation of this approach [16]. It was stated, that the application of numerical solution techniques to the integral equation in the left half of the p-plane may result in a highly ill-conditioned set of linear equations

[17]. An unexpected phenomenon was observed in [18]. Some of the generalized forces for strongly decaying motions in high subsonic flow reveal a looping behaviour in the complex plane. Testing a new method for solving the lifting surface equation, UEDA [19] stated that "it looks probable that an aerodynamic singularity exists in the left half of the p plane when the flow becomes high subsonic".

Another interesting approach to the approximation of transfer functions was given by STARK [20]. He proposed an expression for the lift deficiency function in the time domain and assumed that this function is independent of the deflection mode of the wing. Laplace transform of his deficiency function possesses branch points $p = 0$ and $p = -\infty$, which are the only singularities of the transfer functions in the entire p plane. This approach leads to a good approximation in the incompressible case, but for non-zero Mach numbers the results were less satisfactory.

The knowledge of analytic properties of aerodynamic transfer functions in the p plane is until now only fragmentary. It is known that the matrix $[\widehat{A}(M,p)]$ cannot have any poles in the right half of the p plane, since the transient aerodynamic response is always stable. It is also known, that for a subsonic flow, the aerodynamic transfer functions have logarithmic branch points $p = 0$ and $p = -\infty$, as a result of the unlimited length of the wake. It is usually expected, that the aerodynamic transfer functions have no poles also in the left half of the p plane [5]. This is true for the exact solution of two-dimensional airfoil in incompressible flow but was newer proved for the compressible case. The problem is additionally complicated by the fact, that the solution of the singular integral equation is ill-posed, but the numerical methods used in the chordwise integration introduce a self-regularization and, after discretization, the resulting set of algebraic linear equations is usually well-conditioned. In the two-dimensional case (of an airfoil) the proof of this statement was given by LIFANOV [21].

The aim of this paper is to investigate the numerical problems which occur in solving the lifting surface equation in the Laplace domain and the analytical properties of the transfer functions in the left half of the p plane (for decaying motion). Particular attention will be paid to the conditioning of the linear algebraic equations obtained by the discretization of the lifting surface equation, and to the identification of singularities of the transfer functions.

2. Lifting surface equation in the Laplace domain

The lifting surface equation in the Laplace domain is the result of an analytic continuation of the kernel of (1.3). Formally, the variable ik should be replaced by the Laplace variable

$$\frac{\widehat{w}(x,y,p)}{U} = \frac{1}{8\pi}\iint\limits_S K(M,x_0,y_0,p)\widehat{c}_p(\xi,\eta,p)\,d\xi\,d\eta. \tag{2.1}$$

The singular kernel (in the case of a flat surface) may be expressed in the form [4, 18]

$$K(M, x_0, y_0, p) = \frac{1}{r^2}\left[\left(1 + \frac{x_0}{R}\right) e^{-pru} - F(pr, u)\right] e^{-px_0}, \tag{2.2}$$

where

$$\begin{aligned} x_0 &= x - \xi, \\ y_0 &= y - \eta, \\ r &= |y_0|, \\ R &= \sqrt{x_0^2 + \beta^2 r^2}, \\ \beta &= \sqrt{1 - M^2}, \\ u &= \frac{MR - x_0}{\beta^2 r}. \end{aligned}$$

The function $F(z, u)$ is defined by means of following integrals

$$F(z, u) = \begin{cases} z \int\limits_u^\infty \left(1 - \dfrac{\eta}{\sqrt{1 + \eta^2}}\right) e^{-z\eta}\, d\eta & \text{for} \quad \mathrm{Re}\,(z) \geq 0, \\ 2e^{zu} + z \int\limits_{-\infty}^u \left(1 + \dfrac{\eta}{\sqrt{1 + \eta^2}}\right) e^{-z\eta}\, d\eta - i\pi z H_1^{(1)}(z) & \\ & \text{for} \quad \mathrm{Re}\,(z) < 0. \end{cases} \tag{2.3}$$

The second expression (for $\mathrm{Re}\,(z) < 0$) may be obtained from the first integral in (2.3), by an appropriate contour deformation.

Only a few papers (e.g. [18, 19, 22]) are known which are devoted to the numerical problems which occur in lifting surface calculations in the Laplace domain.

For small values of $|pru|$ it is convenient to split $F(pr, u)$ into two parts [18]

$$F(z, u) = F_1(z) - F_2(z, u), \tag{2.4}$$

where

$$F_1(z) = z \int\limits_0^\infty \left(1 - \frac{\eta}{\sqrt{1 + \eta^2}}\right) e^{-z\eta}\, d\eta = 1 + z - \frac{\pi}{2} z (H_1(z) - Y_1(z)), \tag{2.5}$$

and the integral $F_2(z, u)$ defines an entire analytic function of the z variable, which may be expanded into a convergent series

$$F_2(z, u) = z \int\limits_0^u \left(1 - \frac{\eta}{\sqrt{1 + \eta^2}}\right) e^{-z\eta}\, d\eta = \sum_{k=0}^\infty \frac{1}{(k + 1)!} g_k(u) z^{k+1}, \tag{2.6}$$

with a recursive relationship for the coefficients

$$g_k(u) = \left(u\left(\sqrt{1+u^2}-u\right) - \frac{k}{k-1}\right)(-u)^{k-1} - \frac{k}{k-1} g_{k-2}(u) \qquad \text{for} \quad k \geq 2$$

and initial terms

$$\begin{aligned} g_0(u) &= 1 - \left(\sqrt{1+u^2}-u\right), \\ g_1(u) &= \ln\left(\sqrt{1+u^2}-u\right) + u\left(\sqrt{1+u^2}-u\right). \end{aligned}$$

The Hankel function $H_1^{(1)}(z)$, Struve function $H_1(z)$ and Bessel function of the second kind $Y_1(z)$ may be calculated with high accuracy on the basis of the series given e.g. in [23]. The accuracy of the series (2.6) is limited by the numerical instability due to the round-off error in actual computation. For calculations performed with double precision, this limit depends on the values of parameters and sufficient accuracy can be achieved only if $|pru| < 6$. For larger values of parameters it is necessary to provide other approximations.

For very large values of $|pr|$ satisfactory results may be obtained from the asymptotic expansion derived by means of integrating by parts the integrals in (2.3)

$$(2.7) \qquad F(z,u) \approx \left(1 - f(u) - \sum_{k=1} f^{(k)}(u)\frac{1}{z^k}\right) e^{-zu} - \begin{cases} 0 & \text{for} \quad \mathrm{Re}\,(z) \geq 0, \\ i\pi z H_1^{(1)}(z) & \text{for} \quad \mathrm{Re}\,(z) < 0, \end{cases}$$

where

$$f^{(0)}(u) = f(u) = \frac{u}{\sqrt{1+u^2}}, \qquad f^{(-1)}(u) = \sqrt{1+u^2},$$

and

$$f^{(k+1)}(u) = -\frac{1}{1+u^2}\left[(2k+1)uf^{(k)}(u) + (k+1)(k-1)f^{(k-1)}(u)\right].$$

The asymptotic series (2.7) is usually divergent and only a limited number of terms can be employed in the calculations.

Very useful in practice is an exponential approximation for the integrands of (2.3)

$$1 - \frac{\eta}{\sqrt{1+\eta^2}} \approx \sum_{k=1}^{12} a_k \exp(-2^k b_0 \eta),$$

proposed by Desmarais. The values of coefficients b_0 and a_k are given in [18]. The resulting rational approximation of $F(pr, u)$ may be used in the range $\pi/4 < |\arg(p)| < 3\pi/4$ and $-\infty < u < \infty$.

3. Discretization of the lifting surface equation

It is possible, independently of the used discretization procedure, to distinguish three steps of calculations of the values of transfer functions for a structure with n degrees of freedom (for a given Mach number M and value of the Laplace variable p):

a. Calculation of the substantial derivative to obtain a N-dimensional approximation of the upwash distribution

$$\underset{N\times 1}{\{\widehat{w}(p)\}} = \Big(\underset{N\times n}{[D_1]} + p\,\underset{N\times n}{[D_2]}\Big)\underset{N\times 1}{\{\widehat{q}(p)\}};\tag{3.1}$$

b. Solution of a linear system of algebraic equations

$$\underset{N\times 1}{\{\widehat{w}(p)\}} = \underset{N\times N}{[K(M,p)]}\,\underset{N\times 1}{\{\widehat{c}_p(p)\}};\tag{3.2}$$

c. Determination of the transforms of the generalized coordinates

$$\underset{N\times 1}{\Big\{\widehat{f}(p)\Big\}} = \underset{n\times N}{[S]}\,\underset{N\times 1}{\{\widehat{c}_p(p)\}}.\tag{3.3}$$

N is the size of the aerodynamic influence coefficients matrix which approximates the integral operator. The vectors $\{\widehat{w}(p)\}$ and $\{\widehat{c}_p(p)\}$ describe the upwash and pressure distributions on the wing surface. In practice, typical values are: $n = 20 \div 30$ and $N \sim$ some hundreds (but always $N \gg n$). The differentiation matrices $[D_1]$, $[D_2]$ are determined by the formula (1.6), and the integration matrix $[S]$ by the definition of the generalized forces (1.7). These constant matrices depend only on the used discretization method. Matrix $[K(M,p)]$ depends also on the Mach number and on the assumed value of p. The evaluation of this matrix is the most time-consuming part of the computation.

Equations (3.2), (3.3) and (3.4) may be put together in the form

$$\underset{n\times 1}{\Big\{\widehat{f}(p)\Big\}} = \underset{n\times N}{[S]}\,\underset{N\times N}{[K(M,p)]^{-1}}\Big(\underset{N\times n}{[D_1]} + p\,\underset{N\times n}{[D_2]}\Big)\underset{N\times 1}{\{\widehat{q}(p)\}}.$$

Hence, the aerodynamic transfer functions matrix is given by the formula

$$\underset{n\times n}{\Big\{\widehat{A}(M,p)\Big\}} = \underset{n\times N}{[S]}\,\underset{N\times N}{[K(M,p)]^{-1}}\Big(\underset{N\times n}{[D_1]} + p\,\underset{N\times n}{[D_2]}\Big).\tag{3.4}$$

If the discretization procedure in the Laplace domain is the same as in the frequency domain (when $p = ik$), then the matrix (3.4) is the result of an exact analytical continuation of the harmonic transfer matrix $[\widehat{A}(M, ik)]$. On the other hand, if the calculations for a harmonic motion are based on the discretized equation (1.3), then the values of analytic functions determined in the entire complex plane p are in fact calculated on the imaginary axis. Therefore, the knowledge of analytic properties of the transfer functions may be useful also in the case, when the calculations are restricted to the imaginary axis only.

In the case of subsonic flow ($M < 1$), the elements of $[K(M,p)]$ have a branch-point in the origin ($p = 0$) and from the expression for the kernel function, it follows that

$$[K(M,p)] = [K(M,0)] + O(p^2 \ln(p)) \qquad \text{for} \quad p \to 0.$$

The transfer functions are holomorphic functions in the complex plane cut along the negative real axis.

Poles of the transfer matrix $[\widehat{A}(M,p)]$ may exist only in those points of the p plane, where the matrix $[K(M,p)]$ is singular

$$\det\Big([K(M,p)]\Big) = 0. \tag{3.5}$$

The number of latent roots of the equation (3.5) may be large or infinite, because the elements of the matrix $[K(M,p)]$ are transcendental functions of p.

4. Condition number and latent roots of the aerodynamic influence coefficients matrix

Most of the calculations in the following analysis were performed for a rectangular wing with an aspect ratio $\Lambda = 3$ in symmetric motion. This wing was also investigated in [20] and [22]. For the discretization, the lifting lines method [4] was used, but some of the calculations were repeated with the doublet-lattice method [2] (with the same or almost the same results).

The sensitivity of the solution of (3.2) to the perturbation of the data

$$\frac{\|\Delta \widehat{c}_p(p)\|}{\|\widehat{c}_p(p)\|} \le \operatorname{cond}[K(M,p)] \frac{\|\Delta \widehat{w}(p)\|}{\|\widehat{w}(p)\|},$$

may be measured by the condition number of $[K(M,p)]$ defined as the product of two matrix norms

$$\operatorname{cond}[K(M,p)] = \|[K(M,p)]\| \cdot \left\|[K(M,p)]^{-1}\right\| \qquad (1 \le \operatorname{cond} \le \infty). \tag{4.1}$$

Logarithm to the base 10 of the condition number can be used to estimate the number of significant digits of the result which can be lost, independently of the accuracy of the method used to solve the linear equations. Hence, if the calculations are performed with double precision, then the matrix is numerical singular when $\log_{10}\left(\mathrm{cond}\,[K(M,p)]\right) > 16$.

The conditions numbers of the matrix $[K(M,p)]$ were calculated for different Mach numbers and for a large set of p-values by means of the SVD algorithms [24] for complex matrices.

Figure 2 shows the results of calculations made for Mach number $M = 0.8$ in a large region $-1.75 \leq \mathrm{Re}\,(p) \leq 0$, $0 \leq \mathrm{Im}\,(p) \leq 3.5$ of the complex p-plane. The size of the aerodynamic model was $N = 10 \times 20 = 200$ elements (10 lifting lines and 20 strips uniformly distributed on the half-span of the wing). It is seen, that the matrix in this region is well-conditioned, although for $\mathrm{Re}\,(p) < -1.0$ the condition number grows very fast. There are also many local "spikes" which may indicate, that in its neighbourhood exist singular points of the matrix $[K(M,p)]$.

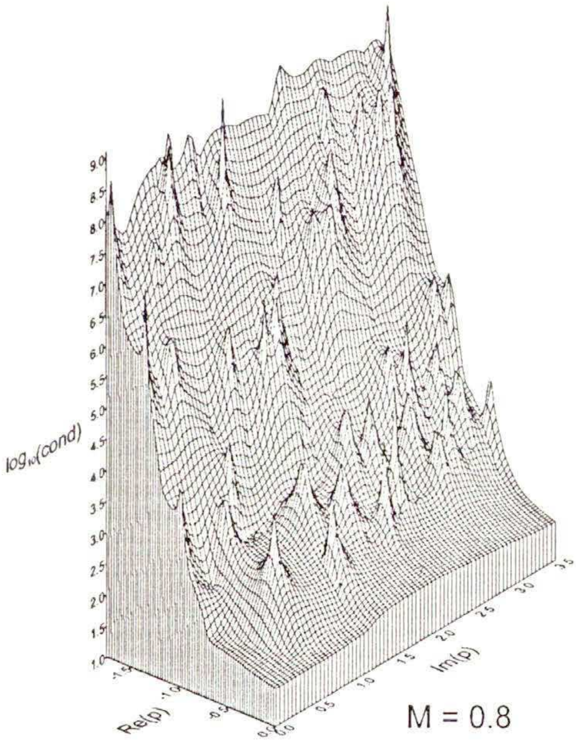

FIG. 2.

Figure 3 shows the same results in the form of a contour map. The latent roots were also calculated on the basis of Eq. (3.5) by means of the Muller method [25]. The results of these calculations are posted on the contour map in the form of black dots. In each of this calculated points $\log_{10}(\mathrm{cond}\,[K(M,p)]) > 16$, hence the matrix is numerically singular. The initial values for the Muller iteration procedure were determined on the basis of the shape of contour lines. The condition

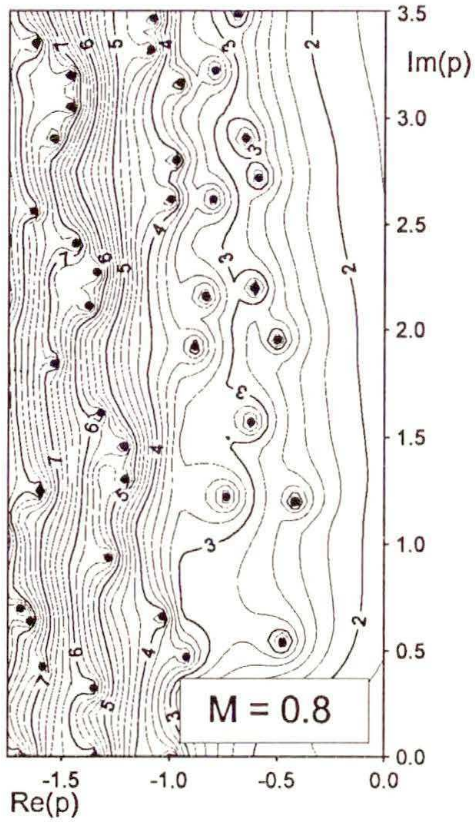

FIG. 3.

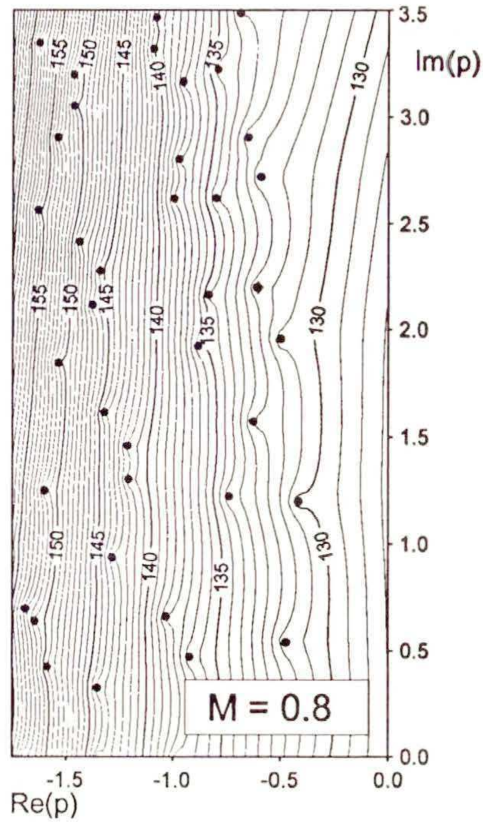

FIG. 4

number grows rapidly only in the vicinity of each root. In a very small region ($|\Delta p| < 10^{-8}$), the determinant of the matrix decreases usually by a factor about 10^{-10}, although its value may be still very large. This is shown in the Fig. 4, where the contour lines correspond to constant values of $\log_{10} |\det[K(M,p)]|$.

The singular points exist for each Mach number in the range $0 < M < 1$. The root distributions at Mach numbers $M = 0.5$, 0.7, 0.9 and 0.95 are shown in Fig. 5. It is seen that, as the Mach number increases, the width of the strip in the left half of the p plane where the matrix $[K(M,p)]$ is well-conditioned, decreases. At the same time, all latent roots move in the direction to the origin. This phenomenon is shown in Fig. 6 where the loci of about 20 selected roots are depicted. The outer ends of these curves correspond to the Mach number $M = 0.5$, and the inner ends to $M = 0.9$.

On the basis of Fig. 5 and Fig. 6 it is possible to formulate a hypothesis that for $M \to 0$, all roots move to infinity and in the incompressible case $M = 0$, there are no roots in the finite part of the plane $|p| < \infty$. On the other side, for $M \to 1$, all roots move to the origin and may significantly influence the behaviour of transfer functions at high subsonic Mach numbers.

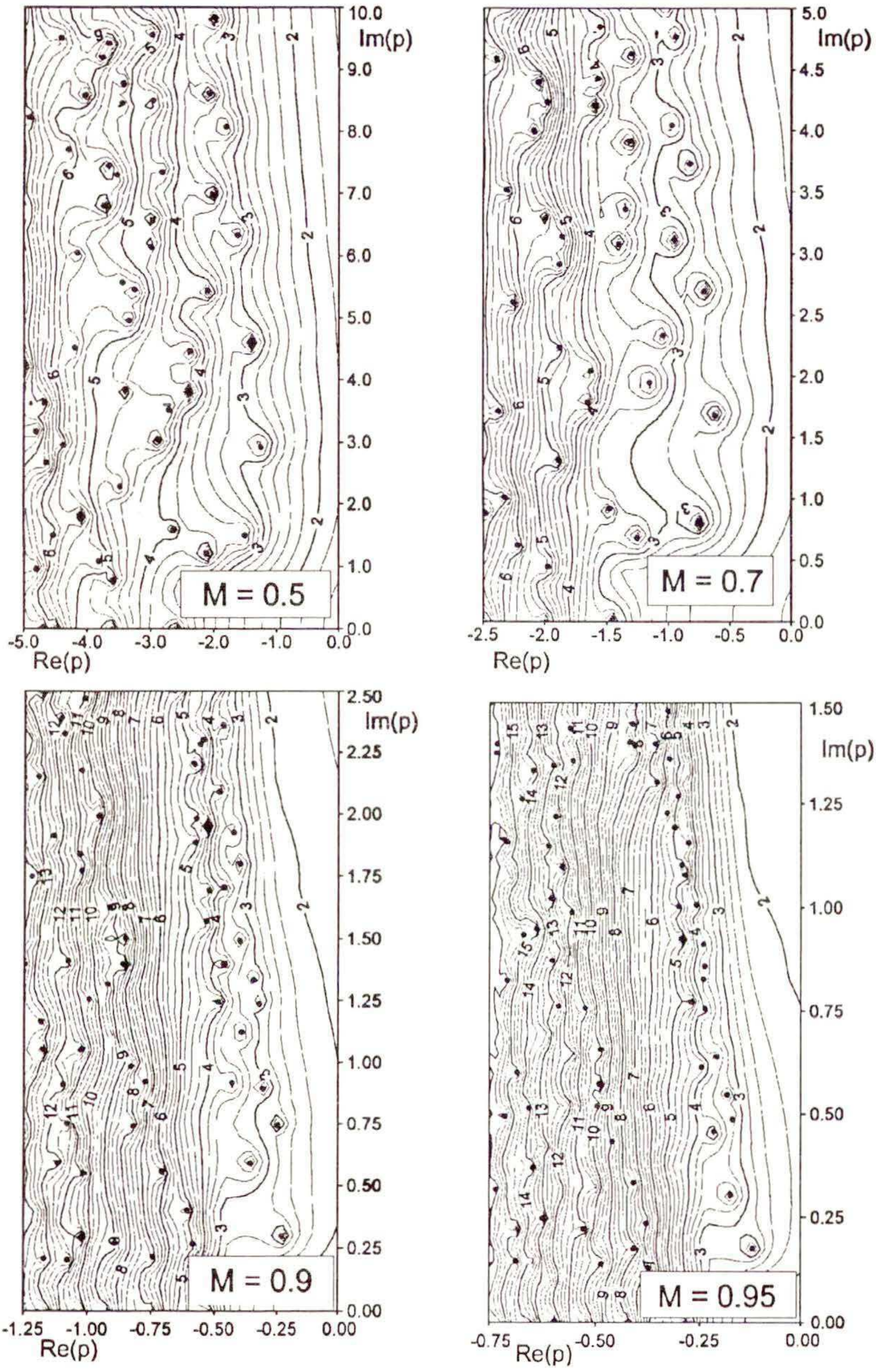

FIG. 5.

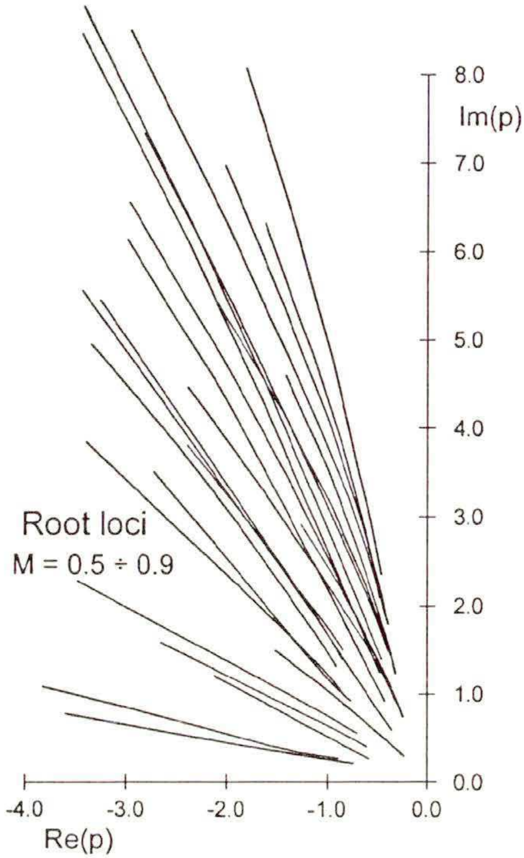

FIG. 6.

It has been found in the example of a rectangular wing, but also for other surface configurations, that the calculated roots of Eq. (3.5) were always simple roots only.

5. The influence of the discretization on the distribution of latent roots

It is not clear if the roots have a physical meaning and are related to the lifting surface equation or if they occur only in numerical calculations and are related to the discretized problem.

Figure 7 shows the influence of the size ($N = 48 \div 437$) of the matrix $[K(M, p)]$ on the distribution of latent roots in the p plane. The calculations were made by means of the lifting lines method, for a rectangular wing, at Mach number $M = 0.8$. It is seen that the differences may be related to the accuracy which may be achieved with the different models. For large values of the frequency coefficient, the pressure distribution is oscillating along the chord (Kutta waves)

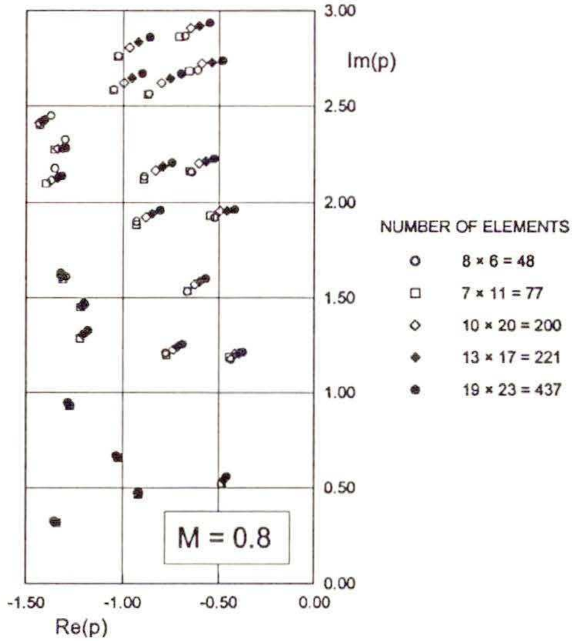

FIG. 7.

and a large number of aerodynamic elements is required at the discretization. It follows, that the number and distribution of latent roots do not depend on the size of matrix $[K(M,p)]$ within the limits of accuracy of the used method.

The most time-consuming part of the procedure to calculate latent roots is the search for a good initial approximation. The results presented in Fig. 7 suggest a practical approach, which may be applied for an arbitrary large N. The process should be divided into a sequence of steps, in which the number of aerodynamic elements increases $N_1 < N_2 < \ldots < N$. The results obtained in each step are used as the initial values for the next step. The choice of the initial approximations for the first step may be not strenuous if N_1 is small enough.

6. Approximation to the transfer matrix in the vicinity of its poles

The resolvent [26] of the matrix $[K(M,p)]$ for a given p has the form

$$\left([K(M,p)] - \lambda[I]\right)^{-1} = \sum_{j=1}^{N} \frac{\{u_j(p)\}\{v_j(p)\}^T}{\lambda_j(p) - \lambda}, \tag{6.1}$$

where the scalar parameter λ is distinct from the eigenvalues $\lambda_j(p)$, $j = 1, 2, ..., N$ of $[K(M,p)]$, while $\{u_j(p)\}$ and $\{v_j(p)\}$ are the right and left eigenvectors associated with $\lambda_j(p)$, and normalised in such a way, that $\{v_j(p)\}^T\{u_j(p)\} = 1$. The relation (6.1) is true also for $\lambda = 0$, because the matrix $[K(M,0)]$ is not singular.

The derivative of an eigenvalue of $[K(M,p)]$ is given by the expression

$$\frac{d}{dp}\lambda_j(p) = \{v_j(p)\}^T \left(\frac{\partial}{\partial p}[K(M,p)]\right)\{u_j(p)\}. \tag{6.2}$$

If the value $p = p_k$ is a latent root of (3.5), then at least one of the eigenvalues $\lambda_j(p_k)$, $j = 1, 2, ..., N$ is equal to zero and in the vicinity of p_k

$$\lambda_j(p) \approx \frac{d}{dp}\lambda_j(p)\bigg|_{p=p_k} \cdot (p - p_k). \tag{6.3}$$

On the basis of (6.1), (6.2) and (6.3) it is possible to obtain an approximation to the matrix inverse $[K(M,p)]^{-1}$ in the proximity of the root $p = p_k$

$$[K(M,p)]^{-1} = \frac{\{u_k\}\{v_k\}^T}{p - p_k} + [\Phi(p)], \tag{6.4}$$

where $[\Phi(p)]$ is a regular function in the vicinity of p_k, while the latent vectors $\{u_k\}$ and $\{v_k\}$ are non-trivial solutions of the sets of homogeneous equations

$$\{v_k\}^T[K(M,p_k)] = 0 \qquad \text{and} \qquad [K(M,p_k)]\{u_k\} = 0, \tag{6.5}$$

normalised in such a way, that

$$\{v_k\}^T \frac{\partial}{\partial p}[K(M,p)]\bigg|_{p=p_k}\{u_k\} = 1. \tag{6.6}$$

It follows from (6.4) that the latent roots of (3.5) usually reflect in poles of the transfer functions (3.4). However, there are two obvious exceptions to this rule.

If $\{s_i\}^T$ is the i-th row of the integration matrix $[S]$ which was defined in (3.3) and $\{s_i\}^T\{u_k\} = 0$, then the latent root p_k is not a pole of the functions (elements) in the i-th row of the transfer matrix $[\widehat{A}(M,p)]$. Similarly, if $\{d_{1j}\}$ and $\{d_{2j}\}$ are the j-th columns of the differentiation matrices $[D_1]$ and $[D_2]$ and at the same time $\{v_k\}^T\{d_{1j}\} = 0$ and $\{v_k\}^T\{d_{2j}\} = 0$, then the latent root p_k is not a pole of the functions in the j-th column of the transfer matrix.

The right latent vector which is a solution of the second homogeneous equation (6.5) determines a pressure distribution. In Fig. 8 and Fig. 9 two examples of such pressure distributions are shown which are associated with two latent roots. It has been numerically proved, that the shapes of these functions do not depend on the number of aerodynamic elements used to the discretization of the integral equation.

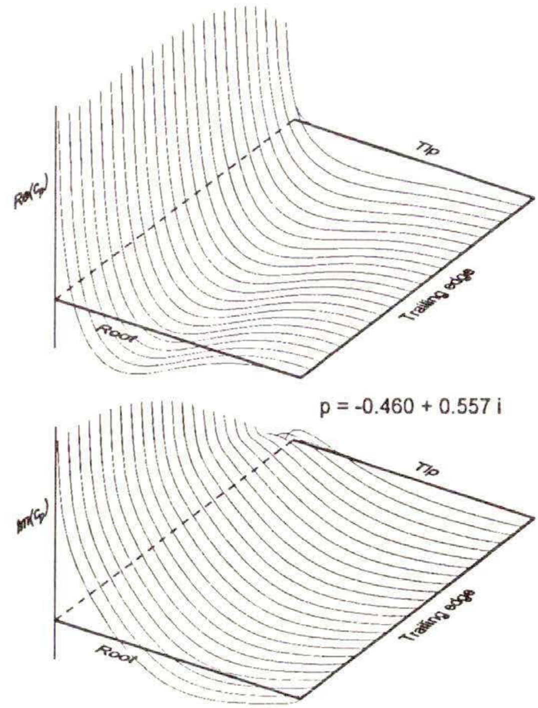
Re(Cp)
Tip
Trailing edge
Root
p = -0.460 + 0.557 i
Im(Cp)
Tip
Trailing edge
Root

FIG. 8.

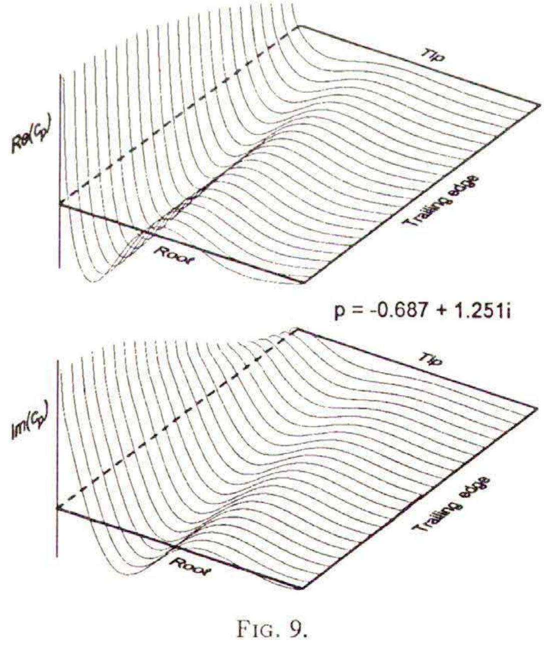
Re(Cp)
Tip
Trailing edge
Root
p = -0.687 + 1.251i
Im(Cp)
Tip
Trailing edge
Root

FIG. 9.

7. Approximation to the transfer functions based on their singularities

The knowledge of the singularities: branch points ($p = 0$ and $p = -\infty$) and poles (latent roots of (3.5)) makes possible an approximate reconstruction of the transfer functions in the entire complex p plane, without any use of analytic continuation (from the imaginary axis).

The solution of the equation may be put in the form

$$\{\widehat{c}_p(p)\} = \left[\widehat{Ae}(p)\right]\{\widehat{w}(p)\}, \tag{7.1}$$

where $\left[\widehat{Ae}(p)\right]$ denotes the matrix $[K(M,p)]^{-1}$.

To simplify the notation, in (7.1) and later in this section, the dependence of the matrices which define the aerodynamic system on the Mach number was not marked explicitly.

It should be emphasised that the relation (7.1) concerns the aerodynamic model only and does not depend on the definition of generalized coordinates used to describe the motion of the structure.

The inverse Laplace transform $\mathcal{L}^{-1}$ applied to (7.1) gives the relation between upwash and pressure distributions in the time domain in the form of a convolution

$$\{c_p(t)\} = [Ae(t)] * \{w(t)\}, \tag{7.2}$$

where the elements of $[Ae(t)]$ are the responses $\{c_p(t)\}$ which result from a unit impulse $\delta(t)$ in the elements of the discretized upwash distribution $\{w(t)\}$. In practice, it is usually more convenient to use inditial functions $[H(t)]$, which are responses to a unit step change in the (discretized) upwash distribution. From (7.2) it follows that

$$\{c_p(t)\} = [H(t)] * \{\dot{w}(t)\}, \tag{7.3}$$

where $\{\dot{w}(t)\}$ is the derivative with respect to time t of the upwash vector $\{w(t)\}$. The inditial functions $[H(t)]$ are related to the hereditary functions $[Ae(t)]$,

$$\left[\widehat{H}(p)\right] = \frac{1}{p}\left[\widehat{Ae}(p)\right] \quad \text{and} \quad [H(t)] = \mathcal{L}^{-1}\left[\widehat{H}(p)\right] = [Ae(t)]*1_+(t), \tag{7.4}$$

where $1_+(t)$ is the unit step function (Heaviside function).

From the final value theorem [6] it follows

$$[H(\infty)] = \lim_{t\to\infty}[H(t)] = \lim_{p\to 0}\left[\widehat{Ae}(p)\right] = \left[\widehat{Ae}(0)\right] = [K(M,0)]^{-1}. \tag{7.5}$$

This limit corresponds to the steady solution (for constant boundary conditions on the surface). In compressible flow ($M \neq 0$), there exists also the limit given by the initial value theorem

$$[D] = \lim_{t\to 0+}[H(t)] = \lim_{p\to\infty}\left[\widehat{Ae}(p)\right] = \left[\widehat{Ae}(\infty)\right], \tag{7.6}$$

which can be calculated directly on the basis of the piston theory [27]

$$c_p(x,y,0+) = \frac{4}{M}\,\frac{w(x,y,0+)}{U} \tag{7.7}$$

(discretization of this relation in the method of the lifting lines is given in the Appendix).

In the incompressible flow ($M = 0$) the limiting values (7.6) do not exist, but

$$[M_A] = \lim_{t\to 0+}\Big([H(t)] * 1_+(t)\Big) = \lim_{p\to\infty}\left(\frac{1}{p}\left[\widehat{A}e(p)\right]\right) \tag{7.8}$$

is the apparent mass matrix, which can be determined on the basis of a simplified model (without wake).

Taking into account the properties of the elements of the matrix $[K(M,p)]$, it is possible to obtain (e.g. [5, 8, 14]) an asymptotic representation

$$\left[\widehat{A}e(p)\right] - \left[\widehat{A}e(0)\right] = O(p^2 \ln p) \qquad \text{for} \quad p \to 0. \tag{7.9}$$

It follows, that in the time domain

$$[H(t)] - [H(\infty)] = O(t^{-2}) \qquad \text{for} \quad t \to \infty. \tag{7.10}$$

The general form of the inditial matrix may be put in the form

$$[H(t)] = [H(\infty)] - [C(t)] + [M_A]\delta(t), \tag{7.11}$$

where the function $[C(t)]$ is usually called the deficiency function, and its asymptotic behaviour is determined by (7.10). The constant matrix $[H(\infty)]$ determines the steady-state limit and may be calculated on the basis of (7.5). The apparent mass matrix $[M_A]$ is involved only for incompressible flow.

This paper is focused on the poles of the transfer functions and their influence on the aerodynamic forces. It was shown that the latent roots of (3.5) appear only when $M > 0$, and therefore, the following analysis will be restricted to the compressible flow when $[M_A] = 0$ and the relations (7.6) and (7.7) may be used.

It is convenient to make a decomposition

$$\left[\widehat{A}e(p)\right] = \left[\widehat{A}e_1(p)\right] + \left[\widehat{A}e_2(p)\right], \tag{7.12}$$

where the first term represents the influence of poles, and on the basis of (6.4) it may be put in the form

$$\left[\widehat{A}e(p)\right] = \sum_k \left(\frac{\{u_k\}\{v_k\}^T}{p - p_k} + \frac{\{u_k^*\}\{v_k^*\}^T}{p - p_k^*}\right), \tag{7.13}$$

where the summation concerns all the (calculated) roots, which exist always as conjugate pairs. It may be assumed, that the second term $[\widehat{A}e_2(p)]$ does not possess any poles and represents the influence of the branch-points $p = 0$ and $p = \infty$.

Similar decomposition of the indicial matrix has the form

$$(7.14) \qquad [H(t)] = [H_1(t)] + [H_2(t)],$$

where on the basis of (7.13) it follows that

$$(7.15) \qquad [H_1(0)] = \lim_{t\to 0+} [H_1(t)] = \lim_{p\to\infty} \left[\widehat{A}e_1(p)\right] = 0,$$

$$(7.16) \qquad [H_1(\infty)] = \lim_{t\to\infty} [H_1(t)] = \lim_{p\to 0} \left[\widehat{A}e_1(p)\right] = \left[\widehat{A}e_1(0)\right],$$

and for $t \to \infty$

$$(7.17) \qquad [H_1(t)] - [H_1(\infty)] = O(e^{\alpha t}) \qquad \text{where} \quad \alpha = \max_k(\mathrm{Re}\,(p_k)) < 0.$$

Hence, the asymptotic behaviour of $[H(t)]$ is determined by $[H_2(t)] - [H_2(\infty)] = O(t^{-2})$.

From (7.6) and (7.15) follows also the limiting value

$$(7.18) \qquad \lim_{t\to 0+} [H_2(t)] = \lim_{t\to 0+} [H(t)] = [D].$$

The deficiency function matrix may be also represented in the form of a sum of two components

$$(7.19) \qquad [C_1(t)] = [H_1(\infty] - [H_1(t)]$$

and

$$(7.20) \qquad [C_2(t)] = [H_2(\infty] - [H_2(t)].$$

The first component is determined by (7.13), but for the second component, only the limiting value is known

$$(7.21) \qquad [C_2(0)] = [H_2(\infty)] - [D] = [K(M,0)]^{-1} - \left[\widehat{A}e_1(0)\right] - [D]$$

and the asymptotic behaviour

$$(7.22) \qquad [C_2(t)] = O(t^{-2}) \qquad \text{for} \quad t \to \infty.$$

Finally, the problem of approximating the response matrix with the use of identified singularities is reduced to the determination of deficiency functions which fulfils the conditions (7.21) and (7.22).

For this purpose it is possible to use a method proposed by STARK [20] in a similar problem. If the deficiency matrix function can be approximated by a scalar function

$$(7.23)\qquad [C_2(t)] = [C_2(0)]\cdot g(t),\qquad \text{where}\quad g(0)=1, \quad \text{and}\quad g(t) = O(t^{-2})\quad \text{for}\quad t\to\infty,$$

then

$$(7.24)\qquad [H(t)] = [H_1(t)] + [H_2(\infty)] - [C_2(0)]\cdot g(t).$$

Taking the Laplace transform of (7.24) and multiplying the result by p we obtain

$$(7.25)\qquad [K(M,p)]^{-1} = \left[\widehat{A}e(p)\right] \approx \left[\widehat{A}e_1(p)\right] - \left[\widehat{A}e_1(0)\right] + [K(M,0)]^{-1} - [C_2(0)]\cdot p\widehat{g}(p),$$

where the matrix $[C_2(0)]$ is given by (7.21), and $\widehat{g}(p) = \mathcal{L}g(t)$.

STARK proposed [20, 28] some forms of the function $g(t)$. The best results were obtained with the set

$$(7.26)\qquad g_m(t) = \left(\frac{a}{a+t}\right)^m \qquad (m = 1, 2, 3, ...),$$

where a is a positive real number which can be chosen in numerical experiments. Laplace transforms of functions (7.26) may be expressed by the exponential integral functions. The conditions (7.23) fulfil the function $g_2(t)$.

8. Conclusions

The numerically calculated aerodynamic forces in the frequency domain are always the values of analytic functions determined in the entire complex plane of the Laplace variable. These functions have poles in the left half of the complex plane, which determine the limits for the approximation by means of rational functions (with analytic continuation from the imaginary axis into the complex plane) and which may significantly influence the aerodynamic forces in the time domain.

1. In the case when the discretizing procedure of the lifting surface equation follows the lifting lines or doublet-lattice methods, the resulting algebraic equations are well-conditioned in the right half of the p-plane and in a strip parallel to the imaginary axis in the left half of the p-plane. The width of this strip decreases with increasing Mach number, but is wide enough for almost all applications. Only for high subsonic flow the problem of conditioning may be severe.

2. In the compressible case ($M \neq 0$), there exist a large (probably infinite) set of latent roots of the aerodynamic coefficients matrix in the left half of the p-plane which reflects (usually) in poles of the transfer functions. The distribution of these latent roots in the p-plane does not depend on the number of aerodynamic elements introduced in the discretization procedure (only small differences were observed which may be related to the accuracy of the results).

3. Also the pressure distributions which correspond to latent vectors of the aerodynamic influence coefficients matrix do not depend on the number of aerodynamic elements introduced in the discretization procedure.

4. For decreasing Mach number $M \to 0$, all latent roots move away from the origin to infinity and probably, for $M = 0$ there are no roots in the finite part of the p-plane. It seems to agree with the results of the STARK method [20] which takes into account only one singularity – the branch-point in the origin. The remarkable accuracy of this method in the incompressible case and less satisfactory results for $M > 0$ may be caused by the influence of the poles of transfer functions.

5. In subsonic flow for $M \to 1$, all latent roots tend to the origin ($p = 0$). The proximity of many poles may cause significant difficulties in the calculation of transfer functions in the range of high subsonic flow.

6. The decomposition of the deficiency function into a part which expresses the influence of latent roots (7.19) and a part influenced by the branch point (7.20) enables the extraction of the part which is responsible for the starting pulse. This agrees with the results of EDWARDS [15] who stated, that the step response function obtained by integrating along the branch cut does not contain the starting pulse.

7. The looping behaviour of some of the generalized forces for strongly decaying motion observed in [18] may be explained as the result of influence of poles of the transfer functions. It may be regarded as an indirect confirmation of the existence of latent roots in the kernel-function results.

The calculations and all considerations presented in this paper concern the aerodynamic model and the results are independent of the choice of generalized coordinates used to describe the motion of the structure.

The approximation to the aerodynamic transfer functions based on the identified singularities and the calculated left and right-hand latent vectors of the influence coefficients matrix avoids the ill-posed analytical continuation from the imaginary axis into the whole p-plane. It may be applied also in regions which contain poles of the transfer functions.

Appendix. Discretization of the piston theory in the lifting lines method

In the lifting lines method of discretization (similarly to many other methods), the pressure distribution on a profile (cross-section of the wing) is approximated by means of a truncated series of functions with appropriate singularities on the leading and trailing edges. The pressure distribution in the piston theory follows the upwash distribution and is a regular, continuos function. Therefore it is not possible to cast the piston theory in the lifting lines discretization scheme exactly. Nevertheless, the approximation can assure the exact values of moments of aerodynamic forces in the case if the upwash distribution is a polynomial of degree less than the number of lifting lines on the cross-section.

The procedure of calculating the (approximate) pressure distribution on the profile $c_p(x)$ on the basis of a known upwash distribution $w(x)/U$ in the lifting lines method [4] consists of the following steps

$$\frac{w(x)}{U} \Rightarrow \{f\} \Rightarrow \{w\} \Rightarrow \{c_p\} \Rightarrow \{a\} \Rightarrow c_p(x). \tag{A.1}$$

The vectors $\{f\}$ and $\{w\}$, as well as $\{c_p\}$ and $\{a\}$ describe in the cross-section the approximate distributions of the upwash and pressure coefficient, respectively. The sizes of these vectors are equal to the number of lifting lines (denoted later by m). The vectors $\{w\}$ and $\{c_p\}$ for all cross-sections of the wing create the vectors in (3.2) and $N = \sum m$.

The pressure distribution on a cross-section is, in the lifting lines method, approximated by a truncated series of Jacobi polynomials

$$c_p(x) = \frac{1}{b_l}\sqrt{\frac{1-x}{1+x}}\sum_{k=0}^{m-1} a_k P_k(x), \tag{A.2}$$

where $2b_l$ is the local chord, the coordinate x is normalised to the interval $-1 < x < 1$ and $P_k(x)$ are polynomials which fulfil the orthogonality condition

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\, P_j(x) P_k(x)\, dx = \delta_{jk}\pi. \tag{A.3}$$

The vector $\{a\}$ of the coefficients a_k is determined for a given pressure distribution by the expression

$$\{a\} = \frac{b_l}{\pi}\int_{-1}^{1} \{P(x)\}\, c_p(x)\, dx. \tag{A.4}$$

The elements of the vector $\{P(x)\}$ are the polynomials $P_k(x)$. The quantities calculated in the lifting lines method from the set of algebraic equations are the strengths of lifting lines (pressure doublets). They are related to the a_k coefficients directly

$$\{c_p\} = [W][P]^T\{a\}, \tag{A.5}$$

where $[W]$ is a diagonal matrix with weight coefficients of the Gauss–Jacobi quadrature, and the elements of the matrix $[P]$ are the values of the polynomials $P_k(x)$ in the nodes of this quadrature.

The upwash distribution is approximated by means of the polynomials $Q_k(x) = P_k(-x)$

$$\frac{w(x)}{U} \approx \frac{1}{\pi} \sum_{k=0}^{n-1} f_k Q_k(x) = \frac{1}{\pi} \{Q(x)\}^T \{f\}, \tag{A.6}$$

where the coefficients f_k are determined by the expression

$$\{f\} = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \{Q(x)\} \frac{w(x)}{U} dx. \tag{A.7}$$

They are next transformed to the form

$$\{w\} = \frac{1}{\pi} [P]^T \{f\} = ([P][W])^{-1} \{f\}. \tag{A.8}$$

The discretized model of the piston theory may be constructed on the basis of the following scheme

$$\{w\} \Rightarrow \{f\} \Rightarrow \frac{w(x)}{U} \Rightarrow c_p(x) \Rightarrow \{a\} \Rightarrow \{c_p\}, \tag{A.9}$$

with the use of the relations (A.8), (A.6), (7.7), (A.4) and (A.5).

$$\{c_p\} = \frac{4b_1}{\pi^2 M} [W][P]^T \left(\int_{-1}^{1} \{P(x)\}\{Q(x)\}^T dx \right) [P][W]\{w\}, \tag{A.10}$$

where

$$\int_{-1}^{1} P_k(x) Q_n(x) \, dx = \begin{cases} \dfrac{1+(-1)^{k+n}}{k+n+1} - \dfrac{1-(-1)^{n-k}}{n-k} & \text{for} \quad n \neq k, \\ \dfrac{2}{2n+1} & \text{for} \quad n = k \end{cases} \tag{A.11}$$

The matrix $[D]$ defined in (7.6) has, in the case of lifting lines method, a quasi-diagonal form and each diagonal block corresponds to a cross-section of the wing and has the form determined by (A.10).

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