Arch. Mech., 48, 4, pp. 765–778, Warszawa 1996

# The stationary Stokes flow through a spherical region with large variations of density and viscosity coefficient

Z. PŁOCHOCKI, B. KAŹMIERCZAK and Z. PERADZYŃSKI (WARSZAWA)

WE ARE INTERESTED in flows of a fluid whose density changes abruptly after entering a certain region in $R^3$. Flows of this kind may be useful in modelling such phenomena as propagating flames. Assuming that the region is a ball we find a closed-form solution for the flow homogeneous at infinity in the Stokes approximation. It is compared with the analytical solution in the Euler approximation. Such solutions can also be used as a test for numerical algorithms solving the flow equations.

## 1. Introduction

FOR GAS SYSTEMS with strong local heat sources (e.g. flames, laser-generated or sustained plasma) there arise at least two important problems concerning the influence of a gas flow on heat exchange processes, and velocity of propagation of the hot region front. In general, such problems are complicated. However, simple hydraulic models of a gas flow through a region with large density variations based on analysis of particular solutions, offer some possibilities of simplification of such problems.

The first such a solution was proposed by GUS'KOV *et al.* [1] as an attempt to study the propagation of plasma front in case of laser-generated plasma. The authors considered a stationary, homogeneous at infinity, inviscid (the Euler approximation, i.e. $\mathrm{Re} \to \infty$) gas flow through a spherical region. The density of the gas is assumed to be constant outside, and also constant but much smaller inside the sphere. The gas is therefore assumed to be incompressible outside and inside the sphere. Such assumptions allow to find an analytical solution of the problem (by dividing the whole flow region into two subregions, finding solutions to the continuity and Euler equations separately in each of them, and then by matching these solutions by means of continuity conditions for densities of mass and momentum fluxes at the surface of the sphere).

Next, Z. PERADZYŃSKI and E. ZAWISTOWSKA [2] treated numerically the same problem for a different Reynolds number, assuming however constant viscosity coefficient in the whole flow region.

The aim of the present paper is to find an analytical solution of this problem in the Stokes approximation ($\mathrm{Re} \to 0$) and to compare it with the analytical solution of the problem in the Euler approximation, and also with the numerical solution mentioned.

## 2. Statement of the problem

Consider a stationary gas flow through the spherical region of radius $R$. The density of the gas is assumed in the form:

$$\tag{2.1} \begin{aligned} \varrho &= \varrho^{\text{int}} + (\varrho^{\text{ext}} - \varrho^{\text{int}})H(\overline{r} - 1), \\ \varepsilon_{\varrho} &:= \frac{\varrho^{\text{int}}}{\varrho^{\text{ext}}} \ll 1, \end{aligned}$$

where $\varrho^{\text{int}}$ and $\varrho^{\text{ext}}$ are constants representing the gas density inside and outside the sphere, respectively; $H(x - x_0)$ is the Heaviside function; and $\overline{r}$ stands for the dimensionless $r$-coordinate in the spherical coordinate system (as referred to the radius $R$). The density variation may be thought as generated by a constant high temperature field inside the sphere and (relatively) low (and also constant) temperature field outside. In such a case, also the viscosity coefficient should be assumed in the form:

$$\tag{2.2} \begin{aligned} \eta &= \eta^{\text{int}} + (\eta^{\text{ext}} - \eta^{\text{int}})H(\overline{r} - 1), \\ \varepsilon_{\eta} &= \frac{\eta^{\text{ext}}}{\eta^{\text{int}}} \ll 1, \end{aligned}$$

where $\eta^{\text{int}}$ and $\eta^{\text{ext}}$ are constants representing the shear viscosity coefficient of the gas inside and outside the sphere, respectively. Since for an ideal gas $\varrho \propto 1/T$ and $\eta \propto \sqrt{T}$, therefore for a gas, which can be approximately treated as an ideal one, we have

$$\tag{2.3} \varepsilon_{\eta} \cong \sqrt{\varepsilon_{\varrho}}\,.$$

The flow at infinity is assumed to be homogeneous. At the sphere surface there are no mass and momentum sources.

In order to find the solution to this problem, the method of dividing the whole region into two subregions is applied. Then, the governing equations for the interior of both subregions, i.e. for $r < R$ and $r > R$, are:

$$\nabla \cdot \mathbf{v} = 0, \qquad \nabla p = \eta \nabla^2 \mathbf{v},$$

where $\mathbf{v}$ and $p$ stand for the velocity vector and pressure, respectively. By introducing the spherical coordinate system $r$, $\varphi$, $\theta$ (with $z$-axis directed along the flow velocity at infinity and centered in the center of the sphere), these equations can be rewritten in the following detailed form:

$$\tag{2.4} \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) = 0,$$

(2.4) [cont.]
$$\frac{1}{\eta}\frac{\partial}{\partial r}p = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}v_r\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}v_r\right) - \frac{2v_r}{r^2} - \frac{2\cos\theta}{r^2\sin\theta}v_\theta - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta},$$
$$\frac{1}{\eta}\frac{\partial}{\partial\theta}p = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}v_\theta\right) - \frac{v_\theta}{r\sin^2\theta} + \frac{2}{r}\frac{\partial v_r}{\partial\theta},$$

where $v_r$ and $v_\theta$ stand for the $r$- and $\theta$-coordinate of the velocity vector, respectively, and the axial symmetry of the flow has been assumed (i.e. $v_\varphi \equiv 0$).

The boundary conditions are assumed in the form:

$$r = \infty: \quad \begin{cases} v_r = v_\infty\cos\theta, \\ v_\theta = -v_\infty\sin\theta, \\ p = p_\infty, \end{cases} \tag{2.5}$$
$$r = 0: \quad |v_r|,\ |v_\theta|, \quad p < \infty,$$

where $v_\infty$ and $p_\infty$ stand for the velocity modulus and pressure at infinity, repectively. In order to match the solutions outside and inside the sphere, the local conservation principles of mass and momentum are used. The equations, which express these conservation principles, are assumed to be valid in the whole space (i.e. – also at the sphere surface). Then the continuity conditions for the $r$-th coordinates of the flux density of mass and that of momentum at the sphere surface read:

$$r = R: \quad [\![\varrho v_r]\!] = 0,$$
$$[\![p - 2\eta\frac{\partial v_r}{\partial r}]\!] = 0, \tag{2.6}$$
$$[\![\eta\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)]\!] = 0,$$

where

$$[\![\psi]\!] := \psi^{\text{ext}}(r = R) - \psi^{\text{int}}(r = R), \tag{2.7}$$

where, in turn, the superscripts ext and int refer to the outside and to the inside of the sphere, respectively.

## 3. Solution

The solution of the problem expressed by Eqs. (2.4)–(2.6) is sought in the form:

$$v_r = v_\infty f(r)\cos\theta, \tag{3.1}$$
$$v_\theta = -v_\infty g(r)\sin\theta.$$

Substituting Eqs. (3.1) into Eq. $(2.4)_1$ one obtains:

$$g = \frac{1}{2} r f' + f, \tag{3.2}$$

where prime denotes the derivative with respect to $r$. Substituting Eqs. (3.1) into Eqs. $(2.4)_{2,3}$ and using Eq. (3.2) one obtains:

$$\frac{1}{\eta v_\infty} \frac{\partial p}{\partial r} = \left( f'' + \frac{4}{r} f' \right) \cos\theta,$$
$$\frac{1}{\eta v_\infty} \frac{\partial p}{\partial \theta} = -\left( \frac{1}{2} r^2 f''' + 3 r f'' + 2 f' \right) \sin\theta.$$

Integrating the latter equation and substituting the result into the former equation we obtain:

$$\frac{1}{\eta v_\infty} p = C_1 + \left( \frac{1}{2} r^2 f''' + 3 r f'' + 2 f' \right) \cos\theta, \tag{3.3}$$

$$r^3 f^{\mathrm{IV}} + 8 r^2 f''' + 8 r f'' - 8 f' = 0, \tag{3.4}$$

where $C_1$ is a constant. The general solution of the latter equation is:

$$f = C_2 + C_3 r^2 + \frac{C_4}{r} + \frac{C_5}{r^3},$$

where $C$ stand for constants. Thus, according to Eqs. (3.1) – (3.3) the solutions of Eqs. (2.4) outside and inside the sphere, which satisfy the boundary conditions as expressed by Eqs. (2.5), may be written in the form (all the constants occuring in the formulae describing the flow in the Stokes approximation will be denoted by tilde, to distinguishing them from the analogous constants in the case of the Euler approximation, which will be discussed later):

$$\begin{aligned}
\overline{r} := \frac{r}{R} > 1: \qquad & v_r^{\mathrm{ext}} = v_\infty \left( 1 + \frac{\widetilde{D}}{\overline{r}} - 2\frac{\widetilde{A}}{\overline{r}^3} \right) \cos\theta, \\
& v_\theta^{\mathrm{ext}} = -v_\infty \left( 1 + \frac{\widetilde{D}}{2\overline{r}} + \frac{\widetilde{A}}{\overline{r}^3} \right) \sin\theta, \\
& p^{\mathrm{ext}} = p_\infty + \frac{\eta^{\mathrm{ext}} v_\infty}{R} \frac{\widetilde{D}}{\overline{r}^2} \cos\theta, \\
\overline{r} := \frac{r}{R} < 1: \qquad & v_r^{\mathrm{int}} = v_\infty (\widetilde{B} + \widetilde{C} \overline{r}^2) \cos\theta, \\
& v_\theta^{\mathrm{int}} = -v_\infty (\widetilde{B} + 2\widetilde{C} \overline{r}^2) \sin\theta, \\
& p^{\mathrm{int}} = \widetilde{E} + \frac{\eta^{\mathrm{ext}} v_\infty}{R} \frac{10\widetilde{C}}{\varepsilon_\eta} \overline{r} \cos\theta.
\end{aligned} \tag{3.5}$$

The constants $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$, $\widetilde{E}$ have to be determined from the continuity conditions as expressed by Eqs. (2.6). In fact, substituting Eqs. (3.5) into Eqs. (2.6) we obtain the following set of equations for the constants considered:

$$
\begin{gathered}
1+\widetilde{D}-2\widetilde{A}=\varepsilon_{\varrho}(\widetilde{B}+\widetilde{C}),\\
\widetilde{E}=p_{\infty},\\
\widetilde{D}-4\widetilde{A}=\frac{2\widetilde{C}}{\varepsilon_{\eta}},\\
2\widetilde{A}=-\frac{\widetilde{C}}{\varepsilon_{\eta}}.
\end{gathered}
$$

It follows immediately from the latter two equations that

$$\widetilde{D}=0, \tag{3.6}$$

and therefore:

$$
\begin{aligned}
1-2\widetilde{A}&=\varepsilon_{\varrho}(\widetilde{B}+\widetilde{C}),\\
\widetilde{A}&=-\frac{\widetilde{C}}{2\varepsilon_{\eta}},
\end{aligned} \tag{3.7}
$$

and

$$\widetilde{E}=p_{\infty}. \tag{3.8}$$

It is seen that we have two equations for three constants: $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$.

Thus, in order to obtain a unique solution we should adopt an additional condition, and the continuity condition of the tangent component of velocity at the surface of the sphere ($r = R$) is assumed:

$$[\![ v_\theta ]\!]=0 \qquad (r=R), \tag{3.9}$$

which leads to the following additional equation:

$$1+\widetilde{A}=\widetilde{B}+2\widetilde{C}. \tag{3.10}$$

From a formal point of view the problem of an additional constant of integration, for which there is no suitable condition, follows naturally from the applied method of dividing the whole flow region into two subregions. From the physical point of view the assumption expressed by Eq. (3.9) may be argued as follows. The expression in $[\![\ ]\!]$ in Eq. $(2.6)_3$ represents the $r\theta$–coordinate of the momentum flux density, which should be a continuous function in the whole flow region (in particular – at $r = R$). The quantities: $\eta$, $v_r$, $v_\theta$ are assumed to be limited. If the function $v_\theta$ was discontinuous (as a function of $r$) at $r = R$, then this coordinate of the momentum flux density would be singular at $r = R$, and this singularity can not be compensated by discontinuities of the other terms. It would denote, that at the boundary between subregions there are some momentum sources (surface tangent forces), which are absent by the assumption. Short discussion of the assumption considered, which is based on the properties of a weak solution of the flow equations in the Stokes approximation, is presented in the Appendix. It may be treated as a formal support for the continuity condition expressed by Eq. (3.9).

Now, solving Eqs. (3.7) and (3.10) we obtain:

$$\begin{aligned}\widetilde{A} &= \frac{1-\varepsilon_\varrho}{2+\varepsilon_\varrho(1+2\varepsilon_\eta)},\\ \widetilde{B} &= \frac{3+4\varepsilon_\eta-2\varepsilon_\eta\varepsilon_\varrho}{2+\varepsilon_\varrho(1+2\varepsilon_\eta)},\\ \widetilde{C} &= -\frac{2\varepsilon_\eta(1-\varepsilon_\varrho)}{2+\varepsilon_\varrho(1+2\varepsilon_\eta)}.\end{aligned}\tag{3.11}$$

Inserting the approximate relation $\varepsilon_\eta \cong \sqrt{\varepsilon_\varrho}$ into the above formulae we may obtain the asymptotic expressions as $\varepsilon_\varrho \to 0$, namely:

$$\begin{gathered}\widetilde{A} \cong \frac{1}{2}\left(1-\frac{3}{2}\varepsilon_\varrho\right), \qquad \widetilde{B} \cong \frac{3}{2}\left(1+\frac{4}{3}\sqrt{\varepsilon_\varrho}\right),\\ \widetilde{C} \cong -\sqrt{\varepsilon_\varrho}\left(1-\frac{3}{2}\varepsilon_\varrho\right) \cong -\sqrt{\varepsilon_\varrho}.\end{gathered}\tag{3.12}$$

On the other hand, by putting $\varepsilon_\eta = 1$ we obtain respectively:

$$\begin{aligned}\widetilde{A} &= \frac{1-\varepsilon_\varrho}{2+3\varepsilon_\varrho} \cong \frac{1}{2}\left(1-\frac{5}{2}\varepsilon_\varrho\right),\\ \widetilde{B} &= \frac{7-2\varepsilon_\varrho}{2+3\varepsilon_\varrho} \cong \frac{7}{2}\left(1-\frac{25}{14}\varepsilon_\varrho\right),\\ \widetilde{C} &= -2\frac{1-\varepsilon_\varrho}{2+3\varepsilon_\varrho} \cong -\left(1-\frac{5}{2}\varepsilon_\varrho\right).\end{aligned}\tag{3.13}$$

Thus, Eqs. (3.5) with Eqs. (3.6), (3.8) and (3.11) represent the solution of the problem expressed by Eqs. (2.4) – (2.6), which is unique in the class of functions specified by Eqs. (3.1).

## 4. Results

From the formulae given in the previous section one may obtain all the information about the flow examined. Examples of two types of such an information will be presented.

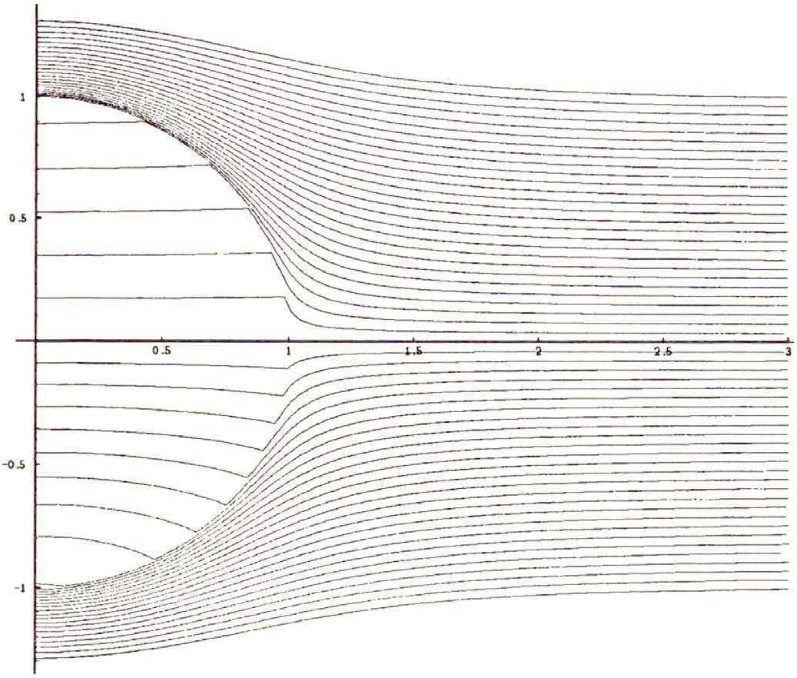


FIG. 1. Streamlines pictures for the flow through the sphere in the Euler (the lower half) and Stokes (the upper half) approximations under the assumptions: $\varepsilon_\eta = \sqrt{\varepsilon_\varrho}$, $\varepsilon_\varrho = 2.5 \times 10^{-2}$.

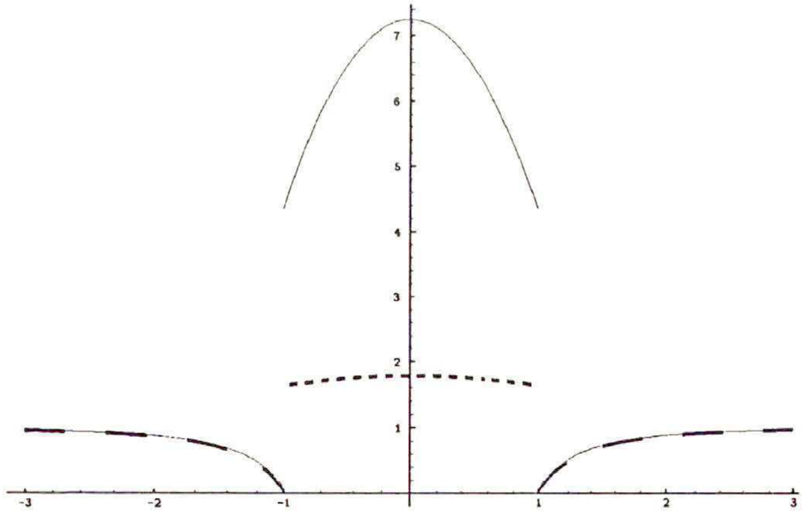


FIG. 2. Dimensionless velocity (as referred to $v_\infty$) at the flow symmetry axis as a function of the dimensionless $z$-coordinate (as referred to $R$) under the same assumptions about $\varepsilon_\eta$ and $\varepsilon_\varrho$ as in the case of Fig. 1 in the Euler (solid line) and Stokes (dashed line) approximations.

The information of the first type concerns the flow fields at a given $\varepsilon_\varrho$. As an example, value $\varepsilon_\varrho = 2.5 \times 10^{-2}$ is assumed as a typical one for the laser-sustained plasma. Thus, the upper half of Fig. 1 presents the streamlines pictures. Figure 2 b (dashed line) presents the dimensionless $z$-coordinate of velocity:

$$\overline{v}_z = \frac{v_r}{v_\infty}\cos\theta - \frac{v_\theta}{v_\infty}\sin\theta$$

at the flow symmetry axis ($\theta = \pi$, 0, respectively) as a function of dimensionless $z$-coordinate ($\overline{z} = (z/R)\cos\theta$), under the same assumptions about $\varepsilon_\eta$ and $\varepsilon_\varrho$ as above. Figure 3 presents the dependence of the dimensionless pressure difference:

$$\overline{\Delta p} = 2\frac{p - p_\infty}{\varrho_\infty v_\infty^2}$$

on the dimensionless $z$-coordinate at the flow symmetry axis under the same assumptions about $\varepsilon_\eta$ and $\varepsilon_\varrho$ as in the case of Fig. 1, where the Reynolds number

$$\mathrm{Re} = \frac{\varrho_\infty v_\infty R}{\eta_\infty}$$

plays the role of the scale factor only.

The information of the second type concerns the characteristics of the flow considered as functions of $\varepsilon_\varrho$, as for example: velocity and pressure on the flow symmetry axis at the center and at the boundary of the sphere (Fig. 4 b, Fig. 5b)(¹):

$$
(4.1)\qquad
\begin{array}{lll}
 & \underline{\varepsilon_\eta = \sqrt{\varepsilon_\varrho}} & \underline{\varepsilon_\eta = 1} \\[1ex]
\overline{v}_z^{\,\mathrm{ext}}(1) = 1 - 2\widetilde{A} & \cong \dfrac{3}{2}\varepsilon_\varrho, & \cong \dfrac{5}{2}\varepsilon_\varrho, \\[1ex]
\overline{v}_z^{\,\mathrm{int}}(1) = \widetilde{B} + \widetilde{C} & \cong \dfrac{3}{2} + \sqrt{\varepsilon_\varrho}, & \cong \dfrac{5}{2} - \dfrac{15}{4}\varepsilon_\varrho, \\[1ex]
\overline{v}_z^{\,\mathrm{int}}(0) = \widetilde{B} & \cong \dfrac{3}{2} + 2\sqrt{\varepsilon_\varrho}, & \cong \dfrac{7}{2} - \dfrac{25}{4}\varepsilon_\varrho, \\[1ex]
[\![\overline{v}_z]\!] = 1 - 2\widetilde{A} - \widetilde{B} - \widetilde{C} & \cong -\dfrac{3}{2} - \sqrt{\varepsilon_\varrho}, & \cong -\dfrac{5}{2} + \dfrac{25}{4}\varepsilon_\varrho, \\[1ex]
\overline{\Delta p}^{\,\mathrm{ext}}(1) = 0 & & \\[1ex]
\dfrac{Re}{20}\overline{\Delta p}^{\,\mathrm{int}}(1) = -\dfrac{\widetilde{C}}{\varepsilon_\eta} & \cong 1 - \dfrac{3}{2}\varepsilon_\varrho, & \cong 1 - \dfrac{5}{2}\varepsilon_\varrho, \\[1ex]
\overline{\Delta p}^{\,\mathrm{int}}(0) = 0 & & \\[1ex]
[\![\overline{\Delta p}]\!] = -\overline{\Delta p}^{\,\mathrm{int}}(1), & &
\end{array}
$$

(¹) Note that the part of the gas flux flowing through the sphere as referred to the flux incoming from infinity is given by $\overline{v}_z^{\,\mathrm{ext}}(1)$.

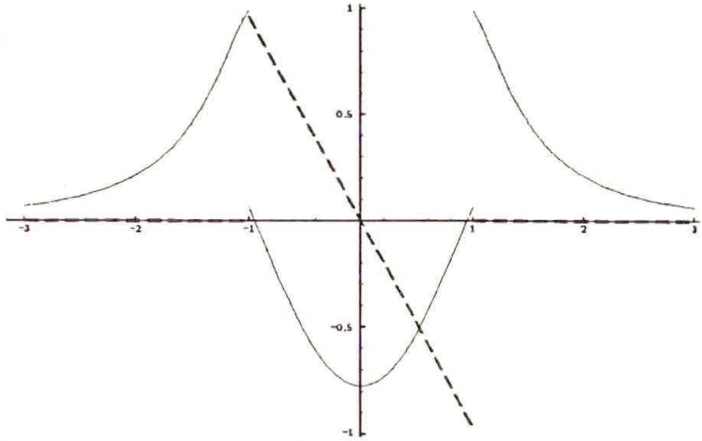


FIG. 3. Scaled relative pressure at the flow symmetry axis for $\varepsilon_\eta = \sqrt{\varepsilon_\varrho}$, $\varepsilon_\varrho = 2.5 \times 10^{-2}$.
solid line – the Euler approximation: $2(p - p_\infty)/(\varrho_\infty v_\infty^2)$,
dashed line – the Stokes approximation: $2(p - p_\infty)/(\varrho_\infty v_\infty^2)(\mathrm{Re})/(20)$.

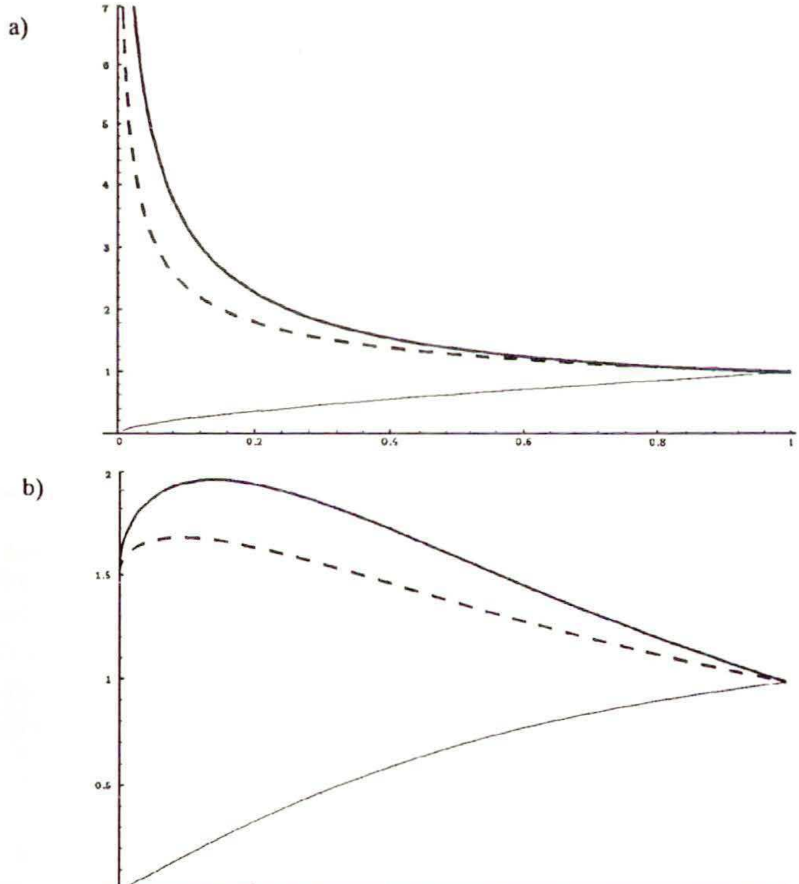


FIG. 4. Dependence of $\overline{v}_z^{\text{ext}}(1)$ (solid line), $\overline{v}_z^{\text{int}}(1)$ (dashed line) and $\overline{v}_z^{\text{int}}(0)$ (bold line) on $\varepsilon_\varrho$ for the flow through the sphere in the Euler (a) and Stokes (b) approximation under the assumption: $\varepsilon_\eta = \sqrt{\varepsilon_\varrho}$.

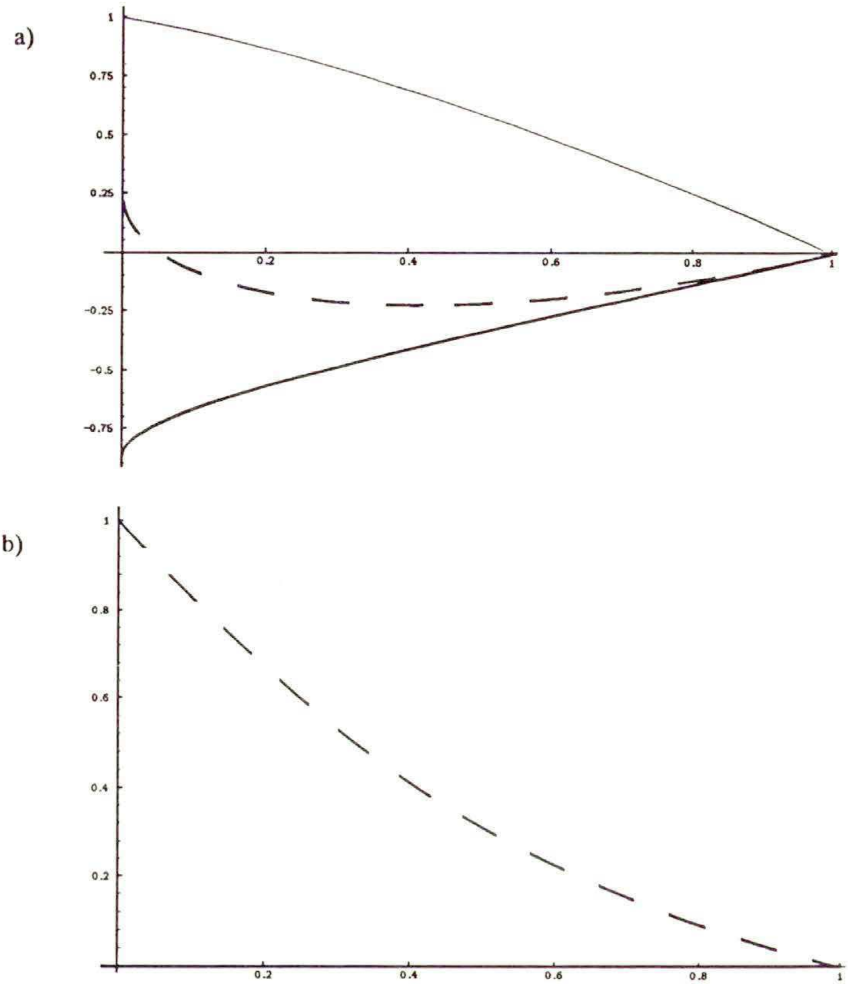


FIG. 5. Dependence of $\overline{\Delta p}^{\text{ext}}(1)$ (solid line), $\overline{\Delta p}^{\text{int}}(1)$ (dashed line) and $\overline{\Delta p}^{\text{int}}(0)$ (bold line) on $\varepsilon_\varrho$ for the flow through the sphere in the Euler (a) and Stokes (b) approximations under the same assumptions about $\varepsilon_\eta$ as in the case of Fig. 3.

where the first column represents the exact formulae, the second one – the asymptotic formulae for small $\varepsilon_\varrho$ under the assumption $\varepsilon_\eta = \sqrt{\varepsilon_\varrho}$, the third one – the asymptotic formulae for small $\varepsilon_\varrho$ under the assumption $\varepsilon_\eta = 1$;

$$\psi(1) := \psi(\theta = \pi,\ r = R),$$
$$\psi(0) := \psi(\theta = \pi,\ r = 0);$$

and $\left[\!\left[\psi\right]\!\right]$ is defined by Eq. (2.7).

## 5. Discussion

The velocity field in the Stokes approximation is, from the qualitative point of view, similar to that in the Euler approximation (Fig. 1). It follows from the fact that the dependence of the velocity coordinates on $r$ and $\theta$ has the same structure in both approximations (Eqs. $(3.5)_{1,2,4,5}$ with Eq. (3.6)). However, quantitative pictures in both cases are different (Fig. 2), because the integration constants $A$, $B$ and $C$ in the Euler approximation (they have no tilde, for distinguishing) are given by different functions of $\varepsilon_\varrho$. Namely, in the case of the Euler approximation they are the solutions of the set (the typing error in the sign is corrected):

$$A = \frac{2-\varepsilon_\varrho - \varepsilon_\varrho B}{4+\varepsilon_\varrho}, \qquad C = \frac{3-(2+\varepsilon_\varrho)B}{4+\varepsilon_\varrho},$$

$$3A(2-A) + 2(1-2A)^2 = -\varepsilon_\varrho C(3B+2C) + 2\varepsilon_\varrho (B+C)^2.$$

Solving this equation set with respect to $A$, $B$, $C$ one may obtain the velocity characteristics in the Euler approximation analogous to those given by Eqs. $(4.1)_{1-4}$ in the case of the Stokes approximation (number errors are corrected) (Fig. 4 a):

$$\begin{aligned}
\overline{v}_z^{\,\mathrm{ext}}(1)_{\mathrm{Eu}} &= 1-2A &&\cong \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\varepsilon_\varrho}, \\
\overline{v}_z^{\,\mathrm{int}}(1)_{\mathrm{Eu}} &= B+C &&\cong \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{2}}\frac{1}{\sqrt{\varepsilon_\varrho}}, \\
\overline{v}_z^{\,\mathrm{int}}(0)_{\mathrm{Eu}} &= B &&\cong -\frac{1}{2} + \sqrt{\frac{3}{2}}\frac{1}{\sqrt{\varepsilon_\varrho}}, \\
\left[\!\left[\overline{v}_z\right]\!\right]_{\mathrm{Eu}} &= 1-2A-B-C &&\cong -\overline{v}_z^{\,\mathrm{int}}(1)_{Eu},
\end{aligned}$$

where the same convention was used as in the case of Eqs. (4.1).

Therefore, from the quantitative point of view the velocity field in the Stokes approximation is remarkably different (especially inside the sphere) as compared to that in the Euler approximation. Generally, one may say, that viscosity forces (when they are dominating over the inertia forces) accommodate the flow, although (inside the sphere) not as much as it follows from the numerical results presented in [2]. For example, the (nondimensional) internal velocity (as referred to $v_\infty$) on the $z$-axis for $\varepsilon_\varrho = 2.5 \times 10^{-2}$ increases parabollically from about 4.35 at $\overline{r} = 1$ to about 7.25 at $\overline{r} = 0$ in the Euler approximation, whereas in the Stokes approximation (under the assumption: $\varepsilon_\eta = \sqrt{\varepsilon_\varrho}$) it increases (also parabolically) from about 1.63 to about 1.78, respectively.

Comparison of the analytical results presented here (under the assumption: $\varepsilon_\eta = 1$) and numerical results presented in [2] for $\mathrm{Re} \to 0$ shows some differences inside the sphere. The numerical results are lower and more weakly depending on $z$-coordinate. For example, the analytical formulae for the internal (dimensionless) velocity (as referred to $v_\infty$) on $z$-axis for $\varepsilon_\varrho = 2.5 \times 10^{-2}$ give the value about 2.46 at $\overline{r} = 1$ and about 3.40 at $\overline{r} = 0$, whereas the values in [2] are about 1.85 and 1.96, respectively.

The pressure field obtained in the Stokes approximation has different structure as compared to that in the Euler approximation, although variations of pressure are relatively small in both of them (Fig. 3). General difference in pressure behaviour is seen by comparing Eqs. $(3.5)_{3,6}$ (with $\widetilde{D} = 0$) and the following formulae for pressure given in [1]:

$$p_{\mathrm{Eu}}^{\mathrm{ext}} = p_\infty + \frac{\varrho v_\infty^2}{2} \frac{A}{\overline{r}^3} \left\{ - \left( 2 + \frac{A}{\overline{r}^3} \right) + 3 \left( 2 - \frac{A}{\overline{r}^3} \right) \cos^2\theta \right\},$$

$$p_{\mathrm{Eu}}^{\mathrm{int}} = p_0 + \varepsilon_\varrho \frac{\varrho v_\infty^2}{2} C\overline{r}^2 \left\{ B + C\overline{r}^2 - (3B + 2C\overline{r}^2)\cos^2\theta \right\},$$

where

$$p_0 = p_\infty - \frac{1}{2}\varrho_\infty v_\infty^2 \left\{ A(2 + A) + \varepsilon_\varrho C(B + C) \right\}.$$

Using these formulae one may obtain the pressure characteristics in the Euler approximation analogous to those given by Eqs. $(4.1)_{5-8}$ in the case of the Stokes approximation (Fig. 5 a):

$$\overline{\Delta p}^{\mathrm{ext}}(1)_{\mathrm{Eu}} = 4A(1 - A) \cong 1 - \frac{3}{8}\varepsilon_\varrho,$$

$$\overline{\Delta p}^{\mathrm{int}}(1)_{\mathrm{Eu}} = -\left\{ A(2 + A) + \varepsilon_\varrho C(3B + 2C) \right\} \cong \frac{1}{4} - \sqrt{\frac{3}{2}}\sqrt{\varepsilon_\varrho},$$

$$\overline{\Delta p}^{\mathrm{int}}(0)_{\mathrm{Eu}} = -\left\{ A(2 + A) + \varepsilon_\varrho C(B + C) \right\} \cong -\frac{7}{8} + \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\varepsilon_\varrho},$$

$$\left[\!\left[ \overline{\Delta p} \right]\!\right]_{\mathrm{Eu}} = 3A(2 - A) + \varepsilon_\varrho C(3B + 2C) \cong \frac{3}{4} + \sqrt{\frac{3}{2}}\sqrt{\varepsilon_\varrho},$$

where the same convention was used as in the case of Eqs. (4.1).

## Appendix

Below we will show that, if a weak solution to the conservation laws exists, then the tangent component of the velocity must be continuous. In a Cartesian system of coordinates the conservation laws for mass and momentum can be written as:

$$\text{(A.1)} \qquad \begin{aligned} \nabla\cdot(\varrho\mathbf{v}) &= 0, \\ \nabla\cdot\sigma_i &= 0, \end{aligned}$$

where $\sigma_i$ is the $i$-th row of the matrix

$$\sigma_{ij} = -\delta_{ij}p + \lambda\delta_{ij}\nabla\cdot\mathbf{v} + \eta\left(\frac{\partial}{\partial x_j}v_i + \frac{\partial}{\partial x_i}v_j\right),$$

$\eta$ is a shear viscosity coefficient, $\lambda = \zeta - \dfrac{2\eta}{3}$ and $\zeta$ is a bulk viscosity coefficient.

Weak formulation can be obtained by multiplying the Eq. (A.1) by smooth test functions and formal integration by parts. Thus, for given $\varrho$, $\eta$ and $\zeta$ we say that $v_i \in L^2_{\mathrm{loc}}$, $i \in \{1,2,3\}$, and a distribution $p \in \mathcal{D}'$ satisfy the system (A.1) in the weak sense, if for all $C_0^\infty(R^3)$ functions $\phi$ and $\psi_i$, $i \in \{1,2,3\}$, we have:

$$\text{(A.2)} \qquad \begin{aligned} \sum_i \int \varrho v_i \phi_{,i}\, dx &= 0, \\ \sum_j \int \sigma_{ij}\psi_{,j}\, dx &= 0, \qquad i \in 1,2,3, \end{aligned}$$

where the integrals are taken over $R^3$. Still, the integration in $(\mathrm{A.2})_1$ must be understood as action of a distribution on $\psi_j$, because $\sigma_{ij}$ is a combination of derivatives of the components of $\mathbf{v}$ and they are, in general, discontinuous. At the beginning let us assume that $\eta$ and $\zeta$ are smooth functions. For the sake of brevity, let us assume that the boundary dividing the regions of different $\varrho$ is a flat surface, e.g. the plane $x_3 = 0$. (In the case of smooth though not flat boundary, the complication would be only technical: curvilinear coordinates and covariant derivatives.) Let us examine the equations for the components of $\sigma_1$ and $\sigma_2$. If we suppose, for example, that $v_1$ is discontinuous while crossing the plane $x_3 = 0$ in the vicinity of the point $x = (0,0,0)$, then there exist bounded continuous functions $A_{11}(x)$, $A_{1p}(x)$ and $A_{13}(x)$ with $A_{13}(0) \neq 0$ such that:

$$\begin{aligned} \sigma_{13}(x) &= \eta(x)A_{13}(x)\delta(x_3) + \{\text{bounded terms}\}, \\ \sigma_{11}(x) &= \lambda(x)A_{11}(x)\delta(x_3) + A_{1p}(x)p + \{\text{bounded terms}\}, \end{aligned}$$

whereas $\sigma_{12}$ is bounded. But then Eq. $(\mathrm{A.2})_1$ cannot be satisfied for functions $\psi_1$. To see this, let us take for example

$$\psi_1 = x_3\,\omega\left(\frac{1}{\varepsilon}x_3\right)\omega\left(\frac{1}{\varepsilon}\sqrt{x_1^2 + x_2^2}\right),$$

where $\omega(y)$ is a $C_0^\infty(R^3)$ function such that $\omega(y) \equiv 1$ for $|y| \le 1$, $0 \le \omega(y) \le 1$ and $\omega(y) \equiv 0$ for $|y| \ge 2$, and choose $\varepsilon$ sufficiently small. So, $v_1$ must be continuous. In the same way we may prove that $v_2$ must be continuous. When the tangent component of $\mathbf{v}$ is continuous, then the distributional sense of derivatives of the components of $\mathbf{v}$ retains its validity even for discontinuous coefficients $\eta$ and $\lambda$

(while crossing the plane $x_3 = 0$). Then, however, the pressure $p$ ceases to be well determined even in the distributional sense, since according to the equation $(A.2)_3$:

$$\int \{(-p + (\lambda + 2\eta)v_{3,3} + S_3)\psi_{3,3} + S_2\psi_{3,2} + S_1\psi_{3,1}\}\, dx = 0,$$

where $S_j$ are bounded, its singular part should be equal to the singular part of the expression $(\lambda + 2\eta)v_{3,3}$. Thus it must be proportional to $(\lambda + 2\eta)\delta(x_3)$ and the last expression is not a well determined distribution (at the boundary surface $x_3 = 0$).

## Acknowledgments

The Authors aknowledge helpful discussion with Dr. J. KURZYNA, Mr. T. LIPNIACKI and Dr. K. PIECHÓR, Institute of Fundamental Technological Research, Warsaw.

## References

1. K.G. GUS'KOV, YU.P. RAYZER and S.T. SURZHIKOV, *On an observed speed of the optical discharge slow propagation* [in Russian], Kvantovaya Elektronika, 17, 7, 937–942, 1990.
2. Z. PERADZYŃSKI and E. ZAWISTOWSKA, *Flow in a region with large density variations*, Arch. Mech., **46**, 6, 881–891, 1994.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

*Received March* 1, 1996.