Travelling wave solutions to model equations of van der Waals fluids

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WE CONSIDER the existence and uniqueness of travelling wave solutions to the model hydrodynamics equations (without capillarity) obtained from a four-velocity kinetic model of van der Waals fluids. We analyze both the Euler and the Navier–Stokes equations. The Euler equations are shown to change their type. The Rankine–Hugoniot conditions are discussed in detail. It is shown that the Hugoniot locus can be disconnected even if the equations are hyperbolic. Using the Navier–Stokes equations we show how to modify the Oleinik–Liu conditions of admissibility of shock waves to such situations. The shock-wave structures are found numerically. In particular, the so-called impending shock splitting is obtained.

1. Introduction

THE VAN DER WAALS fluid is such a hypothetical one whose equation of state reads [1]

$$(1.1) \qquad p(w,T) = \frac{RT}{w-b} - \frac{a}{w^2},$$

where a, b are positive constants characterizing the fluid, $p(w,T)$ is the pressure, R is the gas constant, T is the temperature, and $w > b$ is the specific volume. Now much more sophisticated equations of state are known [2, 3, 4], but Eq. (1.1) is still in use since it describes qualitatively correctly the thermodynamic behaviour of real one-component fluids.

If

$$(1.2) \qquad T > \frac{81}{256} \frac{a}{bR},$$

then the isotherms in the $p-w$ plane are monotonically decreasing convex curves. This is the case of classical gases. The mathematical background is the Lax theory of hyperbolic conservation laws [5].

If

$$(1.3) \qquad \frac{8}{27} \frac{a}{bR} < T < \frac{81}{256} \frac{a}{bR},$$

then the isotherms in the $p-w$ plane are still monotonically decreasing – but they are no longer convex. This occurs in the so-called retrograde or Bethe–Zeldovich–Thompson fluids. Such materials were considered in many papers [6–17].

The third case is when T satisfies

(1.4)
$$\frac{1}{4}\frac{a}{bR} < T < \frac{8}{27}\frac{a}{bR}.$$

The left-hand inequality guarantees that the pressure is positive for all $w >$ b. Now, the isotherms are nonmonotone curves in the $p - w$ plane, and the Euler equations are of mixed hyperbolic-elliptic type. In this case there is no one prevailing theory, and various approaches can be found [18–35]. Closely related problems are met in the theory of elastic rods [36–48].

The equations studied in [4–48] are those of phenomenological thermodynamics. However, at least as fluids are concerned, such a theory cannot describe correctly the structures of neither the shock waves nor the phase boundaries because, in those regions, the gradients of the flow parameters are very large. Hence, the use of kinetic theory seems to be inevitable. Usually one proceeds as follows: the Boltzmann equation is used in the gaseous domain and the fluid bulk is treated as a source (evaporation) or sink (condensation) of particles. References [49–51] represent three of many papers on the topic.

We propose a more radical approach consisting in the use of one kinetic equation both to the liquid and the gaseous phase. Thus, in a sense, we attempt to follow the lines of the van der Waals' philosophy of fluids [1], which is used in the quoted papers [2–35] on liquid-vapour phase transitions. In the papers, one system of hydrodynamic equations with one equation of state suited for liquid-vapour systems is used without any splitting into liquid and gaseous domains. The essential difference between this approach and that of ours consists in that that we want to replace the hydrodynamic description of the system with a kinetic one, and next to compare the results.

The fundamental trouble is the lack of such a universal and fully satisfactory kinetic equation. But this does not mean that there are no models that could be suitable for our purpose. We have chosen the Enskog–Vlasov equation because: i) it is relatively simple; ii) there are some results in [52, 53] suggesting its usefulness. Recently, we showed in [54] that the capillarity equations used in [18–21, 28, 32] can be deduced, at the formal level, from this equation.

Unfortunately, if we want to investigate any flow by means of the Enskog–Vlasov equation, we find it to be too complicated. That is why we elaborated its discrete velocity models (see [55, 56]). In this way we obtain a more tractable system of equations. Basing on the successes of discrete kinetic theory of ideal gases ([57–59]) we hope that this approach will not be a failure in the case of interest.

There are many problems which can be posed. First of all we have to give evidence that our discrete velocity model can be successfully applied to at least some of the phase transition problems. The next question is the relation between the results of our approach and those of [49–51], where kinetic theory was applied to the gaseous phase only.

Another group of problems concerns the connection between the fluid dynamic and kinetic descriptions of phase changes. We know from the theory of the true Boltzmann equation [60] as well as from the theory of its discrete velocity models [57, 61] that the phenomenological fluid dynamics describes correctly the shock wave structure only if the shock is sufficiently weak. In the case under consideration the situation seems to be much more complicated. Namely, in [56] we considered the stagnant phase boundary problem. It turned out that both the model kinetic and generated by it fluid dynamic equations have exactly the same solution. The description of the phase boundary obtained in [56] agrees both with the physics of equilibrium phase transitions and the theoretical analysis of [18], hence it favours our model. But, on the other hand, this result is in contrast with the results of kinetic theory of ideal gases ([57, 60, 61]), because the stagnant phase boundary by no means can be treated as a "weak" shock wave. The explanation of this apparent paradox must be sought in the structure of the local equilibrium, i.e. the Euler equations. In the case of the ideal gases both the true and the model Euler equations are strictly hyperbolic, and the characteristic speeds are either genuinely nonlinear or linearly degenerate in the sense of LAX [5]. It is worth to add that all the existing papers on the hydrodynamic limit of the true Boltzmann [62, 63, 65] or the Enskog equation [64, 65], or else the discrete Broadwell model [66 – 68], and more generally some hyperbolic systems of similar structure as the latter ones [69, 70] make an essential use of the strict hyperbolicity of the local equilibrium conservation equations. Very clearly it is pointed out in [70].

In our problem, as we show it later in this paper, the local equilibrium equations, i.e. the Euler equations, can change type from hyperbolic to elliptic. The question arises: how important is it? This will be discussed in our future papers, but for the time being let us notice that: i) the stagnant phase boundary discussed in [56] is admissible only due to the change of the type of the Euler equations; ii) if the formally deduced local equilibrium equations are elliptic, then they cannot serve as an approximation, as the Knudsen number tends to zero, to the kinetic equations if the latter are strictly hyperbolic. A brilliant example is given in [70]. Hence, the Euler, Navier – Stokes and other equations deduced from the kinetic theory should, with a great caution, be treated as "approximation" to the corresponding kinetic equations.

With the present paper we open systematic studies of various "approximations" to the model kinetic equations. Now we limit ourselves to the Euler and Navier – Stokes equations only, but most of the present results will be used in the future.

In the next Section we classify the Euler equations and give sufficient and necessary criteria for their being of a definite type.

In Sec. 3 we consider shock waves and discuss the solvability of the Rankine – Hugoniot conditions. The properties of these solutions are investigated in Sec. 4.

Section 5 deals with the shock waves in the Navier–Stokes equations. The most important result is Theorem 5.8 stating the sufficient and necessary conditions for existence and uniqueness of the travelling wave solutions to our equations.

In Sec. 6 we give some numerical results concerning the structures of the shock waves discussed in Sec. 5. Our results agree qualitatively with those of [4]. In this way we obtain a consecutive confirmation of usefulness of our model for the qualitative analysis of the dynamic phase changes.

2. Classification of the Euler equations

In the lowest order of approximation to a four-velocity model of the Enskog–Vlasov equation, we obtained in [56] the following system

$$(2.1) \qquad \frac{\partial w}{\partial t} - \frac{\partial u}{\partial x} = 0,$$

$$(2.2) \qquad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0,$$

where $t \geq 0$ is the time, $x \in \mathbb{R}$ is the Lagrangian mass coordinate, u is the velocity, w is the specific volume, and p is the pressure.

The pressure formula reads

$$(2.3) \qquad p = p(w, u) = \frac{1 - u^2}{2(w - b)} - \frac{a}{w^2},$$

where a and b are positive constants; a is the ratio of the mean value of the potential of the attractive tail to the mean kinetic energy, and b can be taken to be unity.

Equations (2.1)–(2.3) form the Euler equations for our model hydrodynamics. We consider them in the following domain:

$$(2.4) \qquad w > b, \qquad \frac{a}{2b} < 1, \qquad \blacksquare$$

The set of (w, u) satisfying (2.4) is denoted by $\mathcal{D}$.

Condition $(2.4)_1$ is obvious: the density $1/w$ does not exceed the close-packing density $1/b$. The remaining constraints result from the physically reasonable demand that the pressure p is positive. Indeed, the immediate consequence of that and (2.3) is

$$(2.5) \qquad \frac{1 - u^2}{2} > \frac{a(w - b)}{w^2}.$$

But for every $w > b$ the following estimates hold

$$0 < \frac{a(w - b)}{w^2} \leq \frac{a}{4b}.$$

Hence, if for some u_0

$$\frac{1 - u_0^2}{2} \leq \frac{a}{4b},$$

then there is $w_0 > b$ such that $p(w_0, u_0) < 0$, contrary to our assumption. Therefore we have to admit only such values of u that $(2.3)_3$ holds.

It we denote

$$(2.6) \qquad T = \frac{1 - u^2}{2},$$

then (2.3) takes the well-known form of the van der Waals equation of state (1.1) provided that T given by (2.6) is interpreted as the temperature.

We rewrite the Euler equations in the matrix form

$$(2.7) \qquad \frac{\partial}{\partial t} \begin{pmatrix} w \\ u \end{pmatrix} + \mathbb{M}(w, u) \cdot \frac{\partial}{\partial x} \begin{pmatrix} w \\ u \end{pmatrix} = 0,$$

where

$$(2.8) \qquad \mathbb{M} = \begin{bmatrix} 0 & -1 \\ -\dfrac{1 - u^2}{2(w - b)^2} + \dfrac{2a}{w^3} & -\dfrac{u}{w - b} \end{bmatrix}.$$

The eigenvalues of $\mathbb{M}$ are called the characteristic speeds. They are solutions of

$$(2.9) \qquad \lambda^2 - \frac{\partial p(w, u)}{\partial u} \lambda + \frac{\partial p(w, u)}{\partial w} = 0,$$

or explicitly

$$(2.10) \qquad \lambda^2 + \frac{u}{w - b} \lambda - \left[\frac{1 - u^2}{2(w - b)^2} - \frac{2a}{w^3} \right] = 0.$$

The system (2.1), (2.2) is called strictly hyperbolic if Eq. (2.9) has two real solutions, and elliptic if both solutions of (2.9) are complex.

We have

LEMMA 1.
i) If

$$(2.11) \qquad \frac{a}{2b} < \frac{27}{37}$$

then for every $(w, u) \in \mathcal{D}$ the Euler equations are strictly hyperbolic;

ii) if

$$(2.12) \qquad \frac{27}{37} < \frac{a}{2b} < \frac{27}{32},$$

then they are hyperbolic-elliptic. The domain $\mathcal{H}$ of hyperbolicity is simply connected and separates the two components of the domain of ellipticity $\mathcal{E}$;

  iii) if

(2.13)
$$\frac{27}{32} < \frac{a}{2b} < 1,$$

then the Euler equations continue to be hyperbolic-elliptic, but the domain of ellipticity $\mathcal{E}$ is simply connected and separates the two components of the domain of hyperbolicity $\mathcal{H}$.

Cases ii) and iii) are shown in Figs. 2 and 3 where the domain of ellipticity is shaded.

P r o o f. Equation (2.10) has two real solutions if and only if

(2.14)
$$\Delta(w, u) = \frac{2 - u^2}{(w - b)^2} - \frac{8a}{w^3}$$

is positive. This is equivalent to

(2.15)
$$u^2 < \left[1 - \frac{4a(w - b)^2}{w^3}\right].$$

However, for any $w > b$

$$0 < \frac{4a(w - b)^2}{w^3} \le \frac{16}{27}\frac{a}{b},$$

and the equality sign takes place for $w = 3b$ only. Therefore, if a/b is such that

$$u^2 < 1 - \frac{a}{2b} < 2\left(1 - \frac{16}{27}\frac{a}{b}\right),$$

then we have i). If

$$0 < 2\left(1 - \frac{16}{27}\frac{a}{b}\right) < u^2 < 1 - \frac{a}{2b},$$

we have ii), and if

$$1 - \frac{16}{27}\frac{a}{b} < 0,$$

we have iii). The proof is complete.

The change of type of the Euler equations is physically interpreted as the phase transition. Case iii) of Lemma 1 is of particular interest since it resembles the situation met in the theory of the true van der Waals fluids (see [18–48]).

PROPOSITION 1.

i) If $(w_0, u_0) \in \mathcal{E}$, and $(w, u) \in \mathcal{E}$, then

(2.16)
$$\frac{p(w, u_0) - p(w_0, u_0)}{w - w_0} > \frac{u_0^2}{4(w - b)(w_0 - b)}.$$

ii) If $(w_0, u_0) \in \mathcal{H}$, $(w, u) \in \mathcal{H}$, and the interval $\langle (w_0, u_0), (w, u) \rangle \subset \mathcal{H}$, then

(2.17)
$$\frac{p(w, u_0) - p(w_0, u_0)}{w - w_0} < \frac{u_0^2}{4(w - b)(w_0 - b)}.$$

P r o o f. First, let us notice that if $(w_0, u_0) \in \mathcal{E}$, $(w, u) \in \mathcal{E}$ then the interval $\langle (w_0, u_0), (w, u_0) \rangle \subset \mathcal{E}$. On the other hand, if $(w_0, u_0) \in \mathcal{H}$, $(w, u_0) \in \mathcal{H}$ then, in general, this is not true, and that is why we have to strengthen the assumptions in the hyperbolic Case ii).

Secondly, let us notice that the left-hand sides of (2.16) and (2.17) are symmetric in their arguments w and w_0. Therefore it is enough to prove these inequalities in the case of $w > w_0$ only. We have

$$\frac{p(w, u_0) - p(w_0, u_0)}{w - w_0} = \frac{1}{w - w_0} \int_{w_0}^{w} \frac{\partial}{\partial \zeta} p(\zeta, u_0) \, d\zeta.$$

CASE i)

For every $w_0 \leq \zeta \leq w$, we have $\Delta(\zeta, u_0) < 0$. Therefore

(2.18)
$$\frac{\partial}{\partial \zeta} p(\zeta, u_0) > \frac{1}{4} \left(\frac{\partial p(\zeta, u_0)}{\partial u} \right)^2 - \frac{u_0^2}{4(\zeta - b)^2}.$$

Hence

$$\frac{p(w, u_0) - p(w_0, u_0)}{w - w_0} > \frac{u_0^2}{4(w - w_0)} \int_{w_0}^{w} \frac{d\zeta}{(\zeta - b)^2} - \frac{u_0^2}{4(w - b)(w_0 - b)},$$

and (2.16) is proved.

To prove (2.17) we proceed in a similar way, the only difference being that in (2.18) it is necessary to change the direction of the inequality sign. The proof is complete.

3. The shock speed problem

A discontinuous solution

(3.1)
$$(w, u)(x, t) = \begin{cases} (w_l, u_l) & \text{for } x < st, \\ (w_r, u_r) & \text{for } x > st, \end{cases}$$

of Eq. (2.1), with shock speed s, is called a shock wave. Here, (w_l, u_l) and (w_r, u_r) are some constant values. To simplify the notation we write (w, u) for (w_l, u_l) or (w_r, u_r), and (w_0, u_0) for (w_r, u_r) or (w_l, u_l), respectively. These values have to satisfy the Rankine–Hugoniot conditions

(3.2)
$$sw + u = sw_0 + u_0,$$
$$-su + p(w, u) = -su_0 + p(w_0, u_0).$$

Eliminating u, and making use of (2.3) we obtain an equation for $s = s(w; w_0, u_0)$ which reads

(3.3) $$\frac{w^2 + w_0 - 2b}{2(w - b)} s^2 + \frac{u_0}{w - b} s - \left[\frac{1 - u_0^2}{2(w_0 - b)(w - b)} - \frac{a(w + w_0)}{w_0^2 w^2} \right] = 0.$$

This equation has two real solutions if and only if

(3.4) $$D(w; w_0, u_0) = \frac{(w + w_0 - 2b) - (u_0^2(w - b)}{(w_0 - b)(w - b)^2} - \frac{2a(w + w_0 - 2b)(w + w_0)}{w_0^2 w^2(w - b)}$$

is positive.

LEMMA 2. Let (w_0, u_0) be such that $p(w_0, u_0) > 0$. Then the set

$$\left\{ w > b \; : \;\; D(w; w_0, u_0) < 0 \right\}$$

is either empty or it is a finite interval contained in $\{x \in \mathbb{R} : \; x > b\}$.

P r o o f. We rewrite $D(w; w_0, u_0)$ in the form

$$D(w; w_0, u_0) = \frac{1}{w^2(w - b)^2} P_3(w - b),$$

where $P(x)$ is the polynomial of grade three.

$$P_3(x) = 2p(w_0, u_0)x^3 + \left[1 + \frac{2b(1 - u_0^2)}{w_0 - b} - \frac{4a}{w_0} \right] x^2$$
$$+ \left[2b \left(1 - \frac{a}{b} \right) + \frac{b^2(1 - u_0^2)}{w_0 - b} + \frac{2ab}{w_0^2} \right] x + b^2.$$

Since $P_3(0) = b^2 > 0$, $P_3(x) > 0$ for sufficiently large positive x, and $P_3(x) < 0$ for sufficiently large negative x, then this polynomial can take negative values in the domain $x > 0$ in a finite interval only. The proof is complete.

In principle, we could make use of the theory of the cubic polynomials to get the precise answer to the question of the sign of $P_3(x)$. Unfortunately, in our

case, the coefficients in $P_3(x)$ are so complicated that we are unable to draw any conclusions. Therefore, we present only partial answers to the question of the sign of $D(w; w_0, u_0)$.

LEMMA 3.

i) If

$$(3.5) \qquad \frac{p(w,0) - p(w_0,0)}{w - w_0} > 0,$$

then $D(w; w_0, u_0) < 0$;

ii) if

$$(3.6) \qquad \frac{p(w,u_0) - p(w_0,u_0)}{w - w_0} < 0,$$

then $D(w; w_0, u_0) > 0$.

P r o o f. We have the following identity

$$D(w; w_0, u_0) = -2\left[\frac{p(w,u_0) - p(w_0,u_0)}{w - w_0} + \frac{w_0 - b}{w - b}\frac{p(w,0) - p(w_0,0)}{w - w_0}\right].$$

The assertion follows immediately from the above and the estimate

$$(3.7) \qquad \frac{p(w,u_0) - p(w_0,u_0)}{w - w_0} \geq \frac{p(w,0) - p(w_0,0)}{w - w_0}$$

for $w > b$, $w_0 > b$, $u_0^2 < 1$. The proof is complete.

COROLLARY 1. If $0 < u_0^2 < 1 - 16a/27b$, then for every $w > b$, $w_0 > b$, $D(w; w_0, u_0) > 0$.

P r o o f. If the assumption is satisfied, then $\partial p(w, u_0)/\partial w < 0$ for every $w > b$. Hence, (3.6) holds. The proof is complete.

LEMMA 4. If $(w_0, u_0) \in \mathcal{E}$, $(w, u_0) \in \mathcal{E}$, and $w > w_0$, then $D(w; w_0, u_0) < 0$.

P r o o f. We write

$$(3.8) \qquad D(w; w_0, u_0) = \frac{u_0^2}{(w - b)^2} - \frac{w + w_0 - 2b}{2(w - b}\left[4\frac{p(w,u_0) - p(w_0,u_0)}{w - w_0}\right].$$

Making use of (2.16) we obtain

$$D(w; w_0, u_0) \leq -\frac{u_0^2(w - w_0)}{2(w_o - b)(w - b} < 0.$$

The proof is complete.

LEMMA 5.

i) There are such pairs $(w_0, u_0) \in \mathcal{E}$, $(w, u_0) \in \mathcal{E}$ with $w < w_0$ that $D(w; w_0, u_0) > 0$;

ii) also, there are other pairs $(w_0, u_0) \in \mathcal{E}$, $(w, u_0) \in \mathcal{E}$, $w < w_0$ such that $D(w; w_0, u_0) < 0$.

P r o o f. Let $w_0 > b$ be such that

$$(3.9) \qquad 0 < 2 - \frac{8a(w_0 - b)^2}{w_0^3} < 1 - \frac{a}{2b},$$

and let $\varepsilon > 0$ be sufficiently small. We take

$$(3.10) \qquad u_0^2 = 2 + \varepsilon - \frac{8a(w_0 - b)^2}{w_0^3}.$$

Of course, then $\Delta(w_0, u_0) < 0$. Using (3.7) in (3.4) we obtain

$$(3.11) \qquad D(w; w_0, u_0) = -\frac{w - w_0}{w - b} \left[\frac{1}{(w_0 - b)(w - b)} \right.$$
$$\left. - \frac{2a\,[(3w_0 - 4b)(w - w_0) + 2w_0(2w_0 - 3b)]}{w_0^3 w^3} \right] - \frac{\varepsilon}{(w_0 - b)(w - b)}.$$

Owing to $D(w_0; w_0, u_0) < 0$, there is $\overline{w}$ such that $b < \overline{w} < w_0$ and

$$(3.12) \qquad D(\overline{w}; w_0, u_0) = 0,$$

and $D(w; w_0, u_0) < 0$ for $\overline{w} < w \leq w_0$. From (3.11), (3.12) we obtain

$$(3.13) \qquad \overline{w} = w_0 - \frac{\varepsilon(w_0 - b)}{1 - \dfrac{4a(w_0 - b)^2(2w_0 - 3b)}{w_0^4}} + O(\varepsilon^2).$$

We assume additionally that

$$(3.14) \qquad 1 - \frac{4a(w_0 - b)^2(2w_0 - 3b)}{w_0^4} > 0$$

for, of course, sufficiently small $\varepsilon > 0$ and $\overline{w} < w_0$.

Let us evaluate $\Delta(\overline{w}, u_0)$. Using (3.10) and (3.13) in (2.14) we obtain

$$\Delta(\overline{w}, u_0) = -\varepsilon \frac{1 - \dfrac{12ab(w_0 - b)^2}{w_0^4}}{1 - \dfrac{4a(w_0 - b)^2(2w_0 - 3b)}{w_0^4}} + O(\varepsilon^2).$$

If there is w_0 satisfying (3.9), (3.14) and

(3.15)
$$1 - \frac{12ab(w_0 - b)^2}{w_0^4} < 0,$$

then we obtain i), since it is enough to take $(w, u_0) \in \mathcal{E}$ such that $w < \overline{w}$. On the other hand, if there is w_0 such that

(3.16)
$$1 - \frac{12ab(w_0 - b)^2}{w_0^4} > 0,$$

then we obtain ii) because then there is $\overline{w} < w < w_0$ satisfying our demands.

Hence, it remains to show that there is w_0 satisfying (3.9), (3.14), (3.15), and that there is, possibly different from the previous one, another w_0 satisfying (3.9), (3.14), and (3.16).

The positive answers are readily available by noticing that (3.14) and (3.15) can be rewritten as

(3.14′)
$$1 - \frac{4a(w_0 - b)^2}{w_0^3} - \frac{4a(w_0 - b)^2(w_0 - 3b)}{w_0^4} > 0,$$

(3.15′)
$$1 - \frac{4a(w_0 - b)^2}{w_0^3} + \frac{4a(w_0 - b)^2(w_0 - 3b)}{w_0^4} < 0;$$

whereas (3.14), (3.16) can be rewritten in the form (3.14′) and

(3.16′)
$$1 - \frac{4a(w_0 - b)^2}{w_0^3} + \frac{4a(w_0 - b)^2(w_0 - 3b)}{w_0^4} > 0.$$

The proof is complete.

LEMMA 6. If $(w_0, u_0) \in \mathcal{H}$, $(w, u_0) \in \mathcal{H}$, $w \leq w_0$, and the interval $\langle (w, u_0), (w_0, u_0) \rangle \subset \mathcal{H}$, then $D(w; w_0, u_0) > 0$.

P r o o f. Use (3.8) and (2.17).

LEMMA 7. If
$$\frac{27}{37} < \frac{a}{2b} < \frac{27}{32}$$

then there is u_0 such that $\Delta(w, u_0) > 0$ for every $w > b$, and there are $(w_0, u_0) \in \mathcal{H}$, $(w, u_0) \in \mathcal{H}$, $w > w_0$ such that $D(w; w_0, u_0) < 0$.

P r o o f. Let $\varepsilon > 0$ be sufficiently small. Let us take

(3.17)
$$w_0 = b(3 - \varepsilon),$$

and

$$(3.18) \qquad u_0^2 = 2 - \frac{a(w_0 - b)(w_0 + b)^2}{bw_0^3}.$$

Then $0 < u_0^2 < 1 - (a/2b)$, and

$$\Delta(w, u_0) = \frac{2}{(w - b)^2}\left[1 - \frac{4a(w - b)^2}{w^3}\right] - \frac{u_0^2}{(w - b)^2}$$

$$\geq \frac{1}{(w - b)^2}\left[2 - \frac{32a}{27b} - u_0^2\right] = \frac{16a\varepsilon^2}{81bw_0^3(w - b)}\left(1 - \frac{7}{12}\varepsilon\right) > 0$$

for ε sufficiently small.

Next, we rewrite $D(w; w_0, u_0)$ in the form

$$(3.19) \qquad D(w; w, u) = \frac{1}{2(w_0 - b)(w - b)^2}\left[(w + w_0 - 2b)\right.$$

$$\left. - u_0^2\left(2 - u_0^2 - \frac{a(w_0 - b)(w_0 + b)^2}{bw_0^3}\right) + (w_0 - w)u_0^2\right]$$

$$+ \frac{a(w + w_0 - 2b)\left[(w_0 - b)w - 2bw_0\right]^2}{2bw_0^3w^2(w - b)^2}.$$

We take also

$$(3.20) \qquad w = \frac{2bw_0}{w_0 - b}.$$

Then

$$w - w_0 = \varepsilon b\frac{3 - \varepsilon}{2 - \varepsilon} > 0.$$

Inserting (3.17), (3.18), and (3.20) into (3.19) one gets

$$D(w; w_0, u_0) = -\frac{\varepsilon bw_0 u_0^2}{2(w_0 - b)^2(w - b)^2} < 0.$$

The proof is complete. The profiles $s(w; w_0, u_0, a)$ for some values of w_0, u_0 and a are shown in Fig. 1. We can see that these profiles depend very strongly on the values of the three parameters. They can be nonmonotonic or even undefined for some values of the specific volume w. Also, the change of sign of $s(w; w_0, u_0, a)$ is noticeable. On the other hand, the profile in Fig. 1c is very much like that in the case of ideal gases, despite the fact that now the system of the Euler equations is hyperbolic-elliptic, and the domain of hyperbolicity is disconnected.
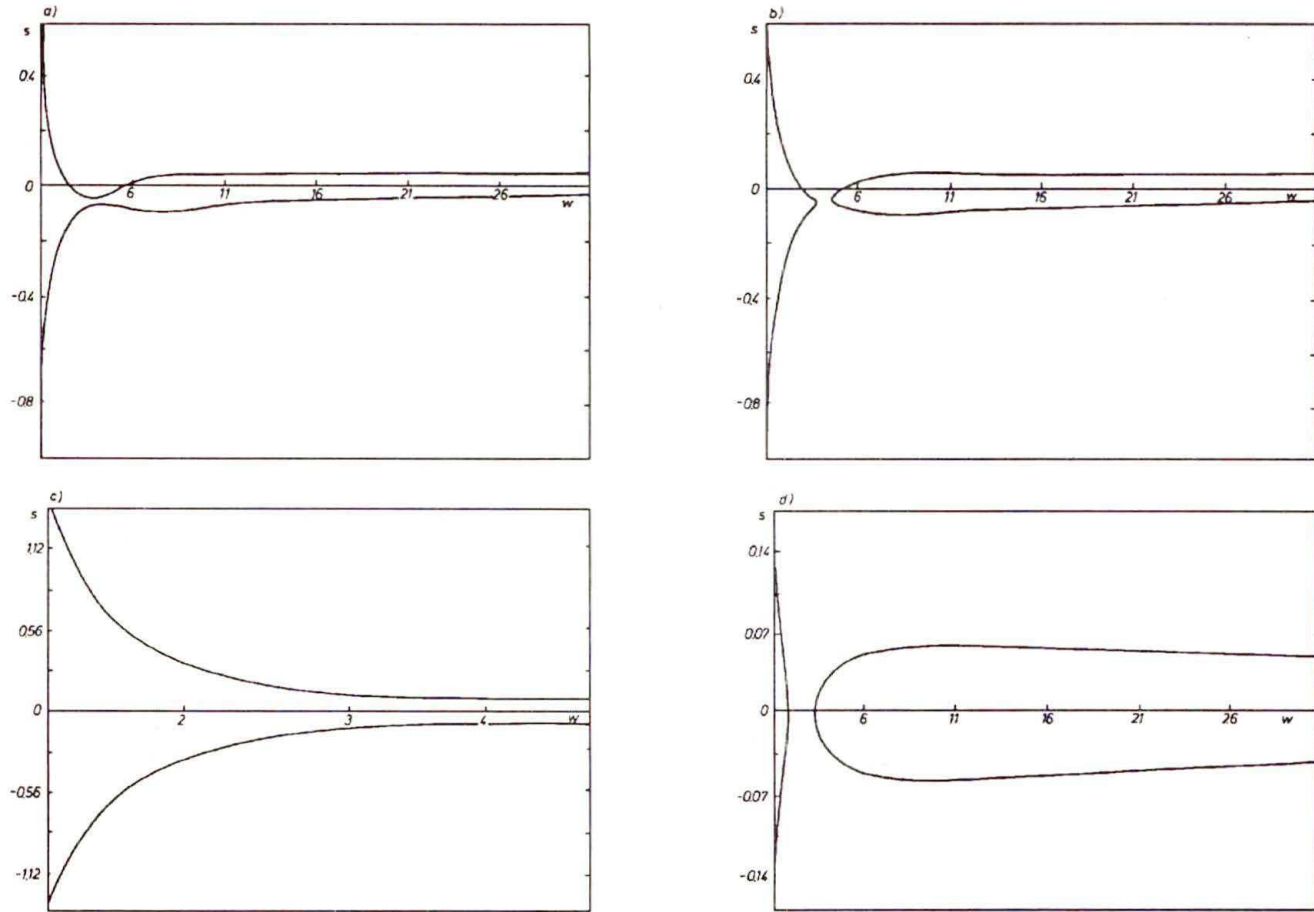
[687]

FIG. 1. The function $s(w; w_0, u_0, a)$ versus w for some values of w_0, u_0, a. a) $a = 1.65$, $w_0 = 2.25$, $u_0 = 0.25$; b) $a = 1.65$, $w_0 = 2.3$, $u_0 = 0.25$; c) $a = 1.9$, $w_0 = 1.7$, $u_0 = 0$; d) $a = 1.9$, $w_0 = 6.5$, $u_0 = 0$.

4. Properties of shock speed and the Hugoniot locus

In this Section we investigate properties of the shock speed $s(w; w_0, u_0)$ assuming, of course, its existence.

PROPOSITION 2. Let $\bar{s} = s(\bar{w}; w_0, u_0)$ be given. If

$$(4.1) \qquad \bar{w} > \frac{bw_0}{w_0 - b},$$

and

$$(4.2) \qquad 0 < s^2(\bar{w}; w_0, u_0) < \frac{a(w_0\bar{w} - b(w_0 + \bar{w}))^2}{2bw_0^3\bar{w}^3},$$

then there are exactly two $\bar{w}_1 > b$, $\bar{w}_2 > b$, $\bar{w}_1 \neq \bar{w}$, $\bar{w}_2 \neq \bar{w}$ such that

$$(4.3) \qquad s(\bar{w}_1; w_0, u_0) = s(\bar{w}_2; w_0, u_0) = s(\bar{w}; w_0, u_0).$$

In other words, any value of s can be taken at most three times.

P r o o f. Let $\bar{s} = s(\bar{w}; w_0, u_0)$ be given. Then the following identity is true

$$(4.4) \qquad \frac{\bar{w} + w_0 - 2b}{2(\bar{w} \cdot b)}\bar{s}^2 + \frac{u_0}{\bar{w} - b}\bar{s} - \left[\frac{1 - u_0^2}{2(w_0 - b)(\bar{w} - b)} - \frac{a(\bar{w} + w_0)}{w_0^2\bar{w}^2}\right] = 0.$$

Now, let us consider Eq. (3.3) with $s = \bar{s}$, but with unknown w. Using the identity (4.4) to eliminate $u_0\bar{s}$ from Eq. (3.3) we obtain

$$(4.5) \qquad (w - \bar{w})\left\{\bar{s}^2(w - b)^2 + 2\left[\bar{s}^2 b - \frac{a(\bar{w}w_0 - b(\bar{w} + w_0))}{w_0^2\bar{w}^2}\right](w - b)\right.$$
$$\left. + b^2\left[\bar{s}^2 + \frac{2a(\bar{w} + w_0)}{w_0^2\bar{w}^2}\right]\right\} = 0.$$

One solution is trivial: $w = \bar{w}$. This equation has two other solutions if and only if (4.2) holds. These two solutions are of the same sign. They are positive if additionally

$$(4.6) \qquad \bar{s}^2 < \frac{a(\bar{w}w_0 - b(\bar{w} + w_0))}{bw_0^2\bar{w}^2}.$$

This condition is not contradictory if the term on the right-hand side is positive. In turn, it happens if and only if (4.1) holds. We show now that (4.1), (4.2) imply (4.6). Indeed,

$$\frac{a(w_0\bar{w} - b(w_0 + \bar{w})^2)}{2bw_0^3\bar{w}^3} = \frac{1}{2}\left(1 - \frac{b}{w_0} - \frac{b}{w}\right)\frac{a(w_0\bar{w} - b(w_0 + \bar{w}))}{bw_0^2\bar{w}^2}$$
$$< \frac{a(w_0\bar{w} - b(w_0 + \bar{w}))}{bw_0^2\bar{w}^2}.$$

The proof is complete.

PROPOSITION 3. If $\Delta(w_0, u_0) > 0$, then the equation

$$(4.7) \qquad s(w; w_0, u_0) = \lambda(w_0, u_0)$$

has at least one solution, namely $w = w_0$. If additionally

$$(4.8) \qquad w_0 > 2b,$$

and

$$(4.9) \qquad \lambda^2(w_0, u_0) < \frac{a(w_0 - 2b)^2}{2bw_0^4},$$

then there are two other solutions w_1, w_2 satisfying $w_1 > b$, $w_2 > b$, $w_1 \neq w_0$, $w_2 \neq w_0$.

P r o o f. Since $D(w_0; w_0, u_0) = \Delta(w_0, u_0)$, then $w = w_0$ is a solution of (4.7). If (4.8) and (4.9) hold, then $\overline{w} = w_0$, and $\overline{s} = \lambda(w_0, u_0)$ satisfy (4.1), (4.2). Therefore, making use of Proposition 2 we obtain the second thesis. The proof is complete.

LEMMA 8. If

$$(4.10) \qquad \frac{d}{dw} s(w; w_0, u_0)\Big|_{w=\overline{w}} = 0,$$

then $\overline{w}$ satisfies

$$(4.11) \qquad \overline{w} > \frac{2bw_0}{w_0 - b}.$$

Moreover, there is exactly one $\widehat{w} \neq \overline{w}$ such that $s(\widehat{w}; w_0, u_0) = s(\overline{w}; w_0, u_0)$; $\widehat{w}$ is given by

$$(4.12) \qquad \widehat{w} = \frac{bw_0\overline{w}}{\overline{w}(w_0 - b) - 2bw_0} > b.$$

P r o o f. Differentiating Eq. (3.3) with respect to w we obtain

$$(4.13) \qquad \left[\frac{w + w_0 - 2b}{w - b}s + \frac{u_0}{w_0 - b}\right]\frac{ds}{dw} = \frac{w_0 - b}{2(w - b)^2}s^2 + \frac{u_0 s}{(w - b)^2}$$
$$- \left[\frac{1 - u_0^2}{2(w_0 - b)(w - b)^2} - \frac{a(w + 2w_0)}{w_0^2 w^3}\right].$$

We use Eq. (3.3) to eliminate $u_0 s$ and obtain

$$\left[\frac{w + w_0 - 2b}{w - b}s + \frac{u_0}{w_0 - b}\right]\frac{ds}{dw} = -\frac{1}{2(w - b)}\left\{s^2 - \frac{2a\left[(w_0 - b)w - 2bw_0\right]}{w_0^2 w^3}\right\}.$$

Hence, (4.10) holds if and only if

$$(4.14) \qquad s^2(\overline{w}; w_0, u_0) = \frac{2a\left[(w_0 - b)\overline{w} - 2bw_0\right]}{w_0^2 \overline{w}^3}.$$

The right-hand side is positive for

$$\overline{w} \geq \frac{2bw_0}{w_0 - b}$$

only.

Inserting (4.14) into Eq. (4.5) we find easily that it has one double solution $w = \overline{w}$, the third one is given by (4.12). The proof is complete.

Let (w_0, u_0) be given. The Hugoniot locus $H(w_0, u_0)$ is defined as the set of all states $(w, u) \in \mathcal{D}$ which satisfy (3.2) for some real s. For any (w_0, u_0) and $w > b$, if $D(w; w_0, u_0) \geq 0$, $H(w_0, u_0)$ consists of two branches $H_{\pm}(w_0, u_0)$, and each of them is defined by

$$H_{\pm}(w_0, u_0) = \Big\{(w, u) : \quad u = u_0 - s_{\pm}(w; w_0, u_0)(w - w_0)\Big\},$$

where

$$(4.16) \qquad s_{\pm}(w; w_0, u_0) = \frac{w - b}{w + w_0 - 2b}\left[-\frac{u_0}{w - b} \pm \sqrt{D(w; w_0, u_0)}\right].$$

Of course, $(w_0, u_0) \in H_+(w_0, u_0) \cap H_-(w_0, u_0)$. However, there can be other states (w, u) belonging both to $H_+(w_0, u_0)$ and $H_-(w_0, u_0)$. As it is seen from (4.16), it occurs if $D(w; w_0, u_0) = 0$, $w \neq w_0$. Then $H(w_0, u_0)$ forms loops. Also, let us notice that the Hugoniot locus can be disconnected.

The shapes of the Hugoniot loci for a few values of w_0, u_0, and a are shown in Figs. 2 and 3. Figure 2 presents them for the case when the domain of hyperbolicity is connected. As we can see, the curves can be either connected or disconnected. In the latter case they can form loops, and enter the domain of ellipticity where the speed of sound is complex.

In Fig. 3, four examples of the Hugoniot loci are given for the case of disconnected domain of hyperbolicity. The ineresting thing is that they can traverse the domain of ellipticity. Also, loops to the right (Fig. 3b) or to the left (Fig. 3d) of the point (w_0, u_0) can be formed. In Fig. 3c the point (w_0, u_0) belongs to the domain of ellipticity. In this case, the Hugoniot locus consists of three components: the left-hand branch, the sole point (w_0, u_0), and the right-hand branch.

We have to add that these do not exhaust all possible interesting situations.
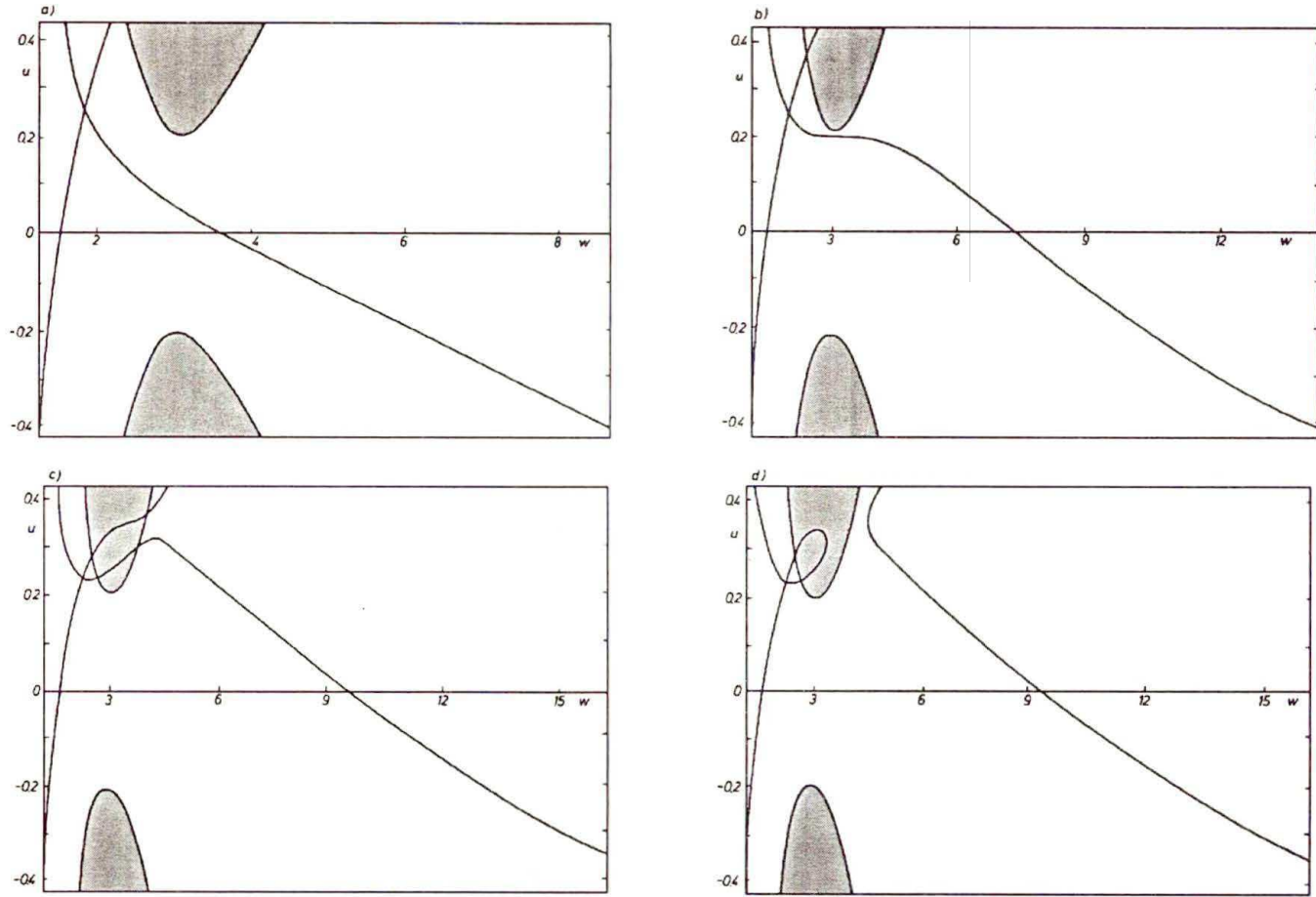
We establish two auxiliary results.

FIG. 2. The Hugoniot loci for some values of w_0, u_0. The domain of hyperbolicity is connected. a) $a = 1.65$, $w_0 = 1.7$, $u_0 = 0.25$; b) $a = 1.65$, $w_0 = 1.95$, $u_0 = 0.25$; c) $a = 1.65$, $w_0 = 2.25$, $u_0 = 0.25$; d) $a = 1.65$, $w_0 = 2.3$, $u_0 = 0.25$.
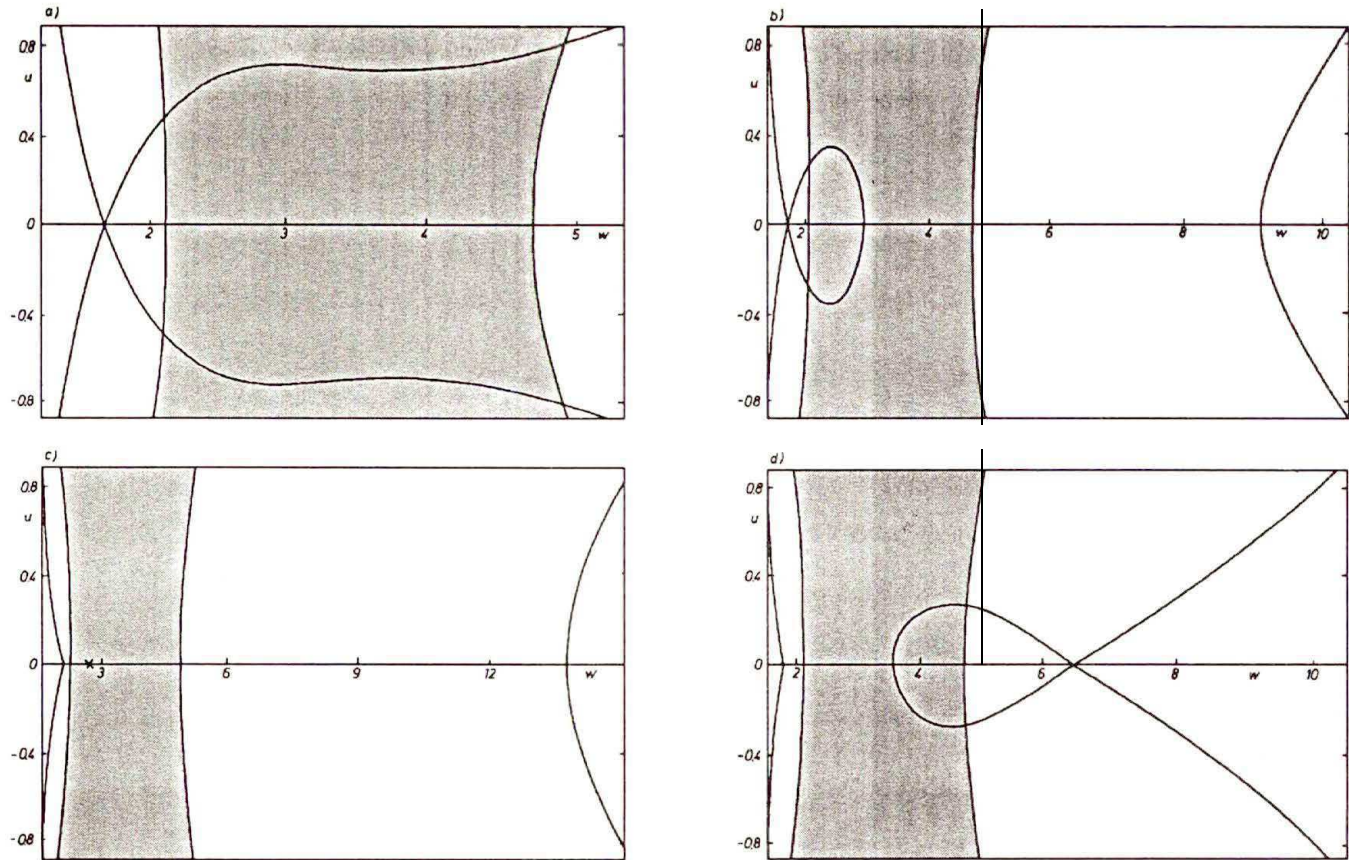
[169]

FIG. 3. The Hugoniot loci for some values of w_0, u_0. The domain of ellipticity is connected. a) $a = 1.9$, $w_0 = 1.7$, $u_0 = 0$;
b) $a = 1.9$, $w_0 = 1.8$, $u_0 = 0$; c) $a = 1.9$, $w_0 = 2.5$, $u_0 = 0$; The cross $\times$ denotes the position of (w_0, u_0) in the domain of ellipticity.
d) $a = 1.9$, $w_0 = 6.5$, $u_0 = 0$.

Proposition 4. Let $s = s(w; w_0, u_0)$ be a solution of Eq. (3.3). Then

$$(4.17) \qquad (w - w_0)\left\{ \frac{w_0 - b}{2(w - b)^2}s^2 + \frac{u_0 s}{(w - b)^2} \right.$$
$$\left. - \left[\frac{1 - u_0^2}{2(w_0 - b)(w - b)^2} + \frac{a(w + 2w_0)}{w_0^2 w^3} \right] \right\}$$
$$= -\left\{ \frac{(w - b)^2 + (w_0 - b)^2}{2(w - b)^2}s^2 + \frac{u_0 s(w_0 - b)}{(w - b)^2} - \left[\frac{1 - u_0^2}{2(w - b)^2} - \frac{2a}{w^3} \right] \right\}.$$

P r o o f. Let $\mathcal{L}$ be the left-hand side of (4.17). We have trivially: $\mathcal{L} = \mathcal{L} - 0$, and substituting the left-hand side of (3.3) for zero we obtain the right-hand side of (4.17). The proof is complete.

Proposition 5. If $(w, u) \in H(w_0, u_0)$, then

$$(4.18) \qquad [s(w; w_0, u_0) - \lambda_+(w, u)] [s(w; w_0, u_0) - \lambda_-(w, u)]$$
$$= \frac{(w - b)^2 + (w_0 - b)^2}{2(w - b)^2}s^2 + \frac{su_0(w_0 - b)}{(w - b)^2} - \left[\frac{1 - u_0^2}{2(w - b)^2} - \frac{2a}{w^3} \right],$$

where $\lambda_\pm(w, u)$ are the solutions of (2.10)

$$\lambda_\pm(w, u) = \frac{1}{2}\left(-\frac{u}{w - b} \pm \sqrt{\Delta(w, u)} \right).$$

P r o o f. Setting $u = u_0 - s(w - w_0)$ in Eq. (2.10) we obtain

$$(4.19) \qquad \lambda_+(w, u) + \lambda_-(w, u) = -\frac{u_0 - s(w - w_0)}{w - b},$$

and

$$(4.20) \qquad \lambda_+(w, u)\lambda_-(w, u) = \frac{s^2(w - w_0)^2}{2(w - b)^2} - \frac{su_0(w - w_0)}{(w - b)^2} - \left[\frac{1 - u_0^2}{2(w - b)^2} - \frac{2a}{w^3} \right].$$

Equation (4.18) is an immediate consequence of these identities. The proof is complete.

Lemma 9. Let $D(w; w_0, u_0) > 0$.

i) If $(w, u) \in H_+(w_0, u_0)$, then

$$(w - w_0)\frac{ds_+}{dw} > 0 \qquad \left(\text{respectively}: \quad (w - w_0)\frac{ds_+}{dw} < 0 \right)$$

if and only if

$$\lambda_-(w, u) < s_+(w; w_0, u_0) < \lambda_+(w, u),$$

(respectively: $s_+(w; w_0, u_0) < \lambda_-(w, u)$ or $s_+(w; w_0, u_0) > \lambda_+(w, u)$).

ii) If $(w, u) \in H_-(w_0, u_0)$, then

$$(w - w_0) \frac{ds_-}{dw} > 0 \qquad \left(\text{respectively}: \quad (w - w_0) \frac{ds_-}{dw} < 0 \right)$$

if and only if $s_-(w; w_0, u_0) < \lambda_-(w, u)$ or $s_-(w; w_0, u_0) > \lambda_+(w, u)$
(respectively: $\lambda_-(w, u) < s_-(w; w_0, u_0) < \lambda_+(w, u)$).

P r o o f. Owing to (4.13), (4.17), and (4.18) we have

$$(4.21) \qquad \left[\frac{w + w_0 - 2b}{w - b} s + \frac{u_0}{w - b} \right] \frac{ds}{dw} = -\frac{(s - \lambda_+)(s - \lambda_-)}{w - w_0}.$$

But on $H_+(w_0, u_0)$

$$\frac{w + w_0 - 2b}{w - b} s + \frac{u_0}{w - b} = \sqrt{D(w; w_0, u_0)}.$$

Therefore on $H_+(w_0, u_0)$

$$\sqrt{D(w; w_0, u_0)} (w - w_0) \frac{ds_+}{dw} = -(s_+ - \lambda_-)(s_+ - \lambda_+).$$

Assertion i) is an immediate consequence of this identity. To prove ii) we use

$$\sqrt{D(w; w_0, u_0)} (w - w_0) \frac{ds_-}{dw} = (s_- - \lambda_-)(s_- - \lambda_+)$$

on $H_-(w_0, u_0)$. The proof is complete.

LEMMA 10. If

$$(4.22) \qquad \frac{d}{dw} s(w; w_0, u_0) \Big|_{w = \overline{w}} = 0,$$

then $s(\overline{w}; w_0, u_0) = \lambda_+(\overline{w}, \overline{u})$ or $\lambda_-(\overline{w}, \overline{u})$, where $(\overline{w}, \overline{u}) \in H(w_0, u_0)$.

Conversely, if for some $w = \overline{w} \neq w_0$ with $(\overline{w}, \overline{u}) \in H(w_0, u_0)$, $D(\overline{w}; w_0, u_0) > 0$, and $s(\overline{w}; w_0, u_0) = \lambda_+(\overline{w}, \overline{u})$ or $\lambda_-(\overline{w}, \overline{u})$, then (4.22) holds.

P r o o f. The first part of the Assertion is a consequence of (4.20) and the Assumption. Conversely, if $s = \lambda_+$ or $s = \lambda_-$ then $ds/dw = 0$, since $D > 0$. The proof is complete.

LEMMA 11. Let $(w, u) \in H(w_0, u_0)$. Then

$$(4.23) \qquad \frac{du}{dw} = -\frac{1}{2} \frac{s^2 - \lambda_- \lambda_+}{s - \dfrac{\lambda_- + \lambda_+}{2}},$$

where $s = s(w; w_0, u_0)$; $\lambda_\pm = \lambda_\pm(w, u)$.

P r o o f. Differentiating

$$u = u_0 - s(w - w_0)$$

we obtain

$$\frac{du}{dw} = -(w - w_0)\frac{ds}{dw} - s.$$

Next, making use of (4.21) we get

$$\frac{du}{dw} = -\frac{\dfrac{w_0 - b}{w - b}s^2 + s\left(\dfrac{u_0}{w - b} + \lambda_- + \lambda_+\right) - \lambda_-\lambda_+}{2s + \dfrac{u}{w - b}}.$$

Applying (4.19) we obtain (4.23). The proof is complete.

LEMMA 12. Let $D(w; w_0, u_0) > 0$, and let

$$\frac{d}{dw}s(w; w_0, u_0)\Big|_{w=\overline{w}} = 0.$$

Then

i) $s(w; w_0, u_0)$ attains a local minimum at $\overline{w}$, provided that $(w - w_0)\mathbf{r}\cdot\nabla\lambda < 0$ at $(\overline{w}, \overline{u})$;

ii) $s(w; w_0, u_0)$ attains a local maximum at $\overline{w}$, provided that $(w - w_0)\mathbf{r}\cdot\nabla\lambda > 0$ at $(\overline{w}, \overline{u})$.

Here, $\overline{u} = u_0 - s(\overline{w}; w_0, u_0)(\overline{w} - w_0)$, $\mathbf{r} = \mathbf{r}_\pm(w, u)$ is the right eigenvector of the matrix $\mathbb{M}$ corresponding to $\lambda_\pm$ respectively, and $\lambda = \lambda_+$ or λ_- according to whether $s = \lambda_+$ or $s = \lambda_-$.

P r o o f. Differentiating (4.21) and using the Assumption we get

$$(2s - \lambda_- - \lambda_+)\frac{d^2s}{dw^2} = \frac{1}{w - w_0}\left[(s - \lambda_-)\frac{d\lambda_+}{dw} + (s - \lambda_+)\frac{d\lambda_-}{dw}\right].$$

Let $s = \lambda_+$, then the above reduces to

$$\frac{d^2s}{dw^2} = \frac{1}{w - w_0}\frac{d\lambda_+}{dw}.$$

But, making use of (4.23) we obtain

$$\frac{d\lambda_+}{dw} = \frac{\partial\lambda_+}{\partial w} + \frac{\partial\lambda_+}{\partial u}\frac{du}{dw} = \frac{\partial\lambda_+}{\partial dw} - \lambda_+\frac{\partial\lambda_+}{\partial u} = \mathbf{r}_+\cdot\nabla\lambda_+.$$

Hence, at $w = \overline{w}$

Similarly, if $s = \lambda$ at $w = \overline{w}$, then

$$\frac{d^2 s}{dw^2} = \frac{1}{w - w_0} \mathbf{r}_- \cdot \nabla \lambda_- .$$

The proof is complete.

PROPOSITION 6. Given w_r, w_l, u_l with $(w_l, u_l) \in \mathcal{D}$, then $(w_l, u_l) \in H(w_r, u_r)$, and

(4.24) $s(w_l; w_r, u_r) = s(w_r; w_l, u_l)$

where

(4.25) $u_r = u_l - s(w_r; w_l, u_l)(w_r - w_l),$

provided that $s(w_r; w_l, u_l)$ exists.

P r o o f. If $(w, u) \in H(w_r, u_r)$ then

$$u = u_r - s(w; w_r, u_r)(w - w_r).$$

Setting here $w = w_l$, and using (4.25) we get

$$u = u_l - \Big[s(w_r; w_l, u_l) - s(w_l; w_r, u_r) \Big](w_r - w_l).$$

It follows from the above that it is sufficient to show that (4.24) holds in order to have $(w_l, u_l) \in H(w_r, u_r)$. We introduce the shorthands $s_l = s(w_r; w_l, u_l)$ and $s_r = s(w_l; w_r, u_r)$. These quantities satisfy Eq. (3.3) with $w = w_r$, $w_0 = w_l$, $u_0 = u_l$, and $w = w_l$, $w_0 = w_r$, $u_0 = u_r$, respectively. Substituting (4.25) into the equation for s_r, and using Eq. (3.3) for s_l we obtain

$$s_r^2 + \frac{2[u_l - s_l(w_r - w_l)]}{w_r + w_l - 2b} s_r - \frac{3w_l - w_r - 2b}{w_r + w_l - 2b} s_l^2 - \frac{2u_l s_l}{w_r + w_l - 2b} = 0.$$

This equation has two real solutions, one of them is given by (4.25). The proof is complete.

PROPOSITION 7. Given w, w_0, u_0 with $(w_0, u_0) \in \mathcal{D}$. Then

(4.26) $s_\pm(w; w_0, -u_0) = -s_\mp(w; w_0, u_0).$

P r o o f. Equation (4.26) is an immediate consequence of (4.16) and the identity

(4.27) $D(w; w_0, -u_0) = D(w; w_0, u_0).$

The proof is complete.

The graphs of the shock speed as a function of the specific volume w are given in Fig. 1 for a few values of a, b, w_0, and u_0.

5. Travelling waves in the model Navier–Stokes equations

Within the Euler equations, the shock wave is a jump discontinuity propagating along the line $x - st = 0$. Let (w_l, u_l) and (w_r, u_r) be the given states to the left and to the right of the line of discontinuity. They have to satisfy the Rankine–Hugoniot relations. However, it is not enough to accept such a jump as physical. It is well known that some additional conditions have to be imposed. Various ideas were used to formulate such additional admissibility criteria [10–48].

We remind that our principal task is to investigate different approximations to the model kinetic equations of [56], and the Euler equations (2.1) are the last but crucial term in the sequence. Hence no freedom of choice of admissibility criteria is left to us, and we have to turn to the next order approximation, i.e. to the Navier–Stokes equations.

The Navier–Stokes equations read

(5.1)
$$\frac{\partial w}{\partial t} - \frac{\partial u}{\partial x} = 0,$$
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} p(w, u) = \varepsilon \frac{\partial}{\partial x} \left(\mu \frac{\partial u}{\partial x} \right),$$

where t, x, w, u, and $p(w, u)$ are the same as previously, but $\varepsilon\mu$ is the coefficient of viscosity, $\varepsilon > 0$ is a parameter, and $\mu = \mu(w, u)$ is given by

(5.2)
$$\mu(w, u) = \frac{1 - u^2 + 2b^2 w^2 \varrho^2(w)}{8w^2 \varrho}, \qquad \varrho(w) = \frac{w}{w - b}.$$

A travelling wave solution to (5.1) is a solution of the form

(5.3)
$$(w, u)(x, t) = (\widehat{w}, \widehat{u})(z), \qquad z = \frac{x - st}{\varepsilon} \in \mathbb{R},$$

where $s = $ const is the wave-speed, such that

(5.4)
$$\lim_{z \to -\infty} (\widehat{w}, \widehat{u})(z) = (w_l, u_l),$$

(5.5)
$$\lim_{z \to +\infty} (\widehat{w}, \widehat{u})(z) = (w_r, u_r),$$

and

(5.6)
$$\lim_{z \to \infty} \frac{d}{dz} (\widehat{w}, \widehat{u})(z) = (0, 0).$$

A discontiuous solution (3.1) to Eqs. (2.1) is said to be admissible, if Eqs. (5.1) admit a travelling wave solution (5.3)–(5.6) for sufficiently small $\varepsilon > 0$.

We substitute (5.3) into (5.1), perform one integration with respect to z, use the limit conditions (5.4)–(5.6), and obtain

(5.7) $$\widehat{u} = u_l - s(w - w_l),$$

(5.8) $$\mu\frac{d\widehat{u}}{dz} = -s(\widehat{u} - u_l) + p(\widehat{w}, \widehat{u}) - p(w_l, u_l),$$

as well as

(5.9)
$$sw_r + u_r = sw_l + u_l,$$
$$-su_r + p(w_r, u_r) = -su_l + p(w_l, u_l).$$

Equations (5.9) have the form of the Rankine–Hugoniot conditions (3.2), which were discussed thoroughly in the preceding sections.

Using (5.7) to eliminate $\widehat{u}$ from Eq. (5.8) we arrive at the problem:

find a solution to

(5.10) $$s\widehat{\mu}\frac{d\widehat{w}}{dz} + f_l(\widehat{w}) = 0, \qquad \xi \in \mathbb{R},$$

such that

(5.11) $$\lim_{z \to -\infty} \widehat{w}(z) = w_l, \qquad \lim_{z \to +\infty} \widehat{w}(z) = w_r,$$

(5.12) $$\lim_{z \to -\infty} \widehat{w}'(\xi) = \lim_{z \to +\infty} \widehat{w}'(\xi) = 0,$$

where the prime ′ denotes d/dz, and where

(5.13) $$f_l(w) = s^2(w - w_l) + p(w, u_l - s(w - w_l)) - p(w_l, u_l),$$
(5.14) $$f_l(w_l) = f_l(w_r) = 0$$

and $\widehat{\mu} = \mu(\widehat{w}, u_l - s(\widehat{w} - w_l))$. The subscript l in f_l is used to mark that $f(w)$ is related to the left state (w_l, u_l) which is treated as given. We have

LEMMA 13. Problem (5.10)–(5.14) has a unique solution if and only if
$f_l(w) < 0$ between w_l and w_r for $s(w_r - w_l) > 0$,
$f_l(w) > 0$ between w_l and w_r for $s(w_r - w_l) < 0$.

P r o o f. If $w_r > w_l$, then we must have $\widehat{w}'(z) > 0$. Hence, if $s(w_r - w_l) > 0$, then $s\widehat{w}'(z) > 0$, therefore $f(w)$ has to be negative between w_l and w_r. The second case is analyzed in a similar way. The proof is complete.

THEOREM 1. *The problem (5.10)–(5.14) has a unique solution if and only if:*

i) for $s(w_r - w_l) > 0$, the chord joining $(w_l, p(w_l, u_l))$ to $(w_r, p(w_r, u_r))$ lies above the graph of $p(w, u_l - s(w - w_l))$ between w_l and w_r;

ii) for $s(w_r - w_l) < 0$, the chord joining $(w_l, p(w_l, u_l))$ to $(w_r, p(w_r, u_r))$ lies below the graph of $p(w, u_l - s(w - w_l))$ between w_l and w_r.

P r o o f. Rewrite $f(w) < 0$ in the form $p(w, u_l - s(w - w_l)) < s^2(w - w_l) + p(w_l, u_l)$. The case of $f(w) > 0$ is analyzed similarly. The proof is complete.

This theorem reminds the similar ones of [18] or [19] for the isothermal case. The essential difference between the latter case and that of ours is that in the isothermal case p does not depend on s. Therefore changing s we change only the slope of the Rayleigh line, i.e. the chord joining $(w_l, p(w_l))$ to $(w_r, p(w_r))$, and the graph of $p(w)$ remains intact. In our case, when changing s we change not only the slope of the Rayleigh line but the graph of p itself, since $p = p(w, u_l - s(w - w_l))$. Hence, the use of this very intuitive theorem is a little bit troublesome in **the case** under consideration.

The assertions of Lemma 13 and Theorem 1 were essentially independent of the specific form of $p(w, u)$. If $p(w, u)$ is given by (2.3), then we can obtain analytical criteria for existence of the travelling waves. Namely, we have

LEMMA 14. Let $p(w, u)$ be given by (2.3). Then, there is a unique solution to (5.10) – (5.14) if and only if:

i) for $s > 0$:

$$(5.15) \qquad \left(s + \frac{u_l}{w + w_l - 2b}\right)^2 - \frac{(w - b)^2}{(w + w_l - 2b)^2}D(w; w_l, u_l) \le 0$$

between w_l and w_r;

ii) for $s < 0$

$$(5.16) \qquad \left(s + \frac{u_l}{w + w_l - 2b}\right)^2 - \frac{(w - b)^2}{(w + w_l - 2b)^2}D(w; w_l, u_l) \ge 0$$

between w_l and w_r, where $D(w; w_l, u_l)$ is given by (3.4).

P r o o f. Using (2.3) we write

$$f_l(w) = \frac{(w - w_l)w + w_l - 2b}{2(w - b)}\left[\left(s + \frac{u_l}{w + w_l - 2b}\right)^2 - \frac{(w - b)^2}{w + w_l - 2b}D(w; w_l, u_l)\right].$$

From this identity and with the use of Lemma 13 we obtain easily (5.15) and (5.16) by considering separately four cases of $s > 0$, $w_r - w_l > 0$, etc. The proof is complete.

Let us notice now, that the necessary condition for (5.15) to hold is $D(w; w_l, u_l) > 0$ between w_l and w_r. Hence, (w_r, u_r) has to belong to the same component

of $H(w_l, u_l)$ as (w_l, u_l) does. Next, $(5.14)_2$ means that s satisfies Eq. (3.3) with $(w_0, u_0) = (w_l, u_l)$ and $w = w_r$. Thus, $s = s(w_r; w_l, u_l)$ Therefore we can rewrite (5.15) as follows

$$(5.17) \qquad s_-(w; w_l, u_l) \leq s(w_r; w_l, u_l) \leq s_+(w; w_l, u_l)$$

between w_l and w_r.

This is a generalization of the Oleinik–Liu condition [10, 13–15] to the present problem.

Summing up we have

COROLLARY 2. If the wave speed $s = s(w_r; w_l, u_l)$ is strictly positive, then (5.10)–(5.14) has a unique solution if and only if

i) (w_r, u_r) belongs to the same component of $H(w_l, u_l)$ as (w_l, u_l) does;

ii) the Oleinik–Liu condition (5.17) is satisfied.

On the other hand, let us notice that if $s < 0$, then the case when $D(w; w_l, u_l)$ takes negative values is not excluded and (w_r, u_r) and (w_l, u_l) can belong to different components of $H(w_l, u_l)$. If so, then $s_\pm(w; w_l, u_l)$ becomes complex for some values of w between w_l and w_r. Consequently, the Oleinik–Liu condition is violated.

This asymmetry can be understood on physical grounds. Namely, if $s > 0$, then the left-hand state (w_l, u_l) is the state after the wave, and the right-hand state (w_r, u_r) is that before the wave, whereas if $s < 0$, the situation is opposite. We can see that by treating the right-hand state (w_r, u_r) is given. Then, instead of (5.7) we have $\widehat{u} = u_r - s(w - w_r)$, and Eq. (5.10) is replaced by

$$(5.18) \qquad s\mu_r(\widehat{w})\frac{d\widehat{w}}{dz} + f_r(\widehat{w}) = 0,$$

where $\mu_r(w) = \mu(w, u_r - s(w - w_r))$ and

$$(5.19) \qquad f_r(w) = s^2(w - w_r) + p(w, u_r - s(w - w_r)) - p(w_r, u_r),$$
$$(5.20) \qquad f_r(w_l) = f_r(w_r) = 0.$$

Instead of Lemma 14 we have

LEMMA 15. Let $p(w, u)$ be given by (2.2). Then, there is a unique solution to (5.11), (5.12), (5.18)–(5.20) if and only if

i) for $s > 0$

$$(5.21) \qquad \left(s + \frac{u_r}{w + w_r - 2b}\right)^2 - \frac{(w - b)^2}{(w + w_r - 2b)^2} D(w; w_r, u_r) \geq 0$$

between w_l and w_r;

ii) for $s < 0$

$$(5.22) \qquad \left(s + \frac{u_r}{w + w_r - 2b}\right)^2 - \frac{(w - b)^2}{(w + w_r - 2b)^2} D(w; w_r, u_r) \leq 0$$

between w_l and w_r.

We have also

COROLLARY 3. If the speed $s = s(w_l; w_r, u_r)$ is strictly negative, then the problem (5.11), (5.12), (5.18)–(5.20) has a unique solution if and only if

i) (w_l, u_l) belongs to the same component of $H(w_r, u_r)$ as (w_r, u_r) does;

ii) the Oleinik–Liu condition in the form

$$(5.23) \qquad s_-(w; w_r, u_r) \leq s(w_r; w_l, u_l) \leq s_+(w; w_r, u_r)$$

holds between w_l and w_r.

LEMMA 16. Conditions (5.15) and (5.21) are equivalent, as well as (5.16) and (5.22) are.

P r o o f. Let $\mathcal{L}_l$ denote the left-hand side of (5.15) and (5.16), and let $\mathcal{L}_r$ denote the right-hand side of (5.21) and (5.22). Since $s = s(w_r; w_l, u_l)$, then using Eq. (3.3) with $w = w_r$, $(w_0, u_0) = (w_l, u_l)$ to eliminate u_ls from $\mathcal{L}_l$, we obtain

$$\mathcal{L}_l = \frac{w - w_r}{w + w_l - 2b} \left\{ s^2 - \frac{2a[w(w_r w_l - bw_l - bw_r) - bw_l w_r]}{w^2 w_l^2 w_r^2} \right\}.$$

Similarly, using the fact that $s = s(w_l; w_r, u_r)$ (cf. Proposition 6) we can write

$$\mathcal{L}_r = \frac{w - w_l}{w + w_r - 2b} \left\{ s^2 - \frac{2a[w(w_r w_l - bw_l - bw_r) - bw_l w_r]}{w^2 w_l^2 w_r^2} \right\}.$$

Let $\mathcal{L} \leq 0$; then either

$\alpha) \qquad w - w_r \leq 0 \qquad$ and $\qquad s^2 - \dfrac{2a[w(w_r w_l - bw_l - bw_r) - bw_l w_r]}{w^2 w_l^2 w_r^2} \geq 0,$

or

$\beta) \qquad w - w_r \geq 0 \qquad$ and $\qquad s^2 - \dfrac{2a[w(w_r w_l - bw_l - bw_r) - bw_l w_r]}{w^2 w_l^2 w_r^2} \leq 0.$

Let us consider $\alpha)$. The inequality $w < w_r$ implies $w_l < w < w_r$. Hence $w - w_l > 0$. Therefore (5.21) holds. In the case $\beta)$, it must be $w_l - w < 0$, and again we obtain (5.21).

If $\mathcal{L}_r \geq 0$, then we proceed similarly and obtain (5.15). The proof is complete.

To unify our considerations we introduce the following definitions:

$$(w_a, u_a) = \begin{cases} (w_l, u_l) & \text{for } s > 0, \\ (w_r, u_r) & \text{for } s < 0, \end{cases}$$

and

$$(w_b, u_b) = \begin{cases} (w_r, u_r) & \text{for } s > 0, \\ (w_l, u_l) & \text{for } s < 0, \end{cases}$$

and call (w_a, u_a) the state after the wave, whereas (w_b, u_b) the state before the wave.

We have

THEOREM 2. *Equations* (5.1) *admit the unique travelling wave solution* (5.3)−(5.6) *if and only if*

i) (w_b, u_b) *belongs to the same component of* $H(w_a, u_a)$ *as* (w_a, u_a) *does;*

ii) *the Oleinik–Liu condition:*

is satisfied for every w between w_a and w_b.

This theorem is a compilation of Corollaries 2 and 3, and as such it needs no proof.

6. Shock-wave structure

The problem (5.10)−(5.14) with $\mu(w, u)$ given by (5.2) and $\hat{u}$ given by (5.7) admits an explicit solution. To determine it we perform some transformations and substitutions. Let w_r, w_l, u_l be given, and let s denote the shock-wave speed, i.e. $s = s(w_r; w_l, u_l)$.

First, from (5.2) and (5.7) we obtain

$$(6.1) \qquad \hat{\mu}(w) = \frac{1}{8w^3(w-b)} \sum_{i=0}^{4} A_i w^i,$$

where

$$(6.2) \qquad \begin{aligned} A_0 &= b^2\left(1 - (u_l + sw_l)^2\right), \\ A_1 &= -2b\left[1 - (u_l + sw_l)^2 - bs(u_l + sw_l)\right], \\ A_2 &= 1 - (u_l + sw_l)^2 - 4bs(u_l + sw_l) - b^2s^2, \\ A_3 &= 2s\left(u_l + s(w_l + b)\right), \\ A_4 &= 2b^2 - s^2. \end{aligned}$$

Hence, $\hat{\mu}(w)$ is a rational function. Also, $f_l(w)$ is such. To show this we make use of (2.3), (5.7), (5.13), and (5.14) and obtain

$$(6.3) \qquad f_l(w) = \frac{(w - w_l)(w - w_r)}{2w^2(w - b)} \left[s^2 w^2 - \frac{2a(w_l w_r - bw_l - bw_r)}{w_l^2 w_r^2} w + \frac{2ab}{w_l w_r} \right].$$

Using (6.1), (6.3) we rewrite Eq. (5.10) in the explicit form

$$(6.4) \qquad \frac{\displaystyle\sum_{i=0}^{4} A_i w_i}{w(w - w_l)(w - w_r)(w^2 - 2\alpha w + \beta)} dw = -4s\, d\xi,$$

where

$$(6.5) \qquad \begin{aligned} \alpha &= \frac{a(w_l w_r - bw_l - bw_r)}{s^2 w_l^2 w_r^2}, \\ \beta &= \frac{2ab}{s^2 w_l w_r} \end{aligned}$$

are constants.

Equation (6.4) can be easily integrated. The result depends significantly on the sign of

$$(6.6) \qquad W = \beta - \alpha^2.$$

CASE I. $W > 0$.

Under this assumption, the equation $f_l(w) = 0$ has exactly two real solutions $w = w_l$ and $w = w_r$. Hence, the general solution of Eq. (6.4) is

$$(6.7) \qquad A \ln w + B \ln |w - w_l| + C \ln |w - w_r| + \frac{1}{2} D \ln(w^2 - 2\alpha w + \beta)$$

$$+ \frac{E}{\sqrt{\beta - \alpha^2}} \arctan \frac{w + \alpha}{\sqrt{\beta - \alpha^2}} = -4sz + \text{integration constant},$$

where

$$A = \frac{A_0}{\beta w_l w_r},$$

$$B = \frac{A_{11} - w_l A_{21}}{(w_r - w_l)(w_l^2 - 2\alpha w_l + \beta)},$$

$$C = \frac{A_{11} - w_r A_{21}}{(w_r - w_l)(w_r^2 - 2\alpha w_r + \beta)},$$

$$A_{11} = A_4 w_l w_r (w_l + w_r) + A_3 w_l w_r - A_1 - A_0 \frac{w_l + w_r}{w_l w_r},$$

$$A_{21} = -A_4(w_l^2 + w_l w_r + w_r^2) + A_3(w_l + w_r) + A_2 - \frac{A_0}{w_l w_r},$$

$$D = A_4 - A - B - C,$$

$$E = A_4(w_l + w_r) + A_3 + 2\alpha A + B(2\alpha - w_l) + C(2\alpha - w_r).$$

The explicit analytic solution (6.7) was used to obtain a series of shock profiles shown in Fig. 4. The input data were so chosen as to receive results resembling those of [4, 7]. Our Fig. 4 is qualitatively similar to Fig. 10 of [4]. In particular, we see that so-called impending shock splitting can be derived from our model equations. The notion of "impending shock splitting" was first introduced in [4] and refers to shocks having two inflexion points instead of one, what is usual.
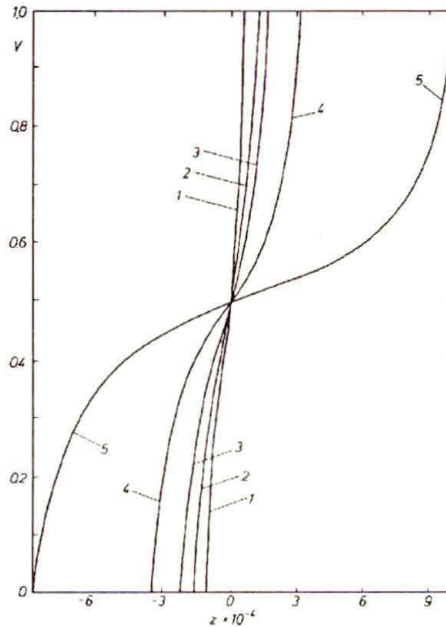

FIG. 4. Impending shock splitting. Normalized profiles $V = (w - w_l)/(w_r - w_l)$ versus $z \times 10^{-4}$, $a = 0.5$, $w_r = 10$, $u_r = 0$; $1 - w_l = 1.623$, $2 - w_l = 1.653$, $3 - w_l = 1.683$, $4 - w_l = 1.713$, $5 - w_l = 1.743$.

We omit all details of how to choose the entry data to obtain such phenomenon since it is fairly well done and explained in [4].

CASE II. $W \leq 0$.

Under this assumption the equation

$$w^2 - 2\alpha w + \beta = 0$$

has two real solutions $w_- \leq w_+$; what means that $f_l(w)$ has two additional zeros $w = w_-$ and $w = w_+$, except the "old" ones $w = w_l$ and $w = w_r$. The existence

of shock connecting (w_l, u_l) to (w_r, u_r) demands w_- and w_+ not to lie between w_l and w_r. The result of integration of Eq. (6.4) depends however on additional detailed relations between the zeros of $f_l(w)$.

i) $w_l \neq w_\pm$, $w_r \neq w_\mp$, $w_- < w_+$. In this case, the general solution of Eq. (6.4) reads

$$A \ln w + B \ln |w - w_l| + C \ln |w - w_r| + D \ln |w - w_-| + E \ln |w - w_+|$$
$$= -4sz + \text{integration constant},$$

where now

$$A = \frac{A_0}{\beta w_l w_r},$$

$$B = -\frac{A_{11} - w_l A_{21}}{(w_l - w_r)(w_l^2 - 2\alpha w_l + \beta)},$$

$$C = -\frac{A_{11} - w_r A_{21}}{(w_l - w_r)(w_r^2 - 2\alpha w_r + \beta)},$$

$$A_{11} = -A_4 w_l w_r (w_l + w_r) - A_3 w_l w_r - A_1 + A_0 \frac{w_l + w_r}{w_l w_r},$$

$$A_{21} = A_4(w_l^2 + w_l w_r + w_r^2) + A_3(w_l + w_r) + A_2 - \frac{A_0}{w_l w_r},$$

$$D = \frac{1}{w_- - w_+} \Big[A_4(w_l + w_r + w_-) + A_3 + A w_+ + B(w_l - w_+) + C(w_r - w_+) \Big],$$

$$E = A_4 - A - B - C - D.$$

This case we illustrate with a series of expansion shock profiles presented in Fig. 5. A shock wave is called expansion shock if the graph of $p(w, u_l - s(w - w_l))$ lies entirely above the Rayleigh line joining (w_l, u_l) with (w_r, u_r). Our Fig. 5 can be treated as a counterpart of Figs. 3 and 7 of [4]. We can notice easily that the shock thickness increases rather than decreases, with strength. Also this phenomenon was discovered first in [4] and it is thoroughly discussed in the cited paper.

ii) Either $w_l = w_\pm$ or $w_r = w_\mp$, or both. In this case we have one-sided or two-sided sonic shocks, i.e. shocks moving at the speed equal to the characteristic speed before or after the shock, or else all three of them are equal. In a situation like that, the asymptotic states of the shock are achieved algebraically rather than exponentially and shock thickness is much greater than in the previous cases. This is very similar to what is discussed in detail in [6]. Therefore we omit a discussion of the case.

Summing up, we can say that our model equations produce results qualitatively similar to those obtained within the framework of the true Navier–Stokes equations.
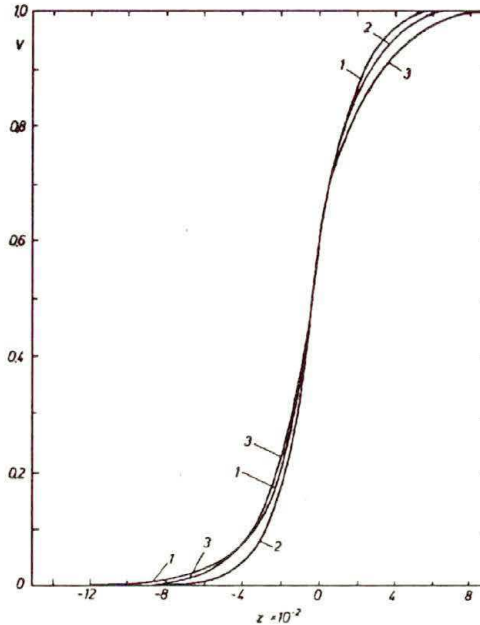
Fig. 5. Rarefaction shocks. Normalized profiles $V = (w - w_l)/(w_r - w_l)$ versus $z \times 10^{-2}$; $a = 0.5$, $w_r = 10$, $u_r = 0.5$; $1 - w_l = 6.5$, $2 - w_l = 7.25$, $3 - w_l = 8.0$.

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