Stability of Couette flow in the wide gap of two circular concentric cylinders with rotating inner cylinder and finite growth rate

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A FINITE-DIFFERENCE SOLUTION for the stability of flow of a viscous fluid in an annular wide-gap space is carried out by taking into account the effects of finite growth rate of the amplification factor σ. The numerical values of the minimum Taylor number Ta_m and the critical Taylor number Ta_c for different values of η and for $\sigma \gtrless 0$ and $\sigma = 0$, respectively, are derived and tabulated. The effects of η and σ on the radial component of disturbance and on the cell patterns are shown. It is observed that for increasing $\sigma (> 0)$, the cell patterns are reduced in size, while for decreasing $\sigma (< 0)$ they are enlarged.

1. Introduction

STABILITY OF AXISYMMETRIC FLOW of a viscous incompressible fluid between two concentric rotating cylinders has been investigated by TAYLOR [6], CHANDRASEK-HAR [1], HARRIS and REID [2], WALOWIT, TSAO and DI PRIMA [7], SOUNDALGEKAR et al. [4], who used different methods and different boundary conditions. The usual mathematical procedure of the stability analysis is to assume that small disturbances are superimposed on the steady motion. These disturbances are assumed to be periodic in the z-direction and proportional to $e^{\sigma t}$, where σ is the amplification parameter or growth rate factor. Then the parameter which governs the stability of the motion is the exponential time factor σ, the motion being stable or unstable according to whether the real part of σ is less than or greater than zero, and when $\sigma = 0$, it is known as the marginal state. Almost all the papers in this field applying the linear stability analysis dealt with $\sigma = 0$, i.e. the marginal state of stability. ROBERTS [3] was the first to study the effects of non-zero values of the growth rate on the Taylor number in the wide-gap Couette flow, and computed the smallest characteristic values of the Taylor number for a given wave number a and the growth rate factor σ, by employing the numerical method given by Harris and Reid. However, the effects of σ on the radial velocity perturbation and on the cell pattern have not been studied or shown graphically. Hence it is now proposed to solve the eigen-value problem of ROBERTS [3] by using a finite-difference technique and to derive the minimum values of the Taylor number, Ta_m, for different values of $\pm\sigma$ of the radial velocity perturbations and the cell-patterns. In the next section, a short account of the finite-difference method is given and numerical values of Ta_m are tabulated for different values of

$\pm\sigma$ and wave number a. In order to verify our results, we have also computed the numerical values of Ta_c, the critical Taylor number for $\sigma = 0$ and critical wave number a_c.

2. Solution to the axisymmetric problem at finite growth rate

The axisymmetric linear stability of Couette flow at finite growth rate, when the outer cylinder is at rest and the two cylinders are separated by a wide gap, can be shown to be governed by the following system of sixth order (e.g. Soundalgekar et al. [5]).

$$(2.1) \qquad (DD^* - a^2 - \sigma)(DD^* - a^2)u = -a^2 Ta \cdot g(x)v,$$

$$(2.2) \qquad (DD^* - a^2 - \sigma)v = u,$$

$$(2.3) \qquad u = v = Du = 0 \qquad \text{at} \quad x = 0, 1.$$

Here u, v are the radial and azimuthal components of the disturbances, a is the axial wave number, σ is the growth rate and Ta is the Taylor number. They are defined as follows:

$$d = R_2 - R_1, \qquad x = \frac{r - R_1}{d}, \qquad g(x) = 1 - x,$$

$$(2.4) \qquad D = \frac{d}{dx}, \qquad D^* = \frac{d}{dx} + \frac{1 - \eta}{\xi}, \qquad a = \lambda d,$$

$$\eta = R_1/R_2, \qquad \xi = \eta + (1 - \eta)x, \qquad Ta = \frac{4\eta^2 d^4}{1 - \eta^2}\left(\frac{\Omega_1}{\nu}\right)^2,$$

where R_1 and R_2 are the radii of the inner and outer cylinders, respectively, and Ω_1 is the rate of rotation of the inner cylinder. Our Ta is equivalent to $2Ta_R$ where Ta_R is the Taylor number defined by Roberts or Chandrasekhar, as

$$Ta_R = \frac{2\eta^2 d^4}{1 - \eta^2}\left(\frac{\Omega_1}{\nu}\right)^2.$$

3. Method of solution

By solving the above eigenvalue problem defined by Eqs. (2.1)–(2.3), we determine the smallest characteristic value of the Taylor number, denoted by Ta_m, for given wave number a and σ. To solve Eqs. (2.1)–(2.2) by the finite-difference technique, we first expand these as follows:

$$(3.1) \qquad \left[D^4 + 2kD^3 - \left(3k^2 + 2a^2 + \sigma\right)D^2 + \left(3k^3 - 2ak - \sigma k\right)D\right.$$
$$\left. + (2a^2k^2 - 3k^4 + a^4 + \sigma(k^2 + a^2))\right]u = -a^2 \cdot Ta \cdot g(x)v,$$

$$(3.2) \qquad \left[D^2 + kD - (k^2 + a^2 + \sigma)\right]v = u, \qquad k = \frac{1 - \eta}{\xi}.$$

We write the derivatives in terms of central differences and rearrange the terms. Then Eqs. (3.1) and (3.2) reduce to

$$(3.3) \qquad m_i U_{i+2} + m_2 U_{i+1} + m_3 U_i + m_4 U_{i-1} + m_5 U_{i-2} = -h^4 a^2 \mathrm{Ta} \cdot g(x) \cdot V_i,$$

$$(3.4) \qquad C_1 V_{i+1} + C_2 V_i + C_3 V_{i-1} = h^2 U_i,$$

where

$$
\begin{aligned}
m_1 &= 1 + hk, \\
m_2 &= -4 - 2hk - h^2(3k^2 + 2a^2 + \sigma) + \frac{1}{2} h^3 (3k^3 - 2a^2 k - \sigma k), \\
m_3 &= 6 + 2h^2(3k^2 + 2a^2 + \sigma) + h^4(2a^2 k^2 - 3k^4 + a^4 + \sigma k^2 + \sigma a^2), \\
(3.5) \qquad m_4 &= -4 + 2hk - h^2(3k^2 + 2a^2 + \sigma) - \frac{1}{2} h^3 (3k^3 - 2a^2 k - \sigma k), \\
m_5 &= 1 - hk, \\
C_1 &= 1 + \frac{1}{2} hk, \\
C_2 &= -2 - h^2(k^2 + a^2 + \sigma), \\
C_3 &= 1 - \frac{1}{2} hk.
\end{aligned}
$$

The suffix i stands for the pivotal point under consideration. The step length $h = 1/N$, where N is the number of intervals into which the range $[0,1]$ is divided. The boundary conditions (2.3) imply that

$$
(3.6) \qquad
\begin{aligned}
U_0 &= V_0 = U_N = V_N = 0, \\
U_{-1} &= U_1, \qquad U_{N+1} = U_{N-1}.
\end{aligned}
$$

Equations (3.3) and (3.4) with conditions (3.6) can be written in matrix notation as

$$
(3.7) \qquad
\begin{aligned}
A_1 \overline{U} &= \mathrm{Ta}\, B_1 \overline{V}, \\
A_2 \overline{V} &= \overline{u},
\end{aligned}
$$

where A_1, A_2 and B_1 are the coefficient matrices of order $n \times n$, and $n = N - 1$.

Equations (3.7) can be combined into an eigenvalue equation of the form

$$(3.8) \qquad (C - \mathrm{Ta}\, I)\overline{V} = 0.$$

The eigenvalues are computed by using the QR algorithm for $\sigma = 0$, ± 0.5, ± 1, ± 1.5, ± 2.0 and are listed in Table 1 for $\eta = 0.85, 0.5, 0.1$ which corresponds to

the wide gap case. For $\sigma = 0$, the marginal state, the eigenvalues Ta_c are known as critical values of the Taylor number which corresponds to lowest values of the wave number a and are denoted by a_c, the critical wave number.

We observe from Table 1 that for the marginal state of stability ($\sigma = 0$), the critical values of the Taylor number and wave number increase by increasing the width of the gap between two concentric cylinders. However, when the amplification factor $\sigma(> 0)$ is increasing, there is also an increase in the minimum values of Taylor number, and opposite is the case when the amplification factor $\sigma(< 0)$ is decreasing; then the minimum values of Ta viz. Ta_m also decrease.

Table 1. Values of a_c, Ta_c ($\sigma = 0$) and a_m, Ta_m ($\sigma \neq 0$).

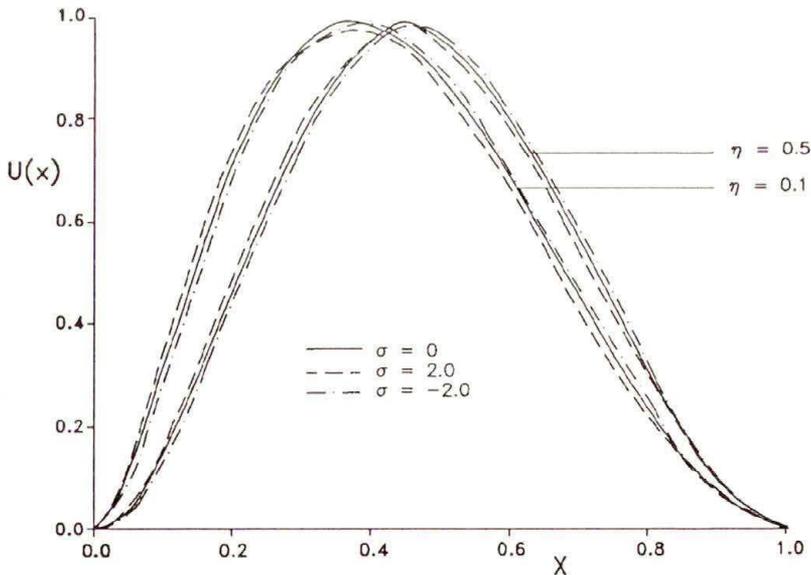|			η 0.85 0.5			a_c 3.130 3.162		Ta_c 3802 6194		$\text{Ta}_{c(R)}$ 3805 6199
	η/σ	-2.0	-1.0	-0.5	-0.1	0.5	1.0	1.5	2.0	
0.85	a_m	3.00	3.068	3.100	3.124	3.159	3.188	3.215	3.241	
	Ta_m	3233	3515	3658	3773	3947	4097	4243	4393	
0.5	a_m	3.033	3.100	3.132	3.156	3.191	3.219	3.247	3.273	
	Ta_m	5289	5737	5964	6148	6425	6659	6895	7134	
0.1	a_m	3.217	3.280	3.310	3.333	3.367	3.395	3.422	3.449	
	Ta_m	56216	60318	62394	64065	66592	68716	70855	73011	

FIG. 1. Radial velocity $U(x)$.

The radial component of velocity disturbance $U(x)$ is shown in Fig. 1 for $\eta = 0.5$, 0.1 and for $\sigma = 0, \pm 2.0$. It is observed that $U(x)$ increases near the inner cylinder and decreases near the outer cylinder when $\sigma = 2.0$ as compared to that at the onset of instability ($\sigma = 0.0$), and opposite is the case when $\sigma = -2.0$. The cell patterns are shown for $\eta = 0.5$ and 0.1 for $\sigma = 0, \pm 2.0$ in Figs. 2–7. It is observed from these figures that the cells get reduced in size for $\sigma = 2.0$ and get enlarged in size for $\sigma = -2.0$ as compared to those at the onset of instability.
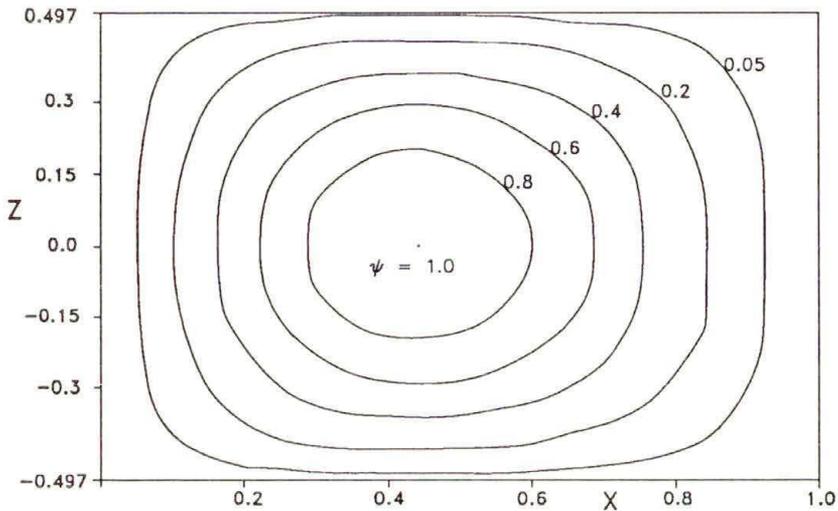
FIG. 2. Cell patterns at the onset of instability for $\eta = 0.5$ and $\sigma = 0.0$, $\psi = U(x) \cos aZ$.
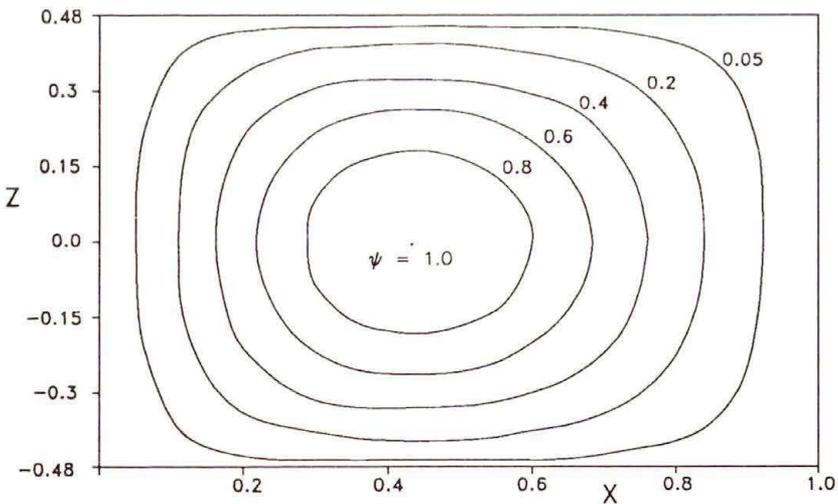

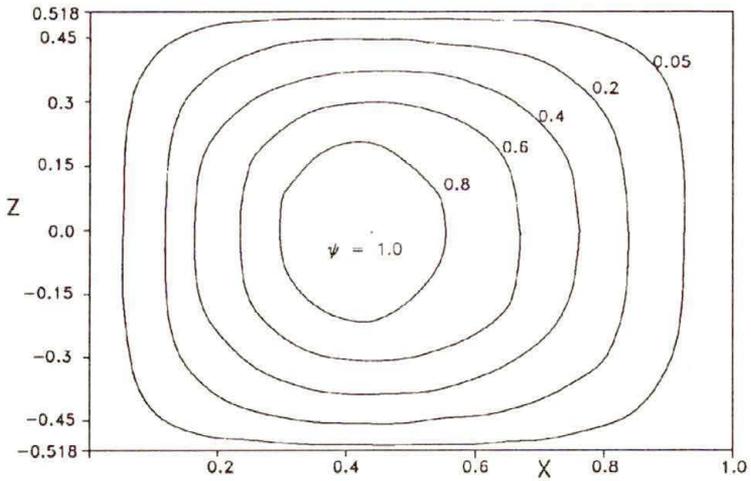FIG. 3. Cell pattern instability for $\eta = 0.5$ and $\sigma = 2.0$.

FIG. 4. Cell pattern instability for $\eta = 0.5$ and $\sigma = -2.0$.

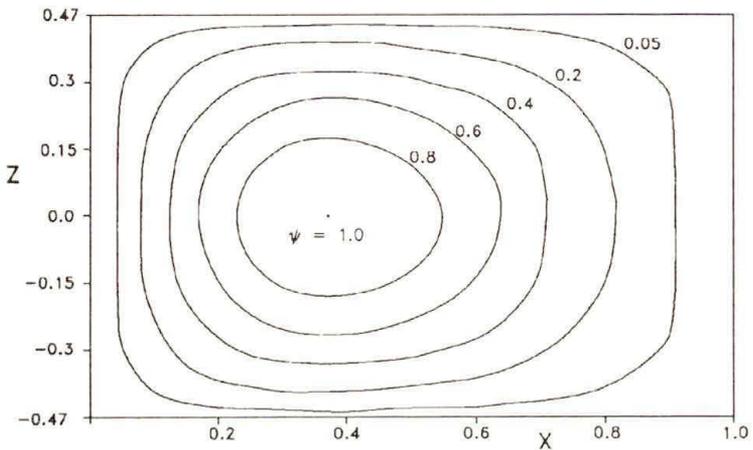

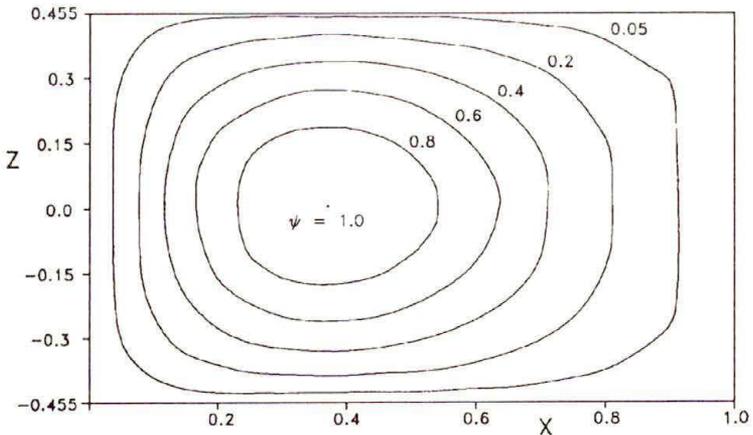FIG. 5. Cell patterns at the onset of instability for $\eta = 0.1$ and $\sigma = 0.0$, $\psi = U(x)\cos aZ$.

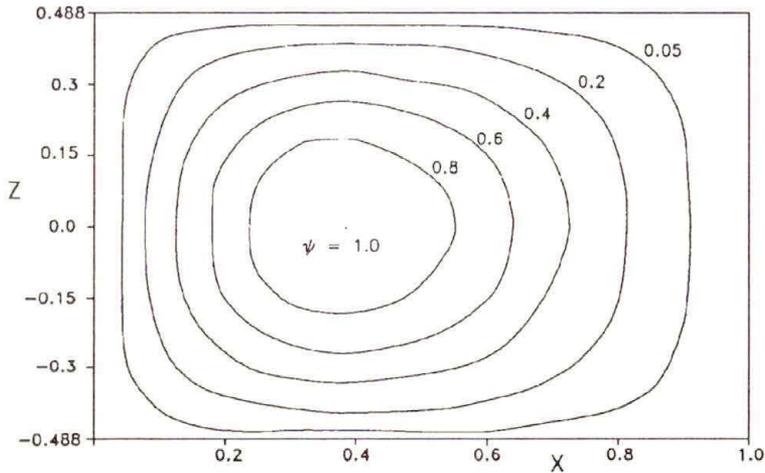FIG. 6. Cell patterns for $\eta = 0.1$ and $\sigma = 2.0$.

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FIG. 7. Cell pattern for $\eta = 0.1$ and $\sigma = -2.0$

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