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Outlooks in Saint Venant theory
Part II. Torsional rigidity, shear-stress "and all that" in the torsion of cylinders with section of variable thickness

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WE EXTEND the perturbative procedure developed in [7] to the case of Saint Venant Cylinders with sections of variable thickness. In this way we are able to generalize the Kelvin and Bredt formulas for torsional rigidity of open and closed sections, respectively. We recover all the results available in technical literature. In particular we deduce an explicit analytical expression for warping function in the cases of open sections of triangular shape [17] and of the closed section studied using numerical methods by WANG [18].

1. Introduction

IN A RECENT PAPER [7] the authors tried to use a "perturbative development" [5] to generalize the well known Bredt formulas in the theory of thin hollow elastic beams. This development is possible for sections of the Saint Venant Cylinders (SVC) constructed from a given curve (the mean curve) as the union of its homotopic curves. The perturbation parameter ε is related to the thickness of the sections. However in [7] the particular homotopic transformation used allows only for the consideration of sections of constant thickness.

Here we want to overcome this limitation by generalizing the results found in [3] and use a similar procedure, but allowing the homotopic transformation to shift along the normal and the tangent directions both depending on the curvilinear coordinate along the inner curve of the sections.

We recover all the classical formulas found by BREDT [1] (see also VLASOV [2]) considering terms of first order in ε in the development. The new procedure we propose in the present paper is general enough to be applied, for instance, to SVC whose doubly connected cross-sections are bounded by ellipses, the case being out of the scope of applicability of the previous ones. In this way we can check our perturbation method on the exact solutions (available in the literature, see [4]) of Saint Venant torsion problem for the homothetic elliptic cross-sections. Moreover, we can give an approximate expression for the warping field in the case of the tubolar section of WANG (cf. [18]) and for the thin isosceles triangle [17].

For the reasons expounded in DELL'ISOLA and RUTA [7] we choose to state the Saint Venant torsion problem in terms of the Prandtl stress function ϕ.

Let $\mathcal{D}$ be the cross-section of the SVC, and let us distinguish two cases: closed sections and open sections. In both cases D can be represented as follows: $\mathcal{D} = \mathcal{D}_1 \backslash \mathcal{D}_0$, where $\mathcal{D}_i$, $i = 0, 1$, are simply connected domains, $\mathcal{D}_0 \subset \mathcal{D}_1$ and $\partial\mathcal{D}_0 \cap \partial\mathcal{D}_1 = \emptyset$ but, in the case of open sections we have $D_0 = \emptyset$.

Prandtl function ϕ is the solution of the following elliptic boundary value problem:

$$
\begin{aligned}
\Delta\phi + 2 &= 0 \quad \text{in } \mathcal{D} \subset \Pi, \\
\phi &= 0 \quad \text{on } \partial\mathcal{D}_1, \\
\phi &= \overline{\phi} \quad \text{on } \partial\mathcal{D}_0, \\
\oint_{\partial\mathcal{D}_0} \nabla\phi \cdot n &= -2A_{\partial\mathcal{D}_0}.
\end{aligned} \tag{1.1}
$$

Here Π is a plane, Δ is the Laplace operator, ∇ is the gradient operator, n is the outer normal of the domain $\mathcal{D}_0$, and $A_{\partial\mathcal{D}_0}$ is its area. The value of ϕ on $\partial\mathcal{D}_0$, $\overline{\phi}$, is an arbitrary constant to be determined from the integral condition $(1.1)_4$.

We will assume that the Prandtl function ϕ [6] can be expanded in terms of ε:

$$
\phi = \sum_{k=0}^{\infty} \phi_k \varepsilon^k \tag{1.2}
$$

in this way we get a hierarchy of ordinary differential equations for the coefficient ϕ_k, which allow us to generalize the well-known Bredt formulas.

Once we have found the expansion for the Prandtl function, we can calculate the corresponding one for the torsional rigidity R, the warping w and the tangent stress t using the following formulas [8, 9, 10]:

$$
\begin{aligned}
&R = 2G \int_{\mathcal{D}_1} \phi + A_{\partial\mathcal{D}_0}\overline{\phi}, \\
&\nabla w(y) = -\tau\left(*\nabla\phi(y) + *(y - o)\right), \qquad t = -G\tau *\nabla\phi,
\end{aligned} \tag{1.3}
$$

where $o \in \Pi$, $*$ is the $\pi/2$-rotation operator in Π, $y \in \mathcal{D}$, G is the modulus of elasticity in shear and τ is the angle of twist.

To this end, we will try the formal expansions also of all the other quantities appearing in the Saint Venant torsion theory in terms of the small parameter ε (for an accurate analysis of these slightly heuristic procedure see NAYFEH [5]):

$$
R = \sum_{n=0}^{\infty} R_n \varepsilon^n, \qquad w(s,z) = \sum_{n=0}^{\infty} w_n(s,z)\varepsilon^n, \qquad t(s,z) = \sum_{n=0}^{\infty} t_n(s,z)\varepsilon^n, \tag{1.4}
$$

thus obtaining, in a very straightforward manner, all the known formulas of the technical theories as terms of the first order in ε. We can find all the terms of higher order in ε and here we quote the next non-zero corrections to these.

2. Families of cross-sections

Let $\Gamma_0 :\ [0,l] \to \Pi$ be the curve of equation

$$r_0 :\ s \mapsto r_0(s). \tag{2.1}$$

We will consider two cases: closed sections and open sections. In the first case we identify the two extrema $0 \sim l$ (we will identify s with the arc-length of the curve Γ_0, thus l will be the length of Γ_0).

Starting from Γ_0, we will consider a family of domains, parameterized by ε. The domain D_ε is obtained as the union of the curves Γ_z: $s \in [0,l] \to \Pi$, with $z \in [0,1]$, z-lifted from Γ_0 by the scalar fields (δ_1, δ_2): $\left(f_{,x} = \dfrac{df(x)}{dx}\right)$

$$r(s,z) = r_0(s) + z\varepsilon\left(\delta_1(s) r_{0,s} - \delta_2(s) {*r_{0,s}}(s)\right), \qquad D_\varepsilon = \bigcup_{z\in[0,1]} (\Gamma_z). \tag{2.2}$$

In this way $\partial\mathcal{D} := \Gamma_0 \bigcup \Gamma_1$ for closed sections while, of course, in the case of open sections we cannot obtain, by means of this procedure, the whole boundary of the domain D because we loose the edges z-lifted from the two distinct points $0, l$.

For these reasons we must assume that for open section the expansion is valid only far away from the ending edges. The expansion we obtain in this paper is an "outer" expansion to be matched with an "inner" one (see NAYFEH [5]) accounting for some edge effect.

We can think of $\delta(s) = \sqrt{\delta_1^2 + \delta_2^2}$ as of a thickness of the section in the point of coordinate s measured along Γ_0, and we will call $(\Gamma_0, \delta_1(s), \delta_2(s))$ the "shape" of the section.

In the following we will consider the cylinder of section $\mathcal{D} = \mathcal{D}_1 \backslash \mathcal{D}_0$ whose boundary is $\partial\mathcal{D} = \Gamma_0 \bigcup \Gamma_1$. Γ_0 is a closed curve for closed SVC sections and an open curve for open SVC sections. In the latter case we have $D_0 = \emptyset$.

Considering the couple (s, z) as a coordinate system on D_ε, we get the following holonomic basis (when not necessary we omit the explicit s-dependence of the various functions)

$$\begin{aligned} e_1(s,z) &= \frac{\partial r}{\partial s} = r_{0,s}\left(1 + z\varepsilon(\delta_{1,s} + \delta_{2,s}K)\right) + z\varepsilon * r_{0,s}(K\delta_1 - \delta_{2,s}),\\ e_2(s,z) &= \frac{\partial r}{\partial z} = \varepsilon(r_{0,s}\delta_1 - *r_{0,s}\delta_2) \end{aligned} \tag{2.3}$$

($K(s)$ is the curvature of Γ_0, $i = 1,2$) and the following metric-tensor:

$$g^{ij} = \frac{1}{g}\begin{pmatrix} \varepsilon^2(\delta_1^2 + \delta_2^2) & -(\varepsilon\delta_1 + z\varepsilon^2(\delta_1\delta_{1,s} + \delta_2\delta_{2,s})) \\ -(\varepsilon\delta_1 + z\varepsilon^2(\delta_1\delta_{1,s} + \delta_2\delta_{2,s})) & (1 + z\varepsilon(\delta_{1,s} + K\delta_2))^2 + z^2\varepsilon^2(\delta_{2,s} - K\delta_{1,s})^2 \end{pmatrix}, \tag{2.4}$$

where $g = \varepsilon^2 \left[\varepsilon z \left(\delta_1\delta_{2,s} - \delta_{1,s}\delta_2 - K(\delta_1^2 + \delta_2^2)\right) - \delta_2\right]^2$ is the determinant of the metric tensor.

For the sake of completeness we quote here the expression of the gradient and Laplacian that will be used in the following [11, 12]:

$$
\begin{aligned}
&\nabla\phi = g^{ij}\phi_{,i}e_j\,,\\
(2.5)\qquad &\Delta\phi = g^{ij}\left(\frac{\partial^2\phi}{\partial x^i\partial x^j} - \frac{\partial\phi}{\partial x^h}\left\{\begin{matrix} h\\ i\ j\end{matrix}\right\}\right) = \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial x_i}\sqrt{g}g^{ij}\phi_{,i}\right),
\end{aligned}
$$

$\left\{\begin{matrix} h\\ i\ j\end{matrix}\right\}$ are the Christoffel symbols, $i, j, h = 1, 2$; $x_1 = s$, $x_2 = z$.

3. Formal expansion of the Prandtl function

Using (1.2) and $(2.5)_2$ Eq. (1.1) becomes:

$$
\begin{aligned}
(3.1)\qquad \sum_{n=0}^{\infty}\Big\{\ &\varepsilon^n A\phi_{n,zz}\\
&+ \varepsilon^{n+1}(B_1\phi_{n,z} + B_2\phi_{n,zz} + B_3\phi_{n,sz})\\
&+ \varepsilon^{n+2}\left[C_1\phi_{n,z} + C_2\phi_{n,zz} + C_3\phi_{n,s} + C_4\phi_{n,sz} + C_5\phi_{n,ss}\right]\\
&+ \varepsilon^{n+3}\left[D_1\phi_{n,z} + D_2\phi_{n,zz} + D_3\phi_{n,s} + D_4\phi_{n,sz} + D_5\phi_{n,ss}\right]\Big\}\\
&= -2\varepsilon^2\left[\varepsilon z\left(\delta_1\delta_{2,s} - \delta_{1,s}\delta_2 - K(\delta_1^2 + \delta_2^2)\right) - \delta_2\right]^3.
\end{aligned}
$$

Here

$$
\begin{aligned}
A &= -\delta_2\,,\\
B_1 &= K(\delta_1^2 - \delta_2^2) - 2\delta_1\delta_{2,s}\,,\\
B_2 &= z\delta_1\delta_{2,s} - 3z\delta_2\delta_{1,s} - 3zK\delta_2^2 - zK\delta_1^2\,,\\
B_3 &= 2\delta_1\delta_2\,,\\
C_1 &= z\Big(-2K^2(\delta_1^2\delta_2 + \delta_2^3) - K(\delta_1^2\delta_{1,s} + 3\delta_2^2\delta_{1,s} + 2\delta_1\delta_2\delta_{2,s}) - 2\delta_1\delta_{1,s}\delta_{2,s} - 2\delta_2\delta_{2,s}^2\\
&\qquad - K_{,s}\left(\delta_1^3 + \delta_1\delta_2^2 + \delta_1^2\delta_{2,ss} + \delta_2^2\delta_{2,ss}\right)\Big),\\
C_2 &= z^2\Big(-3K^2\delta_2(\delta_1^2 + \delta_2^2) - \delta_{1,s}K(2\delta_1^2 - 6\delta_2^2) + \delta_{1,s}(-3\delta_{1,s}\delta_2 + 2\delta_1\delta_{2,s})\\
&\qquad + \delta_2\delta_{2,s}(4\delta_1 K - \delta_{2,s})\Big),\\
C_3 &= \delta_1\left(2\delta_1\delta_{2,s} - \delta_1^2 K - \delta_2^2 K - 2\delta_2\delta_{1,s}\right),\\
C_4 &= 2z\left(\delta_1^3 K + \delta_1\delta_2^2 K + 2\delta_1\delta_2\delta_{1,s} - \delta_1^2\delta_{2,s} + \delta_2^2\delta_{2,s}\right),
\end{aligned}
$$

$$
\begin{aligned}
C_5 &= -\delta_1^2\delta_2 - \delta_2^3\,,\\
D_1 &= -z^2\left(\delta_1^2+\delta_2^2\right)\delta_1^2 k^3 - \delta_2^2 K^3 - 3\delta_2\delta_{1,s}K^2 - 2\delta_{1,s}^2 K + 3\delta_1\delta_{2,s}K^2 - 2\delta_{2,s}^2 K\\
&\qquad - \left(\delta_1\delta_{1,s} + \delta_2\delta_{2,s}\right)K_{,s} + \delta_1\delta_{1,ss}K - \delta_{1,ss}\delta_{2,s} + \delta_2\delta_{2,ss}K + \delta_{1,s}\delta_{2,ss}\,,\\
D_2 &= z^3\Big(-\delta_1^4 K^3 - 2\delta_1^2\delta_2^2 K^3 - \delta_2^4 K^3 - 3\delta_1^2\delta_{1,s}\delta_{2,}K^2 - 3\delta_2^3\delta_{1,s}K^2 - \delta_1^2\delta_{1,s}^2 K\\
&\qquad -3\delta_{1,s}^2\delta_2^2 K - \delta_{1,s}^3\delta_2 + 3\delta_1^3\delta_{2,s}K^2 + 3\delta_1\delta_2^2\delta_{2,s}K^2 + 4\delta_1\delta_2\delta_{1,s}\delta_{2,s}K\\
&\qquad +\delta_1\delta_{1,s}^2\delta_{2,s} - 3\delta_1^2\delta_{2,s}^2 K - \delta_2^2\delta_{1,s}\delta_{2,s}^2 + \delta_2^2\delta_{2,s}^2 K + \delta_1\delta_{2,s}^3\Big),\\
D_3 &= z\Big(-2\delta_1\delta_{1,s}^2\delta_2 + 2\delta_1^2\delta_{1,s}\delta_{2,s} - 2\delta_{1,s}\delta_{2,s}\delta_2^2 + 2\delta_1\delta_2\delta_{2,s}^2 + \delta_1^4 K_{,s} + 2\delta_1^2\delta_2^2 K_{,s}\\
&\qquad +\delta_2^4 K_{,s} + \delta_1^2\delta_2\delta_{1,ss} + \delta_{1,ss}\delta_2^3 - \delta_1^3\delta_{2,ss} - \delta_1\delta_2^2\delta_{2,ss}\Big),\\
D_4 &= 2z^2\left(\delta_1^2 K + \delta_2^2 K + \delta_{1,s}\delta_2 - \delta_1\delta_{2,s}\right)\left(\delta_1\delta_{1,s} + \delta_2\delta_{2,s}\right),\\
D_5 &= z\left(\delta_1^2 + \delta_2\right)\left(-\delta_1^2 K - \delta_2^2 K + \delta_1\delta_{2,s} - \delta_{1,s}\delta_2\right).
\end{aligned}
$$

3.1. Closed section

Noticing that $(2.5)_1$ $\nabla\phi\cdot n|_{z=0} = -\dfrac{\delta_1}{\delta_2}\phi_{,s} + \dfrac{1}{\delta_2}\phi_{,z}$ and using (1.2), we get for condition $(1.1)_4$

$$
\sum_{n=0}^{\infty}\varepsilon^n \oint_{\Gamma_0}\left\{-\varepsilon\frac{\delta_1}{\delta_2}\phi_{,s} + \frac{1}{\delta_2}\phi_{,z}\right\} = -2\varepsilon A_0\,. \tag{3.2}
$$

In this way we get for the first three terms of the ε-expansion of the Prandtl function:

$$
\begin{aligned}
&\phi_{0,zz}(z,s) = 0, && \phi_0(0,s) = \overline{\phi}_0\,,\\
&\phi_0(1,s) = 0, && \oint_{\Gamma_0}\frac{1}{\delta_2}\overline{\phi}_0 = 0,\\[2ex]
&\phi_{1,zz}(z,s) = 0, && \phi_1(0,s) = \overline{\phi}_1\,,\\
&\phi_1(1,s) = 0, && \oint_{\Gamma_0}\frac{1}{\delta_2}\overline{\phi}_1 = 2A_0\,,\\[2ex]
&A\phi_{2,zz} + B_1\phi_{1,z} = 2\delta_2^3\,, && \phi_2(0,s) = \overline{\phi}_2\,,\\
&\phi_2(1,s) = 0, && \oint_{\Gamma_0}\frac{1}{\delta_2}\overline{\phi}_2 = -\oint_{\Gamma_0}\frac{A_0}{I_0}J - \delta_2\,,
\end{aligned}
\tag{3.3}
$$

with $I_0 = \oint_{\Gamma_0}\delta_2^{-1}$, $I(s) = \int_0^s \delta_2^{-1}$, $J = \dfrac{2\delta_1\delta_{2,s} + \left(\delta_2^2 - \delta_1^2\right)K}{\delta_2^2}$.

Solving Eqs. (3.3) we get

$$\tag{3.4}
\begin{aligned}
&\phi_0(s,z) = 0, \qquad \phi_1(s,z) = \frac{2A_0}{I_0}(1-z),\\
&\phi_2(s,z) = \frac{(1-z)}{I_0}\left\{\oint\limits_{\Gamma_0}\delta_2 - \frac{A_0}{I_0}\oint\limits_{\Gamma_0} J\right\} + (z^2-z)\frac{2A_0}{I_0}\left\{J - 2\delta_2^2\right\}.
\end{aligned}$$

3.2. Open sections

In this case we have (up to ε^4)

$$\tag{3.5}
\begin{gathered}
\phi_{0,zz}(z,s) = 0, \qquad \phi_0(0,s) = 0, \qquad \phi_0(1,s) = 0,\\
\phi_{1,zz}(z,s) = 0, \qquad \phi_1(0,s) = 0, \qquad \phi_1(1,s) = 0,\\
A\phi_{2,zz} = 2\delta_2^3, \qquad \phi_2(0,s) = 0, \qquad \phi_2(1,s) = 0,\\
A\phi_{3,zz} - B_1\phi_{2,z} + B_2\phi_{2,zz} + B_3\phi_{2,sz} = 6z\delta_2^2\left[\delta_2\delta_{1,s} - \delta_1\delta_{2,s} + K(\delta_2^2+\delta_1^2)\right],\\
\phi_3(0,s) = 0, \qquad \phi_3(1,s) = 0,
\end{gathered}$$

from which

$$\tag{3.6}
\begin{aligned}
&\phi_0 = 0, \qquad \phi_1 = 0, \qquad \phi_2 = \delta_2^2(z-z^2),\\
&\phi_3 = \frac{1}{6}g(s)(z^3-z) + \frac{1}{2}f(s)(z^2-z),
\end{aligned}$$

with the following notations:

$$g(s) = -\delta_2\left[6\delta_{2,s}\delta_1 + 2K(\delta_1^2+\delta_2^2)\right], \qquad f(s) = \delta_2\left[6\delta_{2,s}\delta_1 + K(\delta_2^2-\delta_1^2)\right].$$

4. Torsional rigidity, warping and shear stress

Using formulas (1.3) and the expansions (1.4), we obtain the following results.

4.1. Closed sections

$$\tag{4.1}
\begin{aligned}
&R_0 = 0, \qquad R_1 = \frac{4GA_0^2}{I_0},\\
&R_2 = \frac{4GA_0}{I_0^2}\left\{I_0\oint\limits_{\Gamma_0}\delta_2 - \frac{A_0}{2}\oint J\right\}.
\end{aligned}$$

For the warping

$$\frac{w_0(s,z)}{\tau} = 2A_0\frac{I(s)}{I_0} - \int_0^s r_0 \times r_{0,s}\,,$$

(4.2)
$$\frac{w_1(s,z)}{\tau} = \frac{A_0}{I_0}\int_0^s J\delta_2^2 + \frac{I(s)}{I_0}\left\{\oint_{\Gamma_0}\delta_2 - \frac{A_0}{I_0}\oint_{\Gamma_0} J\right\} - \int_0^s \delta_2$$
$$+z\left\{2\frac{A_0}{I_0}\frac{\delta_1}{\delta_2} - \delta_1 * r_0\cdot r_{0,s} + \delta_2 r_0\cdot r_{0,s}\right\},$$

and finally for the tangential stress

$$\frac{t_0(s,z)}{G\tau} = \left(\frac{t_{0s}}{G\tau}, \frac{t_{0z}}{G\tau}\right) = \left(2A_0\frac{I_0}{\delta_2}, 0\right),$$

(4.3)
$$\left(\frac{t_{1s}}{G\tau}, \frac{t_{1z}}{G\tau}\right) = \left(2\frac{A_0}{I_0}\left[J/2 + z(\delta_2\delta_{1,s} - \delta_1\delta_{2,s})\right] + \delta_2(2z-1)\right.$$
$$\left. + \frac{1}{\delta_2 I_0}\left\{\oint_{\Gamma_0}\delta_2 - \frac{A_0}{I_0}\oint_{\Gamma_0} J\right\},\ 2\delta_1 A_0\frac{I_0}{\delta_2}\right).$$

The values R_1, w_0 and t_0 are the usual ones quoted in the literature [14, 15, 16]; they are due to BREDT [1]. We emphasize that for the rather general cross-sections considered here, the first non-zero contribution to the z-component of the shearing stress is of the first order in ε. This means that the procedure proposed by Bredt in deducing his formulas (in which this z-component is assumed as vanishing), cannot be applied for the sections considered in the present paper, being valid only for the class of sections dealt with in [7].

4.2. Open sections

We find for the torsional rigidity:

$$R_0 = 0, \qquad R_1 = 0,$$
$$R_2 = 0, \qquad R_3 = \frac{G}{3}\oint_{\Gamma_0}\delta_2^3,$$

(4.4)
$$R_4 = \frac{1}{12}\oint_{\Gamma_0}\left\{\delta_2^2\left[\delta_{1,s}\delta_2 - \delta_{2,s}\delta_1 + K(\delta_1^2+\delta_2^2)\right]\right\}$$
$$-\frac{1}{12}\oint_{\Gamma_0}\delta_2\left(f(s) + \frac{g(s)}{2}\right).$$

For the warping

$$
\text{(4.5)}\qquad
\begin{aligned}
\frac{w_0(s,z)}{\tau} &= -\int_0^s r_0 \times r_{0,s}\,,\\
\frac{w_1(s,z)}{\tau} &= z\left(r_0\cdot r_{0,s}\delta_2 - r_0 \times r_{0,s}\delta_1\right) - \int_0^s \delta_2\,,
\end{aligned}
$$

and finally for the tangential stress

$$
\text{(4.6)}\qquad
\begin{aligned}
\frac{t_0(s,z)}{G\tau} &= \left(\frac{t_{0s}}{G\tau}, \frac{t_{0z}}{G\tau}\right) = (0,\ 0),\\
\left(\frac{t_{1s}}{G\tau}, \frac{t_{1z}}{G\tau}\right) &= \left(-\delta_2(1-2z),\ 0\right).
\end{aligned}
$$

As in this case we do not consider the effect due to the "short ends" of the section, it seems reasonable that there is no influence of the edge affect up to the fourth order, at least in connection with torsional rigidity, but this needs more investigation.

5. Conclusions and perspectives

In this final section we consider some applications of the results found in the previous ones. The first application concerns the torsion of a section bounded by two ellipses: in particular we find the expression for torsional rigidity available in the literature for sections bounded by homothetic ellipses. As a second application we find the warping field for a section studied by WANG [18] (who used a rather sophisticated numerical method): we are able to supply a simple explicit polynomial perfectly matching his numerical results.

Finally as a third application, we recover the results found in [17] concerning torsion of the cylinder whose cross-section is an isosceles triangle, under the assumption that its base is much shorther than its altitude.

5.1. Section bounded by two non-homothetic ellipses

Let $\mathcal{D}$ be the section enclosed between two non-homothetic ellipses Γ_0 and Γ_1 whose parametric representations are, respectively:

$$
\text{(5.1)}\qquad
\begin{aligned}
r_0 &: \ [0,2\pi] \to \Pi, \quad r_0 = (a\cos\varphi,\ b\sin\varphi),\\
r &: \ [0,2\pi] \to \Pi, \quad r = (ka\cos\varphi,\ (k+q)b\sin\varphi);
\end{aligned}
$$

we get for the torsional rigidity

$$
(5.2)\qquad
\begin{aligned}
R_1 &= 2G\pi a^3 b^3 \frac{q}{p},\\
R_2 &= G\pi a^3 b^3 \frac{q^2}{p^2}\left\{\frac{2(b^2-a^2)(1-k)}{q}(1+c)+2a^2+c\left[\frac{b^2}{2}-\frac{a^2(1-k)}{2(k+q-1)}\right]\right\}
\end{aligned}
$$

with $c=\sqrt{\dfrac{q+k-1}{k-1}}$ and $p=a^2-b^2+c\left[\dfrac{(b^2-a^2)(k-1)+b^2q}{k+q-1}\right]$.

When $q\to 0$ we find

$$
(5.3)\qquad R_1 = 4\pi G\frac{a^3b^3(k-1)}{a^2+b^2},\qquad R_2 = 6\pi G\frac{a^3b^3(k-1)^2}{a^2+b^2}
$$

in agreement with the well-known (exact) formula.

We observe that for fixed a, b and k, the ratio R_2/R_1 is a function of q. Choosing $a = 4$, $b = 2$ and $k = 1.3$ we get

$$
(5.4)\qquad \frac{R_2(k-1)^2}{R_1(k-1)} \simeq 0.135 + 0.292q - 0.091q^2 + 0.122q^3 + O(q^4);
$$

so, for example, with $q = 0.2$ we find $\dfrac{R_2(k-1)^2}{R_1(k-1)} \simeq 20\%$.

5.2. The warping field for a flattened tube

The efficiency of our asympthotic expansion is here tested on a section which is not thin and which was studied by WANG [18] using numerical methods. For a discussion of the limits of the present form of our expansion we refer to [19]. We consider the linear (in z coordinate) terms appearing in the first four terms of the asymptotic expansion for warping, calculated in the particular case examined, thus finding:

$$
(5.5)\qquad
\begin{aligned}
\frac{w}{\tau} &= \left(9-\frac{8}{3(8+3\pi)}\right)\frac{s}{(8+3\pi)}-\frac{1}{2}(1-z)\sin(2s), \qquad \text{if } s\in(0,\pi/4),\\
\frac{w}{\tau} &= \frac{\left[(4s-4-\pi)(384+80\pi-27\pi^2+384z+288\pi z+54\pi^2 z)\right]}{48(8+3\pi)^2},\\
&\qquad \text{if } s\in(\pi/4, 1+\pi/4).
\end{aligned}
$$

It is very easy to check that the contour plots we produce exactly coincide with those given by Wang. Because the (s, z) coordinate-system is meaningful also outside the section and because the Prandtl and warping functions are determined as elementary functions of these coordinates, they can be extended outside of the section. Thus we have a hint about the form of warping for larger sections. The scale is immaterial for the elliptic problem determining warping (see [4]).

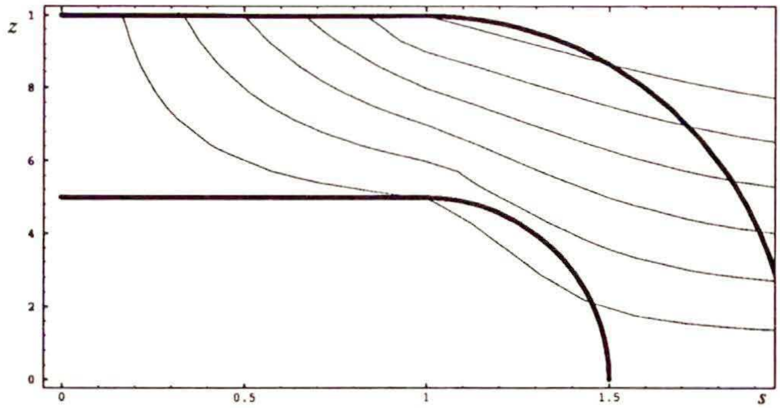

FIG. 1. The figure shows the iso-warping contour lines for the flattened tube studied in [18].

5.3. Warping field of thin triangular cross-sections

It is easy to generate the triangular cross-section considered on page 74 of [17] using the following values of δ_1 and δ_2 expressed as functions of the altitude C and basis h of the triangle:

$$\delta_1 = s\frac{2(h/C)^2}{4+(h/C)^2}, \qquad \delta_2 = s\frac{4(h/C)}{16+(h/C)^2}. \tag{5.6}$$

Using formulas (4.5) we prove the validity of assumption (1.3) p. 6 [17] at the first order of the ratio h/C. The warping field we find at the same order is given by:

$$\frac{w(s,z)}{\tau} = s^2\left(\frac{1}{2}-z\right)\frac{4h/C}{16+(h/C)^2}. \tag{5.7}$$

It is easy to see that Eq. (5.7) coincides with formula (2.19) on p. 75 of [17] modulo a rigid motion.

5.4. Conclusions

Finally we want make a few comments on the results obtained. Despite the fact that our procedure is rather general, it is not capable of reproducing the most general cross-section. Maybe this task can be solved by means of the Conformal Mapping Theory [20].

In [19] are studied some cases in which the proposed expansion does not converge. Therefore – assuming that before diverging the expansion seems to approach reasonably the solution – a regularizing method seems to be necessary to increase its scope of applicability.

On the other hand – from the mathematical point of view – our results seem to open some interesting estimation problems which most likely can be solved using the methods of the papers [21, 22].

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