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Two-dimensional tensor function representations involving third-order tensors

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AMONG THE PHYSICALLY possible infinitely many material symmetries of all kinds in a two-dimensional space, there exist eight kinds, i.e., the isotropy $C_{\infty\nu}$, hemitropy C_{∞}, two symmetries C_1 and C_2 in the oblique system, $C_{1\nu}$ and $C_{2\nu}$ in the rectangular system, and C_3 and $C_{3\nu}$ in the trigonal system, that can be characterized in terms of tensors of orders not higher than three. In this paper, the complete and irreducible representations relative to these eight symmetries are established for scalar-, vector-, second-order tensor- and third-order tensor-valued functions of any finite number of vectors, second-order tensors and third-order tensors. These representations allow to obtain, in the case of two-dimensional problems, general invariant forms of the physical laws; in particular, the constitutive equations involving thrid-order tensors.

1. Introduction

RECENTLY, the complete and irreducible representations in two-dimensional space were established by ZHENG [7] relative to every kind of material symmetry for scalar-,vector- and second-order tensor-valued functions of any finite number of second-order symmetric tensors $\mathbf{A}_1, \ldots, \mathbf{A}_N$ (denoted by $\mathbf{A}_\alpha$), second-order skew-symmetric tensors $\mathbf{W}_1, \ldots, \mathbf{W}_P$ (denoted by $\mathbf{W}_\xi$) and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_M$ (denoted by $\mathbf{v}_\varrho$). In contrast to these general results, complete and irreducible representations for tensor functions involving tensors of order higher than two are much less well understood (PENNISI [4], ZHENG [9], ZHENG and BETTEN [10], BETTEN and HELISCH [1]). In particular, the problem of constructing of general, complete and irreducible tensor function representations which contain any finite number of third-order tensor agencies $\mathbf{T}_1, \ldots, \mathbf{T}_L$ (denoted by $\mathbf{T}_\lambda$), even for $L = 1$, is still open, although its importance can be seen in many modern physical contexts (cf., PENNISI [4]).

ZHENG and BOEHLER [11] have described and classified the physically possible infinitely many material symmetries of all kinds in two dimensions (and also in three dimensions). Among them, there are eight symmetries that can be characterized in terms of vector(s), second-order tensor(s), and/or third-order tensor, as shown below in Table 1.

In this paper, notation is based on the following conventions. We denote by $\mathbf{1}$ the second-order intensity tensor, $\boldsymbol{\varepsilon}$ the permutation tensor (a second-order skew-symmetric tensor), $\mathbf{R}(\theta)$ the rotation tensor of angle θ, $\mathbf{a}$ and $\mathbf{b}$ two unit orthogonal vectors, $\mathbf{R}_\mathbf{b}$ the reflection transformation in $\mathbf{b}$ direction, and

$$\mathbf{P} = \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a} - (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a}). \tag{1.1}$$

Table 1. All kinds of two-dimensional material symmetry that have structural tensors as vector(s), second-order tnsor(s), and/or third-order tensor.

system	Schoenflies symbol	generators of symmetry group	structural tensors
oblique	C_1	$\mathbf{R}(0) = \mathbf{1}$	$\mathbf{a}$, $\boldsymbol{\varepsilon}$ (or $\mathbf{a}$, $\mathbf{b}$)
	C_2	$\mathbf{R}(\pi) = -\mathbf{1}$	$\mathbf{M}$, $\boldsymbol{\varepsilon}$
rectangular	$C_{1\nu}$	$\mathbf{R_b} = \mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b}$	$\mathbf{a}$
	$C_{2\nu}$ = orthotropy	$\mathbf{R}(\pi)$, $\mathbf{R_b}$	$\mathbf{M} = \mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b}$
trigonal	C_3	$\mathbf{R}(2\pi/3)$	$\mathbf{P}$, $\boldsymbol{\varepsilon}$
	$C_{3\nu}$	$\mathbf{R}(2\pi/3)$, $\mathbf{R_b}$	$\mathbf{P}$
circle	C_∞ = hemitropy	$\mathbf{R}(\theta)$ $(0 \le \theta < 2\pi$	$\boldsymbol{\varepsilon}$
	$C_{\infty\nu}$ = isotropy	$\mathbf{R}(\theta)$, $\mathbf{R_b}$ $(0 \le \theta < 2\pi)$	$\mathbf{1}$

The operators $\otimes$, $\cdot$, $:$, and $\vdots$ mean tensor, scalar (or dot), double dot and triple dot products, respectively. Components of vectors and tensors are referred to an orthonormal frame, say $\{\mathbf{e}_i\}$, lower-case Latin indices $(i, j, k, \ldots)$ range from 1 to 2, repeated indices are summed from 1 to 2, and the abbreviations $\mathbf{e}_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathbf{e}_{ijk} = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ are used. The prefix tr indicates trace.

A tensor $\mathbf{H}$ is termed as *irreducible*, if it is a completely symmetric and traceless:

$$(1.2) \qquad H_{ijk\ldots l} = H_{jik\ldots l} = H_{kji\ldots l} = \ldots = H_{ljk\ldots i}\,, \qquad H_{mmk\ldots l} = 0_{k\ldots l}\,,$$

where $0_{k\ldots l}$ corresponds to the zero-tensor of the relevant order. It is well known that any irreducible tensor $\mathbf{H}$ in two-dimensional space has only two independent components (e.g. $H_{111\ldots 1}$ and $H_{211\ldots 1}$). In particular, the relations among the components of an irreducible third-order tensor $\mathbf{T}$ are:

$$(1.3) \qquad T_{122} = T_{212} = T_{221} = -T_{111}\,, \qquad T_{112} = T_{121} = T_{211} = -T_{222}\,.$$

We can decompose any third-order tensor $\mathbf{D}$ into an irreducible third-order tensor $\mathbf{T}$ and three vectors $D_{ill}\mathbf{e}_i$, $D_{lil}\mathbf{e}_i$, $D_{lli}\mathbf{e}_i$ in the form:

$$(1.4) \qquad T_{ijk} = 4D_{ijk} - (3D_{llk} - D_{lkl} - D_{kll})\delta_{ij} - (3D_{ljl} - D_{jll} - D_{llj})\delta_{ik} \\ -(3D_{ill} - D_{lil} - D_{lli})\delta_{jk}\,,$$

where δ_{ij} denotes the Kronecker symbol. An elementary method of reducing tensors of any order to sums of irreducible tensors is described by SPENCER [6] and HANNABUSS [3].

In view of (1.4) we further postulate that the third-order tensors $\mathbf{T}_\lambda$ (i.e., $\mathbf{T}_1, \ldots, \mathbf{T}_L$) are all irreducible. In this paper, we determine the complete and irreducible representations relative to the eight symmetries in Table 1 for scalar-, vector-, second-order tensor- and third-order tensor-valued functions of $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$ and $\mathbf{T}_\lambda$.

2. Isotropic representations in different cases

In order to provide compact procedures of determining complete and irreducible tensor function representations relative to the eight symmetries given in Table 1, we derive in this section representations with respect to different cases of the variables $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$, $\mathbf{T}_\lambda$. The method employed here is described by ZHENG [7, 8].

It is profitable to introduce the following abbreviations and relations:

$$(2.1)\quad \begin{aligned} \mathbf{t}^{\mathbf{A}} &= \mathbf{T}:\mathbf{A} = T_{ijk}A_{jk}\mathbf{e}_i \\ &= (T_{111}\mathbf{e}_1 + T_{211}\mathbf{e}_2)(A_{11} - A_{22}) + 2(T_{211}\mathbf{e}_1 - T_{111}\mathbf{e}_2)A_{12}, \\ \mathbf{t}^{\mathbf{v}} &= \mathbf{T}:(\mathbf{v}\otimes\mathbf{v}) = T_{ijk}x_jx_k\mathbf{e}_i \\ &= (T_{111}\mathbf{e}_1 + T_{211}\mathbf{e}_2)(x_1^2 - x_2^2) + 2(T_{211}\mathbf{e}_1 - T_{111}\mathbf{e}_2)x_1x_2, \\ \mathbf{T}^{\mathbf{v}} &= \mathbf{T}\cdot\mathbf{v} = T_{ijk}x_k\mathbf{e}_{ij} \\ &= (T_{111}x_1 + T_{211}x_2)(\mathbf{e}_{11} - \mathbf{e}_{22}) + (T_{211}x_1 - T_{111}x_2)(\mathbf{e}_{12} + \mathbf{e}_{21}), \\ \mathbf{T}:\mathbf{S} &= T_{ikl}S_{klj}\mathbf{e}_{ij} \\ &= 2(T_{111}S_{111} + T_{211}S_{211})\mathbf{1} + 2(T_{111}S_{211} - T_{211}S_{111})(\mathbf{e}_{12} - \mathbf{e}_{21}), \\ \mathbf{T}\mathbf{W} &= T_{ijl}W_{lk}\mathbf{e}_{ijk} \\ &= [T_{111}(\mathbf{e}_{112} + \mathbf{e}_{121} + \mathbf{e}_{211} - \mathbf{e}_{222}) + T_{211}(\mathbf{e}_{122} + \mathbf{e}_{212} + \mathbf{e}_{221} - \mathbf{e}_{111})]W_{12}, \end{aligned}$$

and

$$\mathbf{T}\vdots\mathbf{S} = T_{ijk}S_{ijk} = 4(T_{111}S_{111} + T_{211}S_{211}),$$

$$(2.2)\quad (\mathbf{x}\cdot\mathbf{t}_{\mathbf{x}})^2 + (\mathbf{x}\cdot\boldsymbol{\varepsilon}\mathbf{t}_{\mathbf{x}})^2 = (\mathbf{T}\vdots\mathbf{T})(\mathbf{x}\cdot\mathbf{x})^3/4,$$

$$(\mathbf{t}_{\mathbf{A}}\cdot\mathbf{A}\,\mathbf{t}_{\mathbf{A}})^2 + (\mathbf{t}_{\mathbf{A}}\cdot\boldsymbol{\varepsilon}\mathbf{A}\,\mathbf{t}_{\mathbf{A}})^2 = (\mathbf{T}\vdots\mathbf{T})^2[2\,\mathrm{tr}\,\mathbf{A}^2 - (\mathrm{tr}\,\mathbf{A})^2]^3/64,$$

where $\mathbf{T}$ and $\mathbf{S}$ denote any two irreducible third-order tensors, and $\mathbf{A}$, $\mathbf{W}$ and $\mathbf{v}$ any second-order symmetric tensor, second-order skew-symmetric tensor and vector, respectively. Let $\mathbf{D}$ be any third-order tensor. The symbol $\{\mathbf{D}\}$ denotes as a set of the following three tensors:

$$(2.3)\quad D_{ijk}\mathbf{e}_{ijk}, \qquad D_{ijk}\mathbf{e}_{jki}, \qquad D_{ijk}\mathbf{e}_{kij},$$

and $\langle\mathbf{D}\rangle$ is the summation of the above three tensors, that is,

$$(2.4)\quad \langle\mathbf{D}\rangle = D_{ijk}(\mathbf{e}_{ijk} + \mathbf{e}_{jki} + \mathbf{e}_{kij}).$$

2.1. Representations when there exists a non-zero vector v among $\mathbf{v}_\varrho$

We can choose an orthonormal frame $\{\mathbf{e}_i\}$ so that $\mathbf{v} = \nu_1\mathbf{e}_1$ with $\nu_1 > 0$. Thus, we can write

$$(2.5)\quad \begin{aligned} &\mathbf{v}\cdot\mathbf{v} \Rightarrow \nu_1, \qquad \mathbf{v} \Rightarrow \mathbf{e}_1, \qquad \mathbf{v}\cdot\mathbf{v}_\varrho \Rightarrow \nu_{\varrho 1} \qquad (\mathbf{v}_\varrho \neq \mathbf{v}), \\ &\mathbf{v}\otimes\mathbf{v},\ \mathbf{1} \Rightarrow \mathbf{e}_{11},\ \mathbf{e}_{22}, \qquad \mathbf{v}\cdot\mathbf{A}_\alpha\mathbf{v},\ \mathrm{tr}\,\mathbf{A}_\alpha \Rightarrow A_{\alpha 11},\ A_{\alpha 22}, \\ &\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v},\ \{\mathbf{v}\otimes\mathbf{1}\} \Rightarrow \mathbf{e}_{111},\ \{\mathbf{e}_{122}\}, \qquad \mathbf{v}\cdot\mathbf{t}_\lambda^{\mathbf{v}} \Rightarrow T_{\lambda 111}. \end{aligned}$$

In this paper, the notation $\mathcal{A} \Rightarrow \mathcal{B}$ means that $\mathcal{B}$ is uniquely determined by $\mathcal{A}$. It is evident that the alternative of $\pm\mathbf{e}_2$ does not affect (2.5). Thus, the undetermined quantities remain

$$\nu_{\varrho 2}\,,\ A_{\alpha 12}\,,\ W_{\xi 12}\,,\ T_{\lambda 211}\,; \qquad \mathbf{e}_2\,; \qquad \mathbf{e}_{12} \pm \mathbf{e}_{21}\,; \qquad \mathbf{e}_{222}\,, \{\mathbf{e}_{112}\}.$$

If all $\nu_{\varrho 2}$, $A_{\alpha 12}$, $W_{\xi 12}$, and $T_{\lambda 211}$ equal zero, we do not need to determine $\mathbf{e}_2$, $\mathbf{e}_{12} \pm \mathbf{e}_{21}$, $\{\mathbf{e}_{112}\}$ and $\mathbf{e}_{222}$. Otherwise, keeping (2.5) in mind, we consider the following cases $1-4$.

CASE 1. $W_{12} \neq 0$ for a tensor $\mathbf{W}$ among $\mathbf{W}_\xi$. We alternate $\pm\mathbf{e}_2$ so that $W_{12} > 0$ and then give

$$\begin{gathered}
\operatorname{tr}\mathbf{W}^2 \Rightarrow W_{12}\,, \qquad \mathbf{W}\mathbf{v} \Rightarrow \mathbf{e}_2\,, \qquad \mathbf{v}\cdot\mathbf{W}\mathbf{v}_\varrho \Rightarrow \nu_{\varrho 2} \qquad (\mathbf{v}_\varrho \neq \mathbf{v}),\\
\mathbf{v}\otimes\mathbf{W}\mathbf{v} + \mathbf{W}\mathbf{v}\otimes\mathbf{v} \Rightarrow \mathbf{e}_{12} + \mathbf{e}_{21}\,, \qquad \mathbf{v}\cdot\mathbf{A}_\alpha\mathbf{W}\mathbf{v} \Rightarrow A_{\alpha 12}\,,\\
\mathbf{W} \Rightarrow \mathbf{e}_{12} - \mathbf{e}_{21}\,, \qquad \operatorname{tr}\mathbf{W}\mathbf{W}_\xi \Rightarrow W_{\xi 12} \qquad (\mathbf{W}_\xi \neq \mathbf{W}),\\
\langle \mathbf{v}\otimes\mathbf{v}\otimes\mathbf{W}\mathbf{v}\rangle \quad (\text{or} \quad \mathbf{W}\mathbf{v}\otimes\mathbf{W}\mathbf{v}\otimes\mathbf{W}\mathbf{v}), \quad \{\mathbf{W}\mathbf{v}\otimes\mathbf{1}\} \Rightarrow \mathbf{e}_{222}\,,\ \{\mathbf{e}_{112}\},\\
\mathbf{v}\cdot\mathbf{W}\mathbf{t}^{\mathbf{v}}_\lambda \qquad (\text{or} \quad \mathbf{v}\cdot\mathbf{W}\mathbf{t}^{\mathbf{W}\mathbf{v}}_\lambda) \Rightarrow T_{\lambda 211}\,.
\end{gathered} \tag{2.6}$$

CASE 2. $u_2 \neq 0$ for a vector $\mathbf{u}$ among $\mathbf{v}_\varrho$. We can select $\pm\mathbf{e}_2$ so that $u_2 > 0$ and then have

$$\begin{gathered}
\mathbf{u}\cdot\mathbf{u} \Rightarrow u_2\,, \qquad \mathbf{u} \Rightarrow \mathbf{e}_2\,, \qquad \mathbf{u}\cdot\mathbf{v}_\varrho \Rightarrow \nu_{\varrho 2} \qquad (\mathbf{v}_\varrho \neq \mathbf{v},\ \mathbf{u}),\\
\mathbf{v}\otimes\mathbf{u} \pm \mathbf{u}\otimes\mathbf{v} \Rightarrow \mathbf{e}_{12} \pm \mathbf{e}_{21}\,, \qquad \mathbf{v}\cdot\mathbf{A}_\alpha\mathbf{u} \Rightarrow A_{\alpha 12}\,, \qquad \mathbf{v}\cdot\mathbf{W}_\xi\mathbf{u} \Rightarrow W_{\xi 12}\,,\\
\langle \mathbf{v}\otimes\mathbf{v}\otimes\mathbf{u}\rangle \quad (\text{or} \quad \mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u}, \ \text{ if } \ u_2^2 \neq 3u_1^2), \quad \{\mathbf{u}\otimes\mathbf{1}\} \Rightarrow \mathbf{e}_{222}\,, \quad \{\mathbf{e}_{112}\},\\
\mathbf{u}\cdot\mathbf{t}^{\mathbf{v}}_\lambda \quad (\text{or} \quad \mathbf{u}\cdot\mathbf{t}^{\mathbf{u}}_\lambda, \ \text{ if } \ u_2^2 \neq 3u_1^2) \Rightarrow T_{\lambda 211}\,.
\end{gathered} \tag{2.7}$$

CASE 3. $A_{12} \neq 0$ for a tensor $\mathbf{A}$ among $\mathbf{A}_\alpha$. By alternating $\pm\mathbf{e}_2$ we can arrive at $A_{12} > 0$ and

$$\begin{gathered}
\operatorname{tr}\mathbf{A}^2 \Rightarrow A_{12}\,, \qquad \mathbf{A} \Rightarrow \mathbf{e}_{12} + \mathbf{e}_{21}, \qquad \operatorname{tr}\mathbf{A}\mathbf{A}_\alpha \Rightarrow A_{\alpha 12} \qquad (\mathbf{A}_\alpha \neq \mathbf{A}),\\
\mathbf{A}\mathbf{v} \Rightarrow \mathbf{e}_2\,, \qquad \mathbf{v}\cdot\mathbf{A}\mathbf{v}_\varrho \Rightarrow \nu_{\varrho 2} \qquad (\mathbf{v}_\varrho \neq \mathbf{v}),\\
\mathbf{v}\otimes\mathbf{A}\mathbf{v} - \mathbf{A}\mathbf{v}\otimes\mathbf{v} \Rightarrow \mathbf{e}_{12} - \mathbf{e}_{21}\,, \qquad \mathbf{v}\cdot\mathbf{A}\mathbf{W}_\xi\mathbf{v} \Rightarrow W_{\xi 12}\,,\\
\langle \mathbf{v}\otimes\mathbf{A}\rangle, \quad \{\mathbf{A}\mathbf{v}\otimes\mathbf{1}\} \Rightarrow e_{222}\,, \quad \{\mathbf{e}_{112}\}, \qquad \mathbf{v}\cdot\mathbf{t}^{\mathbf{A}}_\lambda \Rightarrow T_{\lambda 211}\,.
\end{gathered} \tag{2.8}$$

CASE 4. $T_{211} \neq 0$ for a tensor $\mathbf{T}$ among $\mathbf{T}_\lambda$. Choosing $\pm\mathbf{e}_2$ so that $T_{211} > 0$ can follow

$$\begin{gathered}
\mathbf{T}\,\vdots\,\mathbf{T} \Rightarrow T_{211}\,, \qquad \mathbf{t}^{\mathbf{v}} \Rightarrow \mathbf{e}_2\,, \qquad \mathbf{v}_\varrho\cdot\mathbf{t}^{\mathbf{v}} \Rightarrow \nu_{\varrho 2} \qquad (\mathbf{v}_\varrho \neq \mathbf{v}),\\
\mathbf{T}^{\mathbf{v}} \Rightarrow \mathbf{e}_{12} + \mathbf{e}_{21}\,, \qquad \mathbf{v}\cdot\mathbf{t}^{\mathbf{A}_\alpha} = \operatorname{tr}(\mathbf{A}_\alpha\mathbf{T}^{\mathbf{v}}) \Rightarrow A_{\alpha 12}\,,\\
\mathbf{v}\otimes\mathbf{t}^{\mathbf{v}} - \mathbf{t}^{\mathbf{v}}\otimes\mathbf{v} \Rightarrow \mathbf{e}_{12} - \mathbf{e}_{21}\,, \qquad \mathbf{v}\cdot\mathbf{W}_\xi\mathbf{t}^{\mathbf{v}} \Rightarrow W_{\xi 12}\,,\\
\mathbf{T}, \quad \{\mathbf{t}^{\mathbf{v}}\otimes\mathbf{1}\} \Rightarrow \mathbf{e}_{222}\,, \quad \{\mathbf{e}_{112}\}, \qquad \mathbf{T}\,\vdots\,\mathbf{T}_\lambda \Rightarrow T_{\lambda 211} \qquad (\mathbf{T}_\lambda \neq \mathbf{T}).
\end{gathered} \tag{2.9}$$

2.2. Representations when all vectors $\mathbf{v}_\varrho$ equal zero, there exists a tensor A among $\mathbf{A}_\alpha$ which has two distinct principal values, and there exists a non-zero tensor T among $\mathbf{T}_\lambda$

By selecting $\pm\mathbf{e}_1$ and $\pm\mathbf{e}_2$ as orthogonal principal directions of $\mathbf{A}$, we can express $\mathbf{A} = A_{11}\mathbf{e}_{11} + A_{22}\mathbf{e}_{22}$ and then we have

$$
\begin{aligned}
&\operatorname{tr}\mathbf{A},\ \operatorname{tr}\mathbf{A}^2 \Rightarrow A_{11}\,,\ A_{22}\,, \qquad \mathbf{1},\ \mathbf{A} \Rightarrow \mathbf{e}_{11}\,,\ \mathbf{e}_{22}\,,\\
&\operatorname{tr}\mathbf{A}_\alpha\,,\ \operatorname{tr}\mathbf{A}\mathbf{A}_\alpha \Rightarrow A_{\alpha 11}\,,\ A_{\alpha 22} \qquad (\mathbf{A}_\alpha \neq \mathbf{A}).
\end{aligned}
\tag{2.10}
$$

Note that choices of $\pm\mathbf{e}_1$ and of $\pm\mathbf{e}_2$ do not influence (2.10). Since $\mathbf{T}$ is a non-zero irreducible third-order tensor, without loss of generality we can suppose that $T_{111} > 0$ by alternating $\mathbf{e}_1$ and $\mathbf{e}_2$ and choosing $\pm\mathbf{e}_1$. It follows

$$
\mathbf{T}\vdots\mathbf{T},\ \mathbf{t}^{\mathbf{A}}\cdot\mathbf{A}\,\mathbf{t}^{\mathbf{A}} \Rightarrow T_{111}\,,\ T_{211}^2\,.
\tag{2.11}
$$

For the remaining undetermined quantities:

$$
\nu_{\varrho i}\,,\ A_{\alpha 12}\,,\ W_{\xi 12}\,,\ T_{\lambda i11}\,; \qquad \mathbf{e}_i\,; \qquad \mathbf{e}_{12} \pm \mathbf{e}_{21}\,; \qquad \mathbf{e}_{ijk} \qquad (i,j,k = 1,2),
$$

we consider the following cases i – iv.

CASE i. $W_{12} \neq 0$ for a tensor $\mathbf{W}$ among $\mathbf{W}_\xi$. We choose $\pm\mathbf{e}_2$ so that $W_{12} > 0$ and then we have

$$
\begin{aligned}
&\operatorname{tr}\mathbf{W}^2 \Rightarrow W_{12}\,, \qquad \mathbf{t}^{\mathbf{A}}\cdot\mathbf{A}\,\mathbf{W}\,\mathbf{t}^{\mathbf{A}} \Rightarrow T_{211}\,, \qquad \mathbf{t}^{\mathbf{A}},\ \ \mathbf{W}\,\mathbf{t}^{\mathbf{A}} \Rightarrow \mathbf{e}_i\,,\\
&\mathbf{A}\,\mathbf{W} - \mathbf{W}\,\mathbf{A} \Rightarrow \mathbf{e}_{12} + \mathbf{e}_{21}\,, \qquad \operatorname{tr}\mathbf{A}\,\mathbf{A}_\alpha\mathbf{W} \Rightarrow A_{\alpha 12} \qquad (\mathbf{A}_\alpha \neq \mathbf{A}),\\
&\mathbf{W} \Rightarrow \mathbf{e}_{12} - \mathbf{e}_{21}\,, \qquad \operatorname{tr}\mathbf{W}\,\mathbf{W}_\xi \Rightarrow W_{\xi 12} \qquad (\mathbf{W}_\xi \neq \mathbf{W}),\\
&\mathbf{T},\ \ \mathbf{T}\,\mathbf{W},\ \ \{\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\},\ \ \{\mathbf{W}\,\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\} \Rightarrow \mathbf{e}_{ijk}\,,\\
&\mathbf{T}\vdots\mathbf{T}_\lambda,\ \ \operatorname{tr}(\mathbf{T} : \mathbf{T}_\lambda)\mathbf{W} \Rightarrow T_{\lambda i11} \qquad (\mathbf{T}_\lambda \neq \mathbf{T}).
\end{aligned}
\tag{2.12}
$$

CASE ii. $T_{111}T_{211} \neq 0$. Alternating $\pm\mathbf{e}_2$ can yield $T_{211} > 0$ and then

$$
\begin{aligned}
&\mathbf{T}\vdots\mathbf{T},\ \ \mathbf{t}^{\mathbf{A}}\cdot\mathbf{A}\,\mathbf{t}^{\mathbf{A}} \Rightarrow T_{211}\,, \qquad \mathbf{t}^{\mathbf{A}},\ \ \mathbf{A}\,\mathbf{t}^{\mathbf{A}} \Rightarrow \mathbf{e}_i\,,\\
&\mathbf{t}^{\mathbf{A}} \otimes \mathbf{t}^{\mathbf{A}} \Rightarrow \mathbf{e}_{12} + \mathbf{e}_{21}\,, \qquad \mathbf{t}^{\mathbf{A}}\cdot\mathbf{A}_\alpha\mathbf{t}^{\mathbf{A}} \Rightarrow A_{\alpha 12} \qquad (\mathbf{A}_\alpha \neq \mathbf{A}),\\
&\mathbf{t}^{\mathbf{A}} \otimes \mathbf{A}\,\mathbf{t}^{\mathbf{A}} - \mathbf{A}\,\mathbf{t}^{\mathbf{A}} \otimes \mathbf{t}^{\mathbf{A}} \Rightarrow \mathbf{e}_{12} + \mathbf{e}_{21}\,, \qquad \mathbf{t}^{\mathbf{A}}\cdot\mathbf{A}\,\mathbf{W}_\xi\mathbf{t}^{\mathbf{A}} \Rightarrow W_{\xi 12}\,,\\
&\mathbf{T},\ \ \langle\mathbf{A}\,\mathbf{t}^{\mathbf{A}} \otimes \mathbf{A}\rangle,\ \ \{\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\},\ \ \{\mathbf{A}\,\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\} \Rightarrow \mathbf{e}_{ijk}\,,\\
&\mathbf{T}\vdots\mathbf{T}_\lambda\,,\ \ \mathbf{t}^{\mathbf{A}}\cdot\mathbf{A}\,\mathbf{t}^{\mathbf{A}}_\lambda \Rightarrow T_{\lambda i11} \qquad (\mathbf{T}_\lambda \neq \mathbf{T}).
\end{aligned}
\tag{2.13}
$$

CASE iii. $T_{211} = 0$ and $B_{12} \neq 0$ for a tensor $\mathbf{B}$ among $\mathbf{A}_\alpha$. Selecting $\pm\mathbf{e}_2$ so that $B_{12} > 0$, we have

$$
(2.14)\qquad
\begin{gathered}
\operatorname{tr}\mathbf{B}^2 \Rightarrow B_{12}, \qquad \mathbf{t}^{\mathrm{A}},\ \mathbf{t}^{\mathrm{B}} \Rightarrow \mathbf{e}_i, \\
\mathbf{B} \Rightarrow \mathbf{e}_{12} + \mathbf{e}_{21}, \qquad \operatorname{tr}\mathbf{B}\mathbf{A}_\alpha \Rightarrow A_{\alpha 12} \qquad (\mathbf{A}_\alpha \neq \mathbf{A},\ \mathbf{B}), \\
\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A} \Rightarrow \mathbf{e}_{12} - \mathbf{e}_{21}, \qquad \operatorname{tr}\mathbf{A}\mathbf{B}\mathbf{W}_\xi \Rightarrow W_{\xi 12}, \\
\mathbf{T},\ \ \mathbf{T}(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}),\ \ \{\mathbf{t}^{\mathrm{A}} \otimes \mathbf{1}\},\ \ \{\mathbf{t}^{\mathrm{B}} \otimes \mathbf{1}\} \Rightarrow \mathbf{e}_{ijk}, \\
\mathbf{T}\,\vdots\,\mathbf{T}_\lambda,\ \ \operatorname{tr}(\mathbf{T} : \mathbf{T}_\lambda)\mathbf{A}\mathbf{B} \Rightarrow T_{\lambda i 11} \qquad (\mathbf{T}_\lambda \neq \mathbf{T}).
\end{gathered}
$$

CASE iv. $T_{211} = 0$ and $S_{211} \neq 0$ for a tensor $\mathbf{S}$ among $\mathbf{T}_\lambda$. Selecting $\pm\mathbf{e}_2$ so that $S_{211} > 0$ can follow,

$$
(2.15)\qquad
\begin{gathered}
\mathbf{T}\,\vdots\,\mathbf{S},\ \mathbf{S}\,\vdots\,\mathbf{S} \Rightarrow S_{111},\ S_{211}, \qquad \mathbf{t}^{\mathrm{A}},\ \mathbf{s}^{\mathrm{A}} \Rightarrow \mathbf{e}_i, \\
\mathbf{A}(\mathbf{T} : \mathbf{S}) - (\mathbf{S} : \mathbf{T})\mathbf{A} \Rightarrow \mathbf{e}_{12} + \mathbf{e}_{21}, \\
\operatorname{tr}(\mathbf{T} : \mathbf{S})\mathbf{A}\mathbf{A}_\alpha \Rightarrow A_{\alpha 12} \qquad (\mathbf{A}_\alpha \neq \mathbf{A}), \\
\mathbf{T} : \mathbf{S} - \mathbf{S} : \mathbf{T} \Rightarrow \mathbf{e}_{12} - \mathbf{e}_{21}, \qquad \operatorname{tr}(\mathbf{T} : \mathbf{S})\mathbf{W}_\xi \Rightarrow W_{\xi 12}, \\
\mathbf{T},\ \mathbf{S},\ \{\mathbf{t}^{\mathrm{A}} \otimes \mathbf{1}\},\ \{\mathbf{s}^{\mathrm{A}} \otimes \mathbf{1}\} \Rightarrow \mathbf{e}_{ijk}, \\
\mathbf{T}\,\vdots\,\mathbf{T}_\lambda,\ \mathbf{S}\,\vdots\,\mathbf{T}_\lambda \Rightarrow T_{\lambda i 11} \qquad (\mathbf{T}_\lambda \neq \mathbf{T},\ \mathbf{S}).
\end{gathered}
$$

2.3. Representations when all $\mathbf{v}_\varrho$ are null vectors, all $\mathbf{A}_\alpha$ have not two distinct principal values, and there is a non-zero tensor $\mathbf{T}$ among $\mathbf{T}_\lambda$

Since $\mathbf{A}_\alpha$ have not two distinct principal values, we can express them as $\mathbf{A}_\alpha = (\operatorname{tr}\mathbf{A}_\alpha)\mathbf{1}/2$. It is known from Table 2 of ZHENG and SPENCER [12] that any rotation tensor $\mathbf{R}(\varphi)$ leaves $\mathbf{1}$, $\mathbf{e}_{12} - \mathbf{e}_{21}$, $\mathbf{A}_\alpha = (\operatorname{tr}\mathbf{A}_\alpha/2)\mathbf{1}$ and $\mathbf{W}_\xi = W_{\xi 12}(\mathbf{e}_{12} - \mathbf{e}_{21})$ unaltered. Then, we have the following transformation relations

$$
(2.16)\qquad
\begin{aligned}
&(\mathbf{e}_1 + \imath\mathbf{e}_2) \mapsto \exp(\pm\imath\varphi)(\mathbf{e}_1 + \imath\mathbf{e}_2), \\
&\{(\mathbf{e}_1 + \imath\mathbf{e}_2) \otimes \mathbf{1}\} \mapsto \exp(\pm\imath\varphi)\{(\mathbf{e}_1 + \imath\mathbf{e}_2) \otimes \mathbf{1}\}, \\
&(\mathbf{e}_{11} - \mathbf{e}_{22}) + \imath(\mathbf{e}_{12} + \mathbf{e}_{21}) \mapsto \exp(\pm\imath 2\varphi)[(\mathbf{e}_{11} - \mathbf{e}_{22}) + \imath\mathbf{e}_{12} + \mathbf{e}_{21})], \\
&(\mathbf{e}_{111} - \langle\mathbf{e}_{122}\rangle) + \imath(\langle\mathbf{e}_{112}\rangle - \mathbf{e}_{222}) \\
&\qquad\qquad \mapsto \exp(\pm\imath 3\varphi)[(\mathbf{e}_{111} - \langle\mathbf{e}_{122}\rangle) + \imath(\langle\mathbf{e}_{112}\rangle - \mathbf{e}_{222})],
\end{aligned}
$$

where $\imath = \sqrt{(-1)}$ is the unit imaginary number. Thus, we can rotate $\{\mathbf{e}_i\}$ until $T_{111} > 0$ and $T_{211} = 0$, and then give

$$
(2.17)\qquad \mathbf{T}\,\vdots\,\mathbf{T} \Rightarrow T_{111}, \qquad \mathbf{T} \Rightarrow \mathbf{e}_{111} - \langle\mathbf{e}_{122}\rangle, \qquad \mathbf{T}\,\vdots\,\mathbf{T}_\lambda \Rightarrow T_{\lambda 111} \qquad (\mathbf{T}_\lambda \neq \mathbf{T}).
$$

From (2.16) we can see that $\mathbf{R}(2\pi/3)$ leaves $\mathbf{e}_{111} - \langle\mathbf{e}_{122}\rangle$, $\langle\mathbf{e}_{112}\rangle - \mathbf{e}_{222}$ and $\mathbf{T}_\lambda$ unaltered, but it varies $\mathbf{e}_i$, $\mathbf{e}_{11} - \mathbf{e}_{22}$, $\mathbf{e}_{12} + \mathbf{e}_{21}$ and $\{\mathbf{e}_i \otimes \mathbf{1}\}$ so that they do not need to be determined in view of the isotropy condition. In other words, we only need to determine

$$W_{\xi 12}\,, \quad T_{\lambda 211}\,, \quad \mathbf{e}_{12} - \mathbf{e}_{21}\,, \quad \langle \mathbf{e}_{112} \rangle - \mathbf{e}_{222}\,.$$

If there exists a non-zero tensor $\mathbf{W}$ among $\mathbf{W}_\xi$ or there is a non-zero tensor $\mathbf{S}$ among $\mathbf{T}_\lambda$, we select $\pm \mathbf{e}_2$ so that $W_{12} > 0$ or $S_{211} > 0$, and then we write

$$\text{(2.18)} \qquad \begin{aligned} &\operatorname{tr}\mathbf{W}^2 \Rightarrow W_{12}\,, \qquad \mathbf{W} \Rightarrow \mathbf{e}_{12} - \mathbf{e}_{21}\,, \qquad \operatorname{tr}\mathbf{W}\mathbf{W}_\xi \Rightarrow W_{\xi 12} \quad (\mathbf{W}_\xi \neq \mathbf{W}),\\ &\mathbf{T}\mathbf{W} \Rightarrow \langle \mathbf{e}_{112} \rangle - \mathbf{e}_{222}\,, \qquad \operatorname{tr}(\mathbf{T} : \mathbf{S})\mathbf{W}_\xi \Rightarrow T_{\lambda 211} \quad (\mathbf{T}_\lambda \neq \mathbf{T}), \end{aligned}$$

or

$$\text{(2.19)} \qquad \begin{aligned} &\mathbf{S} \,\vdots\, \mathbf{S} \Rightarrow \mathbf{S}_{211}\,, \qquad \mathbf{T} : \mathbf{S} - \mathbf{S} : \mathbf{T} \Rightarrow \mathbf{e}_{12} - \mathbf{e}_{21}\,, \qquad \operatorname{tr}(\mathbf{T} : \mathbf{S})\mathbf{W}_\xi \Rightarrow W_{\xi 12}\,,\\ &\mathbf{S} \Rightarrow \langle \mathbf{e}_{112} \rangle - \mathbf{e}_{222}\,, \qquad \mathbf{S} \,\vdots\, \mathbf{T}_\lambda \Rightarrow T_{\lambda 211} \quad (\mathbf{T}_\lambda \neq \mathbf{T},\ \mathbf{S}). \end{aligned}$$

Otherwise, if all $T_{\lambda 211}$ and $W_{\xi 12}$ equal zero, we do not need to determine $\mathbf{e}_{12} - \mathbf{e}_{21}$ and $\langle \mathbf{e}_{112} \rangle - \mathbf{e}_{222}$.

2.4. Representation when all vectors $\mathbf{v}_\varrho$ and third-order tensors $\mathbf{T}_\lambda$ are equal to zero

Since the central inversion $-\mathbf{1}$ leaves tensors of even orders unaltered but changes the sign of tensors of odd orders, the isotropy condition requires that any isotropic vector- and third-order tensor-valued functions of second-order tensor $\mathbf{A}_\alpha$ and $\mathbf{W}_\xi$ should be only a zero-vector and a third-order zero-tensor, respectively. The complete and irreducible representations for scalar- and second-order tensor-valued functions of $\mathbf{A}_\alpha$ and $\mathbf{W}_\xi$ can be seen, for example, in Tables 2, 4 and 5 of ZHENG [8].

3. The complete and irreducible tensor function representations

The complete and irreducible representations were established by ZHENG [8] for scalar-, vector-, second-order symmetric and skew-symmetric tensor-valued functions of $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$ with respect to all kinds of symmetry, particularly, to the eight symmetries shown in Table 1. With these known results and the representations derived in the preceding section, we determine in the sequel the complete representations for the eight symmetries shown in Table 1. The irreducibility of the derived representations is verified in the next section.

3.1. Representation for isotropy $C_{\infty v}$

The complete representations for isotropic tensor functions of $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$ and $\mathbf{T}_\lambda$ can be obtained by considering all the cases in Secs. 2.1 – 2.3 i.e., from (2.5) – (2.19), as summarized in Tables 2 and 3.

Table 2. Irreducible function bases.

$C_{\infty\nu}$	$\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{A}^2$, $\operatorname{tr}\mathbf{AB}$, $\operatorname{tr}\mathbf{W}^2$, $\operatorname{tr}\mathbf{ABW}$, $\operatorname{tr}\mathbf{WV}$, $\mathbf{v}\cdot\mathbf{v}$, $\mathbf{v}\cdot\mathbf{Av}$, $\mathbf{v}\cdot\mathbf{AWv}$, $\mathbf{v}\cdot\mathbf{u}$, $\mathbf{v}\cdot\mathbf{Au}$, $\mathbf{v}\cdot\mathbf{Wu}$; $\mathbf{T}\vdots\mathbf{T}$, $\mathbf{t}^{\mathbf{A}}\cdot\mathbf{At}^{\mathbf{A}}$, $\mathbf{t}^{\mathbf{A}}\cdot\mathbf{Bt}^{\mathbf{A}}$, $\mathbf{T}^{\mathbf{A}}\cdot\mathbf{AWt}^{\mathbf{A}}$, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{v}}$, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{A}}$, $\mathbf{v}\cdot\mathbf{Wt}^{\mathbf{v}}$, $\mathbf{u}\cdot\mathbf{t}^{\mathbf{v}}$, $\mathbf{T}\vdots\mathbf{S}$, $\mathbf{t}^{\mathbf{A}}\cdot\mathbf{As}^{\mathbf{A}}$, $\operatorname{tr}(\mathbf{T}:\mathbf{S})\mathbf{AB}$, $\operatorname{tr}(\mathbf{T}:\mathbf{S})\mathbf{W}$	C_∞	$\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{A}^2$, $\operatorname{tr}\mathbf{AB}$, $\operatorname{tr}\mathbf{AB}\boldsymbol{\varepsilon}$, $\operatorname{tr}\boldsymbol{\varepsilon}\mathbf{W}$, $\mathbf{v}\cdot\mathbf{v}$, $\mathbf{v}\cdot\mathbf{Av}$, $\mathbf{v}\cdot\mathbf{A}\boldsymbol{\varepsilon}\mathbf{v}$, $\mathbf{v}\cdot\mathbf{u}$, $\mathbf{v}\cdot\boldsymbol{\varepsilon}\mathbf{u}$; $\mathbf{T}\vdots\mathbf{T}$, $\mathbf{t}^{\mathbf{A}}\cdot\mathbf{At}^{\mathbf{A}}$, $\mathbf{t}^{\mathbf{A}}\cdot\mathbf{A}\boldsymbol{\varepsilon}\mathbf{t}^{\mathbf{A}}$, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{v}}$, $\mathbf{v}\cdot\boldsymbol{\varepsilon}\mathbf{t}^{\mathbf{v}}$, $\mathbf{T}\vdots\mathbf{S}$, $\operatorname{tr}(\mathbf{T}:\mathbf{S})\boldsymbol{\varepsilon}$
$C_{3\nu}$	$\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{A}^2$, $\mathbf{p}^{\mathbf{A}}\cdot\mathbf{Ap}^{\mathbf{A}}$, $\operatorname{tr}\mathbf{AB}$, $\mathbf{p}^{\mathbf{A}}\cdot\mathbf{Bp}^{\mathbf{A}}$, $\operatorname{tr}\mathbf{W}^2$, $\mathbf{p}^{\mathbf{A}}\cdot\mathbf{AWp}^{\mathbf{A}}$, $\operatorname{tr}\mathbf{ABW}$, $\operatorname{tr}\mathbf{WV}$, $\mathbf{v}\cdot\mathbf{v}$, $\mathbf{v}\cdot\mathbf{p}^{\mathbf{v}}$, $\mathbf{v}\cdot\mathbf{Av}$, $\mathbf{v}\cdot\mathbf{p}^{\mathbf{A}}$, $\mathbf{v}\cdot\mathbf{Wp}^{\mathbf{v}}$, $\mathbf{v}\cdot\mathbf{AWv}$, $\mathbf{v}\cdot\mathbf{u}$, $\mathbf{u}\cdot\mathbf{p}^{\mathbf{v}}$, $\mathbf{v}\cdot\mathbf{Wu}$; $\mathbf{T}\vdots\mathbf{T}$, $\mathbf{P}\vdots\mathbf{T}$, $\mathbf{p}^{\mathbf{A}}\cdot\mathbf{At}^{\mathbf{A}}$, $\operatorname{tr}(\mathbf{P}:\mathbf{T})\mathbf{W}$, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{v}}$, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{A}}$, $\mathbf{v}\cdot(\mathbf{P}:\mathbf{T})\mathbf{u}$, $\operatorname{tr}(\mathbf{P}:\mathbf{T})\mathbf{AB}$, $\mathbf{T}\vdots\mathbf{S}$	C_3	$\operatorname{tr}\mathbf{A}$, $\mathbf{p}^{\mathbf{A}}\cdot\mathbf{Ap}^{\mathbf{A}}$, $\mathbf{p}^{\mathbf{A}}\cdot\mathbf{A}\boldsymbol{\varepsilon}\mathbf{p}^{\mathbf{A}}$, $\operatorname{tr}\mathbf{AB}$, $\operatorname{tr}\mathbf{AB}\boldsymbol{\varepsilon}$, $\operatorname{tr}\boldsymbol{\varepsilon}\mathbf{W}$, $\mathbf{v}\cdot\mathbf{p}^{\mathbf{v}}$, $\mathbf{v}\cdot\boldsymbol{\varepsilon}\mathbf{p}^{\mathbf{v}}$, $\mathbf{v}\cdot\mathbf{Av}$, $\mathbf{v}\cdot\mathbf{A}\boldsymbol{\varepsilon}\mathbf{v}$, $\mathbf{v}\cdot\mathbf{u}$, $\mathbf{v}\cdot\boldsymbol{\varepsilon}\mathbf{u}$; $\mathbf{P}\vdots\mathbf{T}$, $\operatorname{tr}(\mathbf{P}:\mathbf{T})\boldsymbol{\varepsilon}$
$C_{2\nu}$	$\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{A}^2$, $\operatorname{tr}\mathbf{MA}$, $\operatorname{tr}\mathbf{AB}$, $\operatorname{tr}\mathbf{W}^2$, $\operatorname{tr}\mathbf{MAW}$, $\operatorname{tr}\mathbf{WV}$, $\mathbf{v}\cdot\mathbf{v}$, $\mathbf{v}\cdot\mathbf{Mv}$, $\mathbf{v}\cdot\mathbf{Av}$, $\mathbf{v}\cdot\mathbf{MWv}$, $\mathbf{v}\cdot\mathbf{u}$, $\mathbf{v}\cdot\mathbf{Mu}$, $\mathbf{v}\cdot\mathbf{Au}$, $\mathbf{v}\cdot\mathbf{Wu}$; $\mathbf{T}\vdots\mathbf{T}$, $\mathbf{t}^{\mathbf{M}}\cdot\mathbf{Mt}^{\mathbf{M}}$, $\mathbf{t}^{\mathbf{M}}\cdot\mathbf{At}^{\mathbf{M}}$, $\mathbf{t}^{\mathbf{M}}\cdot\mathbf{MWt}^{\mathbf{M}}$, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{v}}$, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{M}}$, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{A}}$, $\mathbf{v}\cdot\mathbf{Wt}^{\mathbf{M}}$, $\mathbf{T}\vdots\mathbf{S}$, $\mathbf{t}^{\mathbf{M}}\cdot\mathbf{Ms}^{\mathbf{M}}$, $\operatorname{tr}(\mathbf{T}:\mathbf{S})\mathbf{MA}$, $\operatorname{tr}(\mathbf{T}:\mathbf{S})\mathbf{W}$	C_2	$\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{MA}$, $\operatorname{tr}\mathbf{MA}\boldsymbol{\varepsilon}$, $\operatorname{tr}\boldsymbol{\varepsilon}\mathbf{W}$, $\mathbf{v}\cdot\mathbf{Mv}$, $\mathbf{v}\cdot\mathbf{M}\boldsymbol{\varepsilon}\mathbf{v}$, $\mathbf{v}\cdot\mathbf{u}$, $\mathbf{v}\cdot\boldsymbol{\varepsilon}\mathbf{u}$; $\mathbf{T}\vdots\mathbf{T}$, $\mathbf{t}^{\mathbf{M}}\cdot\mathbf{Mt}^{\mathbf{M}}$, $\mathbf{t}^{\mathbf{M}}\cdot\mathbf{M}\boldsymbol{\varepsilon}\mathbf{t}^{\mathbf{M}}$, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{M}}$, $\mathbf{v}\cdot\boldsymbol{\varepsilon}\mathbf{t}^{\mathbf{M}}$, $\mathbf{T}\vdots\mathbf{S}$, $\operatorname{tr}(\mathbf{T}:\mathbf{S})\boldsymbol{\varepsilon}$
$C_{1\nu}$	$\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{A}^2$, $\mathbf{a}\cdot\mathbf{Aa}$, $\operatorname{tr}\mathbf{AB}$, $\operatorname{tr}\mathbf{W}^2$, $\mathbf{a}\cdot\mathbf{AWa}$, $\operatorname{tr}\mathbf{WV}$, $\mathbf{v}\cdot\mathbf{v}$, $\mathbf{a}\cdot\mathbf{v}$, $\mathbf{a}\cdot\mathbf{Av}$, $\mathbf{a}\cdot\mathbf{Wv}$, $\mathbf{v}\cdot\mathbf{u}$; $\mathbf{T}\vdots\mathbf{T}$, $\mathbf{a}\cdot\mathbf{t}^{\mathbf{a}}$, $\mathbf{a}\cdot\mathbf{t}^{\mathbf{A}}$, $\mathbf{a}\cdot\mathbf{Wt}^{\mathbf{a}}$, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{a}}$, $\mathbf{T}\vdots\mathbf{S}$	C_1	$\mathbf{a}\cdot\mathbf{Aa}$, $\mathbf{b}\cdot\mathbf{Ab}$, $\mathbf{a}\cdot\mathbf{Ab}$, $\operatorname{tr}\boldsymbol{\varepsilon}\mathbf{W}$, $\mathbf{a}\cdot\mathbf{v}$, $\mathbf{b}\cdot\mathbf{v}$; $\mathbf{a}\cdot\mathbf{t}^{\mathbf{a}}$, $\mathbf{b}\cdot\mathbf{t}^{\mathbf{b}}$

In Tables 2 and 3, the following abbreviations are employed:

$$\begin{array}{llll} \mathbf{A} = \mathbf{A}_\alpha\,, & \mathbf{B} = \mathbf{A}_\beta\,, & \mathbf{W} = \mathbf{W}_\xi\,, & \mathbf{V} = \mathbf{W}_\zeta\,, \\ \mathbf{v} = \mathbf{v}_\varrho\,, & \mathbf{u} = \mathbf{v}_\sigma\,, & \mathbf{T} = \mathbf{T}_\lambda\,, & \mathbf{S} = \mathbf{T}_\mu\,, \end{array} \tag{3.1}$$

with $\alpha, \beta = 1, \ldots, N$ and $\alpha < \beta$; $\xi, \zeta = 1, \ldots, M$ and $\xi < \zeta$; $\varrho, \sigma = 1, \ldots, P$ and $\varrho < \sigma$; and $\lambda, \mu = 1, \ldots, L$ and $\lambda < \mu$.

An explanation of the redundancy of one of $\mathbf{u}\cdot\mathbf{t}^{\mathbf{v}}$ and $\mathbf{v}\cdot\mathbf{t}^{\mathbf{u}}$ may be required. Without loss of generality, we set $\mathbf{T} = \mathbf{e}_{111} - \langle\mathbf{e}_{122}\rangle$. Denote by $(\mathbf{v}, \mathbf{u})$ and $(\overline{\mathbf{v}}, \overline{\mathbf{u}})$ two solutions of the equations

$$\mathbf{v}\cdot\mathbf{v},\ \mathbf{u}\cdot\mathbf{u},\ \mathbf{v}\cdot\mathbf{t}^{\mathbf{v}},\ \mathbf{u}\cdot\mathbf{t}^{\mathbf{u}},\ \mathbf{v}\cdot\mathbf{u},\ \mathbf{u}\cdot\mathbf{t}^{\mathbf{v}} = \text{const}. \tag{3.2}$$

Because of $\mathbf{v}\cdot\mathbf{v} = \overline{\mathbf{v}}\cdot\overline{\mathbf{v}}$ and $\mathbf{u}\cdot\mathbf{u} = \overline{\mathbf{u}}\cdot\overline{\mathbf{u}}$, we may assume that $\mathbf{v} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$, $\mathbf{u} = \cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2$, $\overline{\mathbf{v}} = \cos\overline{\theta}\mathbf{e}_1 + \sin\overline{\theta}\mathbf{e}_2$ and $\overline{\mathbf{u}} = \cos\overline{\varphi}\mathbf{e}_1 + \sin\overline{\varphi}\mathbf{e}_2$. The equations $\mathbf{v}\cdot\mathbf{t}^{\mathbf{v}}$, $\mathbf{u}\cdot\mathbf{t}^{\mathbf{u}}$, $\mathbf{v}\cdot\mathbf{u}$, $\mathbf{u}\cdot\mathbf{t}^{\mathbf{v}}$ = constans yield immediately.

$$\begin{array}{ll} \cos 3\theta = \cos 3\overline{\theta}, & \cos 3\varphi = \cos 3\overline{\varphi}, \\ \cos(\theta - \varphi) = \cos(\overline{\theta} - \overline{\varphi}), & \cos(2\theta + \varphi) = \cos(2\overline{\theta} + \overline{\varphi}). \end{array} \tag{3.3}$$

It follows that $\cos(\theta + 2\varphi) = \cos(\overline{\theta} + 2\overline{\varphi})$, i.e., $\mathbf{v}\cdot\mathbf{t}^{\mathbf{u}} = \overline{\mathbf{v}}\cdot\mathbf{t}^{\mathbf{u}}$. Therefore, $\mathbf{v}\cdot\mathbf{t}^{\mathbf{u}}$ is redundant.

In a similar manner we can verify the redundancy of one of $\mathbf{t}^{\mathbf{A}}\cdot\mathbf{B}\,\mathbf{t}^{\mathbf{A}}$ and $\mathbf{t}^{\mathbf{B}}\cdot\mathbf{A}\,\mathbf{t}^{\mathbf{B}}$, and one of $\langle\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{u}\rangle$ and $\langle\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{v}\rangle$.

Table 3. Complete and irreducible tensor-valued function representations.

	vector-valued		
$C_{\infty\nu}$	$\mathbf{v}, \mathbf{A}\mathbf{v}, \mathbf{W}\mathbf{v};\ \mathbf{t}^{\mathbf{A}}, \mathbf{A}\mathbf{t}^{\mathbf{A}}, \mathbf{W}\mathbf{t}^{\mathbf{A}}, \mathbf{t}^{\mathbf{v}}$	C_{∞}	$\mathbf{v}, \boldsymbol{\varepsilon}\mathbf{v};\ \mathbf{t}^{\mathbf{A}}, \mathbf{A}\mathbf{t}^{\mathbf{A}}$
$C_{3\nu}$	$\mathbf{p}^{\mathbf{A}}, \mathbf{A}\mathbf{p}^{\mathbf{A}}, \mathbf{W}\mathbf{p}^{\mathbf{A}}, \mathbf{v}, \mathbf{W}\mathbf{v}, \mathbf{p}^{\mathbf{v}};\ \mathbf{t}^{\mathbf{v}}$	C_3	$\mathbf{v}, \boldsymbol{\varepsilon}\mathbf{v};\ \mathbf{p}^{\mathbf{A}}, \boldsymbol{\varepsilon}\mathbf{p}^{\mathbf{A}}$
$C_{2\nu}$	$\mathbf{v}, \mathbf{M}\mathbf{v}, \mathbf{A}\mathbf{v}, \mathbf{W}\mathbf{v};\ \mathbf{t}^{\mathbf{M}}, \mathbf{M}\mathbf{t}^{\mathbf{M}}, \mathbf{t}^{\mathbf{A}}, \mathbf{W}\mathbf{t}^{\mathbf{M}}$	C_2	$\mathbf{v}, \boldsymbol{\varepsilon}\mathbf{v};\ \mathbf{t}^{\mathbf{M}}, \boldsymbol{\varepsilon}\mathbf{t}^{\mathbf{M}}$
$C_{1\nu}$	$\mathbf{a}, \mathbf{A}\mathbf{a}, \mathbf{W}\mathbf{a}, \mathbf{v};\ \mathbf{t}^{\mathbf{a}}$	C_1	$\mathbf{a}, \mathbf{b}$
	second-order symmetric tensor-valued		
$C_{\infty\nu}$	$\mathbf{1}, \mathbf{A}, \mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A}, \mathbf{v}\otimes\mathbf{v}, \mathbf{v}\otimes\mathbf{W}\mathbf{v} + \mathbf{W}\mathbf{v}\otimes\mathbf{v},$ $\mathbf{v}\otimes\mathbf{u} + \mathbf{u}\otimes\mathbf{v};\ \mathbf{t}^{\mathbf{A}}\otimes\mathbf{t}^{\mathbf{A}}, \mathbf{A}(\mathbf{T}:\mathbf{S}) - (\mathbf{T}:\mathbf{S})\mathbf{A}$	C_{∞}	$\mathbf{1}, \mathbf{A}, \mathbf{A}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\mathbf{A}, \mathbf{v}\otimes\mathbf{v},$ $\mathbf{v}\otimes\boldsymbol{\varepsilon}\mathbf{v} + \boldsymbol{\varepsilon}\mathbf{v}\otimes\mathbf{v}$
$C_{3\nu}$	$\mathbf{1}, \mathbf{A}, \mathbf{p}^{\mathbf{A}}\otimes\mathbf{p}^{\mathbf{A}}, \mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A}, \mathbf{v}\otimes\mathbf{v}, \mathbf{P}^{\mathbf{v}},$ $\mathbf{v}\otimes\mathbf{W}\mathbf{v} + \mathbf{W}\mathbf{v}\otimes\mathbf{v};\ \mathbf{T}^{\mathbf{v}}, \mathbf{A}(\mathbf{P}:\mathbf{T}) - (\mathbf{P}:\mathbf{T})\mathbf{A}$	C_3	$\mathbf{1}, \mathbf{A}, \mathbf{A}\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\mathbf{A}, \mathbf{v}\otimes\mathbf{v},$ $\mathbf{v}\otimes\boldsymbol{\varepsilon}\mathbf{v} + \boldsymbol{\varepsilon}\mathbf{v}\otimes\mathbf{v}$
$C_{2\nu}$	$\mathbf{1}, \mathbf{M}, \mathbf{A}, \mathbf{M}\mathbf{W} - \mathbf{W}\mathbf{M}, \mathbf{v}\otimes\mathbf{v}, \mathbf{v}\otimes\mathbf{u} + \mathbf{u}\otimes\mathbf{v};$ $\mathbf{T}^{\mathbf{v}}, \mathbf{t}^{\mathbf{M}}\otimes\mathbf{t}^{\mathbf{M}}, \mathbf{M}(\mathbf{T}:\mathbf{S}) - (\mathbf{T}:\mathbf{S})\mathbf{M}$	C_2	$\mathbf{1}, \mathbf{M}, \mathbf{M}\boldsymbol{\varepsilon}$
$C_{1\nu}$	$\mathbf{1}, \mathbf{a}\otimes\mathbf{a}, \mathbf{A}, \mathbf{a}\otimes\mathbf{W}\mathbf{a} + \mathbf{W}\mathbf{a}\otimes\mathbf{a}, \mathbf{a}\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{a};\ \mathbf{T}^{\mathbf{a}}$	C_1	$\mathbf{a}\otimes\mathbf{a}, \mathbf{b}\otimes\mathbf{b}, \mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a}$
	second-order skew-symmeric tensor-valued		
$C_{\infty\nu}$	$\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}, \mathbf{W}, \mathbf{v}\otimes\mathbf{A}\mathbf{v} - \mathbf{A}\mathbf{v}\otimes\mathbf{v}, \mathbf{v}\otimes\mathbf{u} - \mathbf{u}\otimes\mathbf{v};$ $\mathbf{t}^{\mathbf{A}}\otimes\mathbf{A}\mathbf{t}^{\mathbf{A}} - \mathbf{A}\mathbf{t}^{\mathbf{A}}\otimes\mathbf{t}^{\mathbf{A}}, \mathbf{v}\otimes\mathbf{t}^{\mathbf{v}} - \mathbf{t}^{\mathbf{v}}\otimes\mathbf{v}, \mathbf{T}:\mathbf{S} - \mathbf{S}:\mathbf{T}$	C_{∞}	$\boldsymbol{\varepsilon}$
$C_{3\nu}$	$\mathbf{p}^{\mathbf{A}}\otimes\mathbf{A}\mathbf{p}^{\mathbf{A}} - \mathbf{A}\mathbf{p}^{\mathbf{A}}\otimes\mathbf{p}^{\mathbf{A}}, \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}, \mathbf{W}, \mathbf{v}\otimes\mathbf{p}^{\mathbf{v}} - \mathbf{p}^{\mathbf{v}}\otimes\mathbf{v},$ $\mathbf{v}\otimes\mathbf{A}\mathbf{v} - \mathbf{A}\mathbf{v}\otimes\mathbf{v}, \mathbf{v}\otimes\mathbf{u} - \mathbf{u}\otimes\mathbf{v};\ \mathbf{P}:\mathbf{T} - \mathbf{T}:\mathbf{P}$	C_3	$\boldsymbol{\varepsilon}$
$C_{2\nu}$	$\mathbf{W}, \mathbf{M}\mathbf{A} - \mathbf{A}\mathbf{M}, \mathbf{v}\otimes\mathbf{M}\mathbf{v} - \mathbf{M}\mathbf{v}\otimes\mathbf{v}, \mathbf{v}\otimes\mathbf{u} - \mathbf{u}\otimes\mathbf{v};$ $\mathbf{v}\otimes\mathbf{t}^{\mathbf{M}} - \mathbf{t}^{\mathbf{M}}\otimes\mathbf{v}, \mathbf{t}^{\mathbf{v}}\otimes\mathbf{M}\mathbf{t}^{\mathbf{v}} - \mathbf{M}\mathbf{t}^{\mathbf{v}}\otimes\mathbf{t}^{\mathbf{v}}, \mathbf{T}:\mathbf{S} - \mathbf{S}:\mathbf{T}$	C_2	$\boldsymbol{\varepsilon}$
$C_{1\nu}$	$\mathbf{a}\otimes\mathbf{A}\mathbf{a} - \mathbf{A}\mathbf{a}\otimes\mathbf{a}, \mathbf{W}, \mathbf{a}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{a};\ \mathbf{a}\otimes\mathbf{t}^{\mathbf{a}} - \mathbf{t}^{\mathbf{a}}\otimes\mathbf{a}$	C_1	$\boldsymbol{\varepsilon}$
	third-order tensor-valued		
$C_{\infty\nu}$	$\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}, \{\mathbf{v}\otimes\mathbf{1}\}, \langle\mathbf{v}\otimes\mathbf{A}\rangle, \{\mathbf{A}\mathbf{v}\otimes\mathbf{1}\}, \langle\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{W}\mathbf{v}\rangle,$ $\{\mathbf{W}\mathbf{v}\otimes\mathbf{1}\}, \langle\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{u}\rangle;\ \mathbf{T}, \langle\mathbf{A}\mathbf{t}^{\mathbf{A}}\otimes\mathbf{A}\rangle, \{\mathbf{t}^{\mathbf{A}}\otimes\mathbf{1}\},$ $\{\mathbf{A}\mathbf{t}^{\mathbf{A}}\otimes\mathbf{1}\}, \mathbf{T}(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}), \mathbf{T}\mathbf{W}, \{\mathbf{W}\mathbf{t}^{\mathbf{A}}\otimes\mathbf{1}\}, \{\mathbf{t}^{\mathbf{v}}\otimes\mathbf{1}\}$	C_{∞}	$\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}, \langle\mathbf{v}\otimes\mathbf{v}\otimes\boldsymbol{\varepsilon}\mathbf{v}\rangle,$ $\{\mathbf{v}\otimes\mathbf{1}\}, \{\boldsymbol{\varepsilon}\mathbf{v}\otimes\mathbf{1}\};\ \mathbf{T}, \mathbf{T}\boldsymbol{\varepsilon},$ $\{\mathbf{t}^{\mathbf{A}}\otimes\mathbf{1}\}, \{\boldsymbol{\varepsilon}\mathbf{t}^{\mathbf{A}}\otimes\mathbf{1}\}$
$C_{3\nu}$	$\mathbf{P}, \langle\mathbf{A}\mathbf{p}^{\mathbf{A}}\otimes\mathbf{A}\rangle, \{\mathbf{p}^{\mathbf{A}}\otimes\mathbf{1}\}, \{\mathbf{A}\mathbf{p}^{\mathbf{A}}\otimes\mathbf{1}\}, \mathbf{P}(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}),$ $\mathbf{P}\mathbf{W}, \{\mathbf{W}\mathbf{p}^{\mathbf{A}}\otimes\mathbf{1}\}, \mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}, \{\mathbf{v}\otimes\mathbf{1}\}, \{\mathbf{p}^{\mathbf{v}}\otimes\mathbf{1}\}, \langle\mathbf{v}\otimes\mathbf{A}\rangle,$ $\{\mathbf{W}\mathbf{v}\otimes\mathbf{1}\}, \mathbf{P}(\mathbf{v}\otimes\mathbf{u} - \mathbf{u}\otimes\mathbf{v}), \mathbf{T}, \{\mathbf{t}^{\mathbf{v}}\otimes\mathbf{1}\}, \{\mathbf{t}^{\mathbf{A}}\otimes\mathbf{1}\}$	C_3	$\mathbf{P}, \mathbf{P}\boldsymbol{\varepsilon}, \{\mathbf{p}^{\mathbf{A}}\otimes\mathbf{1}\},$ $\{\boldsymbol{\varepsilon}\mathbf{p}^{\mathbf{A}}\otimes\mathbf{1}\}, \{\mathbf{v}\otimes\mathbf{1}\},$ $\{\boldsymbol{\varepsilon}\mathbf{v}\otimes\mathbf{1}\}$
$C_{2\nu}$	$\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v}, \langle\mathbf{v}\otimes\mathbf{M}\rangle, \{\mathbf{v}\otimes\mathbf{1}\}, \{\mathbf{M}\mathbf{v}\otimes\mathbf{1}\}, \langle\mathbf{v}\otimes\mathbf{A}\rangle,$ $\{\mathbf{A}\mathbf{v}\otimes\mathbf{1}\}, \langle\mathbf{W}\mathbf{v}\otimes\mathbf{M}\rangle, \{\mathbf{W}\mathbf{v}\otimes\mathbf{1}\};\ \mathbf{T}, \langle\mathbf{M}\mathbf{t}^{\mathbf{M}}\otimes\mathbf{M}\rangle,$ $\{\mathbf{t}^{\mathbf{M}}\otimes\mathbf{1}\}, \{\mathbf{M}\mathbf{t}^{\mathbf{M}}\otimes\mathbf{1}\}, \mathbf{T}\mathbf{W}, \mathbf{T}(\mathbf{M}\mathbf{A} - \mathbf{A}\mathbf{M}),$ $\{\mathbf{W}\mathbf{t}^{\mathbf{M}}\otimes\mathbf{1}\}, \{\mathbf{t}^{\mathbf{A}}\otimes\mathbf{1}\}$	C_2	$\langle\mathbf{v}\otimes\mathbf{M}\rangle, \langle\boldsymbol{\varepsilon}\mathbf{v}\otimes\mathbf{M}\rangle, \{\mathbf{v}\otimes\mathbf{1}\},$ $\{\boldsymbol{\varepsilon}\mathbf{v}\otimes\mathbf{1}\};\ \mathbf{T}, \mathbf{T}\boldsymbol{\varepsilon}, \{\mathbf{t}^{\mathbf{M}}\otimes\mathbf{1}\},$ $\{\boldsymbol{\varepsilon}\mathbf{t}^{\mathbf{M}}\otimes\mathbf{1}\}$
$C_{1\nu}$	$\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}, \{\mathbf{a}\otimes\mathbf{1}\}, \langle\mathbf{a}\otimes\mathbf{A}\rangle, \{\mathbf{A}\mathbf{a}\otimes\mathbf{1}\}, \langle\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{W}\mathbf{a}\rangle,$ $\{\mathbf{W}\mathbf{a}\otimes\mathbf{1}\}, \langle\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{v}\rangle, \{\mathbf{v}\otimes\mathbf{1}\}; \mathbf{T}, \{\mathbf{t}^{\mathbf{a}}\otimes\mathbf{1}\}$	C_1	$\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}, \{\mathbf{a}\otimes\mathbf{1}\},$ $\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b}, \{\mathbf{b}\otimes\mathbf{1}\}$

3.2. Representations for hemitropy $C_\infty(\boldsymbol{\varepsilon})$

From the fact that $\boldsymbol{\varepsilon}$ characterizes the group C_∞ it follows that the hemitropic functions of **A**, **W**, **v** and **T** may be considered as isotropic functions of **A**, **W**, **v**, **T** and $\boldsymbol{\varepsilon}$. Noting that $\boldsymbol{\varepsilon}$ is a non-zero second-order skew-symmetric tensor, from (2.5), (2.6), (2.10) – (2.12), (2.17) and (2.18), by substituting $\boldsymbol{\varepsilon}$ for **W** in (2.6), (2.12) and (2.18), we obtain complete representations for hemitropic tensor functions of $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$ and $\mathbf{T}_\lambda$, as shown in Tables 2 and 3.

3.3. Representations for $C_{3\nu}(\mathbf{P})$

To determine the complete isotropic tensor function representations of $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$, $\mathbf{T}_\lambda$ and **P**, we consider the following cases instead of the cases in Sec. 2.1. First suppose that $P_{211} = 0$. We have the following invariants and form-invariants instead of those in (2.5) – (2.9), respectively.

$$
\begin{aligned}
&\mathbf{v}\cdot\mathbf{v},\ \mathbf{v}\cdot\mathbf{v}_\varrho,\ \mathbf{v}\cdot\mathbf{A}_\alpha\mathbf{v},\ \operatorname{tr}\mathbf{A}_\alpha,\ \mathbf{v}\cdot\mathbf{p}^{\mathbf{v}},\ \mathbf{P}\vdots\mathbf{T}_\lambda;\ \ \mathbf{v};\ \ \mathbf{v}\otimes\mathbf{v},\ \mathbf{1};\ \ \mathbf{P},\ \{\mathbf{v}\otimes\mathbf{1}\},\\
&\operatorname{tr}\mathbf{W}^2,\ \mathbf{v}\cdot\mathbf{W}\mathbf{v}_\varrho,\ \operatorname{tr}\mathbf{W}\mathbf{W}_\xi,\ \mathbf{v}\cdot\mathbf{A}_\alpha\mathbf{W}\mathbf{v},\ \operatorname{tr}(\mathbf{P}:\mathbf{T}_\lambda)\mathbf{W};\ \ \mathbf{W}\mathbf{v};\\
&\qquad\qquad\mathbf{v}\otimes\mathbf{W}\mathbf{v}+\mathbf{W}\mathbf{v}\otimes\mathbf{v};\ \ \mathbf{W};\ \ \mathbf{P}\mathbf{W},\ \{\mathbf{W}\mathbf{v}\otimes\mathbf{1}\},\\
&\mathbf{u}\cdot\mathbf{u},\ \mathbf{u}\cdot\mathbf{v}_\varrho,\ \mathbf{u}\cdot\mathbf{p}^{\mathbf{A}_\alpha},\ \mathbf{v}\cdot\mathbf{W}_\xi\mathbf{u},\ \mathbf{v}\cdot(\mathbf{P}:\mathbf{T}_\lambda)\mathbf{u};\ \ \mathbf{u};\\
&\qquad\qquad\mathbf{P}^{\mathbf{u}},\ \mathbf{v}\otimes\mathbf{u}-\mathbf{u}\otimes\mathbf{v},\ \mathbf{P}(\mathbf{v}\otimes\mathbf{u}-\mathbf{u}\otimes\mathbf{v}),\ \{\mathbf{u}\otimes\mathbf{1}\},\\
&\operatorname{tr}\mathbf{A}^2,\ \operatorname{tr}\mathbf{A}\mathbf{A}_\alpha,\ \mathbf{v}_\varrho\cdot\mathbf{p}^{\mathbf{A}},\ \mathbf{v}\cdot\mathbf{A}\mathbf{W}_\xi\mathbf{v},\ \mathbf{v}\cdot\mathbf{t}^{\mathbf{A}}{}_\lambda;\ \ \mathbf{p}^{\mathbf{A}};\\
&\qquad\qquad\mathbf{A};\ \ \mathbf{v}\otimes\mathbf{A}\mathbf{v}-\mathbf{A}\mathbf{v}\otimes\mathbf{v};\ \ \langle\mathbf{v}\otimes\mathbf{A}\rangle,\ \{\mathbf{p}^{\mathbf{A}}\otimes\mathbf{1}\},\\
&\mathbf{T}\vdots\mathbf{T},\ \mathbf{v}_\varrho\cdot(\mathbf{P}:\mathbf{T})\mathbf{v},\ \mathbf{v}\cdot\mathbf{t}^{\mathbf{A}_\alpha},\ \operatorname{tr}(\mathbf{P}:\mathbf{T})\mathbf{W}_\xi,\ \mathbf{T}\vdots\mathbf{T}_\lambda;\\
&\qquad\qquad\mathbf{t}^{\mathbf{v}};\ \ \mathbf{T}^{\mathbf{v}};\ \ \mathbf{P}:\mathbf{T}-\mathbf{T}:\mathbf{P};\ \ \mathbf{T},\ \{\mathbf{t}^{\mathbf{v}}\otimes\mathbf{1}\}.
\end{aligned}
\tag{3.4}
$$

Second, suppose that $P_{211} \neq 0$. Instead of (2.5) and (2.9), respectively, we have

$$
\begin{aligned}
&\mathbf{v}\cdot\mathbf{v},\ \mathbf{v}\cdot\mathbf{v}_\varrho,\ \mathbf{v}\cdot\mathbf{A}_\alpha\mathbf{v},\ \operatorname{tr}\mathbf{A}_\alpha,\ \mathbf{v}\cdot\mathbf{p}^{\mathbf{v}},\ \mathbf{v}\cdot\mathbf{t}^{\mathbf{v}}_\lambda;\ \ \mathbf{v};\ \ \mathbf{v}\otimes\mathbf{v},\ \mathbf{1};\ \ \mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v},\ \{\mathbf{v}\otimes\mathbf{1}\},\\
&\mathbf{v}_\varrho\cdot\mathbf{p}^{\mathbf{v}},\ \mathbf{v}\cdot\mathbf{p}^{\mathbf{A}_\alpha},\ \mathbf{v}\cdot\mathbf{W}_\xi\mathbf{p}^{\mathbf{v}},\ \mathbf{P}\vdots\mathbf{T}_\lambda;\ \ \mathbf{p}^{\mathbf{v}};\ \ \mathbf{P}^{\mathbf{v}};\ \ \mathbf{v}\otimes\mathbf{p}^{\mathbf{v}}-\mathbf{p}^{\mathbf{v}}\otimes\mathbf{v};\ \ \mathbf{P},\ \{\mathbf{p}^{\mathbf{v}}\otimes\mathbf{1}\}.
\end{aligned}
\tag{3.5}
$$

Finally, from (3.4), (3.5) as well as (2.10) – (2.15), replacing **T** by **P**, we can obtain the complete representations for tensor functions of $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$ and $\mathbf{T}_\lambda$ under $C_{3\nu}$, as shown in Tables 2 and 3.

3.4. Representations for $C_3(\mathbf{P}, \boldsymbol{\varepsilon})$

If there exists a non-zero vector **v** among $\mathbf{v}_\varrho$, we can write, instead of (2.5) and (2.6), the equations

$$
\begin{aligned}
&\mathbf{v}\cdot\mathbf{v}_\varrho,\ \mathbf{v}\cdot\boldsymbol{\varepsilon}\mathbf{v}_\varrho,\ \mathbf{v}\cdot\mathbf{p}^{\mathbf{v}},\ \mathbf{v}\cdot\boldsymbol{\varepsilon}\mathbf{p}^{\mathbf{v}},\ \operatorname{tr}\mathbf{A}_\alpha,\ \mathbf{v}\cdot\mathbf{A}_\alpha\mathbf{v},\ \mathbf{v}\cdot\mathbf{A}_\alpha\boldsymbol{\varepsilon}\mathbf{v},\ \operatorname{tr}\boldsymbol{\varepsilon}\mathbf{W}_\xi,\ \mathbf{P}\vdots\mathbf{T}_\lambda,\\
&\operatorname{tr}(\mathbf{P}:\mathbf{T}_\lambda)\boldsymbol{\varepsilon};\ \mathbf{v},\ \boldsymbol{\varepsilon}\mathbf{v};\ \mathbf{1},\ \mathbf{v}\otimes\mathbf{v},\ \mathbf{v}\otimes\boldsymbol{\varepsilon}\mathbf{v}+\boldsymbol{\varepsilon}\mathbf{v}\otimes\mathbf{v};\ \boldsymbol{\varepsilon};\ \mathbf{P},\ \mathbf{P}\boldsymbol{\varepsilon},\ \{\mathbf{v}\otimes\mathbf{1}\},\ \{\boldsymbol{\varepsilon}\mathbf{v}\otimes\mathbf{1}\},
\end{aligned}
\tag{3.6}
$$

where the obviously redundant invariant $\mathbf{v}\cdot\mathbf{v}$ has been removed because of the identity $(\mathbf{v}\cdot\mathbf{v})^3 = (\mathbf{v}\cdot\mathbf{p}^{\mathrm{v}})^2 + (\mathbf{v}\cdot\mathbf{p}^{\mathrm{v}})^2$ according to (2.2). Setting $\mathbf{T} = \mathbf{P}$ and $\mathbf{W} = \boldsymbol{\varepsilon}$ in (2.10)–(2.12), (2.17) and (2.18) together with (3.6), we arrive at the complete representations under C_3, as shown in Tables 2 and 3.

3.5. Representations for $C_{2\nu}(\mathbf{M})$

To determine the complete isotropic tensor functions of $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$, $\mathbf{T}_\lambda$ and $\mathbf{M}$, we consider the following cases instead of the cases in Sec. 2.1. First, suppose that $M_{12} = 0$. We have the following invariants and form-invariants instead of those in (2.5)–(2.9), respectively.

$$\begin{aligned}
&\mathbf{v}\cdot\mathbf{v},\ \mathbf{v}\cdot\mathbf{v}_\varrho,\ \mathbf{v}\cdot\mathbf{M}\mathbf{v},\ \mathrm{tr}\,\mathbf{M}\mathbf{A}_\alpha,\ \mathrm{tr}\,\mathbf{A}_\alpha,\ \mathbf{v}\cdot\mathbf{t}^{\mathbf{M}}{}_\lambda;\ \ \mathbf{v};\ \ \mathbf{M},\ \mathbf{1};\ \ \langle\mathbf{v}\otimes\mathbf{M}\rangle,\ \{\mathbf{v}\otimes\mathbf{1}\},\\
&\mathrm{tr}\,\mathbf{W}^2,\ \mathbf{v}\cdot\mathbf{W}\mathbf{v}_\varrho,\ \mathrm{tr}\,\mathbf{W}\mathbf{W}_\xi,\ \mathrm{tr}\,\mathbf{M}\mathbf{A}_\alpha\mathbf{W},\ \mathbf{v}\cdot\mathbf{W}\mathbf{t}^{\mathbf{M}}{}_\lambda;\ \ \mathbf{W}\mathbf{v};\\
&\qquad\qquad \mathbf{M}\mathbf{W}-\mathbf{W}\mathbf{M};\ \ \mathbf{W};\ \ \langle\mathbf{W}\mathbf{v}\otimes\mathbf{M}\rangle,\ \{\mathbf{W}\mathbf{v}\otimes\mathbf{1}\},\\
(3.7)\quad &\mathbf{u}\cdot\mathbf{u},\ \mathbf{u}\cdot\mathbf{v}_\varrho,\ \mathbf{v}\cdot\mathbf{A}_\alpha\mathbf{u},\ \mathbf{v}\cdot\mathbf{W}_\xi\mathbf{u},\ \mathbf{u}\cdot\mathbf{t}^{\mathbf{M}}{}_\lambda;\ \ \mathbf{u};\ \ \mathbf{v}\otimes\mathbf{u}\pm\mathbf{u}\otimes\mathbf{v};\ \ \langle\mathbf{u}\otimes\mathbf{M}\rangle,\ \{\mathbf{u}\otimes\mathbf{1}\},\\
&\mathrm{tr}\,\mathbf{A}^2,\ \mathbf{v}\cdot\mathbf{A}\mathbf{v}_\varrho,\ \mathrm{tr}\,\mathbf{A}\mathbf{A}_\alpha,\ \mathrm{tr}\,\mathbf{M}\mathbf{A}\mathbf{W}_\xi,\ \mathbf{v}\cdot\mathbf{t}^{\mathbf{A}}{}_\lambda;\ \ \mathbf{A}\mathbf{v};\\
&\qquad\qquad \mathbf{A};\ \ \mathbf{M}\mathbf{A}-\mathbf{A}\mathbf{M};\langle\mathbf{v}\otimes\mathbf{A}\rangle,\{\mathbf{A}\mathbf{v}\otimes\mathbf{1}\},\\
&\mathbf{T}\vdots\mathbf{T},\ \mathbf{v}_\varrho\cdot\mathbf{t}^{\mathbf{M}},\ \mathbf{v}\cdot\mathbf{t}_{\mathbf{A}_\alpha},\ \mathbf{v}\cdot\mathbf{W}_\xi\mathbf{t}^{\mathbf{M}},\ \mathbf{T}\vdots\mathbf{T}_\lambda;\ \ \mathbf{t}^{\mathbf{M}};\ \ \mathbf{T}^{\mathrm{v}};\ \ \mathbf{M}\mathbf{T}^{\mathrm{v}}-\mathbf{T}^{\mathrm{v}}\mathbf{M};\ \ \mathbf{T},\ \{\mathbf{t}^{\mathbf{M}}\otimes\mathbf{1}\}.
\end{aligned}$$

Second, suppose that $M_{12} \neq 0$. Instead of (2.5) and (2.7), respectively, we have

$$\begin{aligned}
&\mathbf{v}\cdot\mathbf{v},\ \mathbf{v}\cdot\mathbf{v}_\varrho,\ \mathbf{v}\cdot\mathbf{A}_\alpha\mathbf{v},\ \mathbf{v}\cdot\mathbf{M}\mathbf{v},\ \mathrm{tr}\,\mathbf{A}_\alpha,\ \mathbf{v}\cdot\mathbf{t}^{\mathrm{v}}{}_\lambda;\ \ \mathbf{v};\ \ \mathbf{v}\otimes\mathbf{v},\ \mathbf{1};\ \ \mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v},\ \{\mathbf{v}\otimes\mathbf{1}\},\\
(3.8)\quad &\mathrm{tr}\,\mathbf{M}\mathbf{A}_\alpha,\ \mathbf{v}\cdot\mathbf{M}\mathbf{v}_\varrho,\ \mathbf{v}\cdot\mathbf{M}\mathbf{W}_\xi\mathbf{v},\ \mathbf{v}\cdot\mathbf{t}^{\mathbf{M}}{}_\lambda;\ \ \mathbf{M}\mathbf{v};\ \ \mathbf{M};\\
&\qquad\qquad \mathbf{v}\otimes\mathbf{M}\mathbf{v}-\mathbf{M}\mathbf{v}\otimes\mathbf{v};\ \ \langle\mathbf{v}\otimes\mathbf{M}\rangle,\ \{\mathbf{M}\mathbf{v}\otimes\mathbf{1}\}.
\end{aligned}$$

Finally, from (3.7), (3.8) as well as (2.9)–(2.14), replacing $\mathbf{A}$ by $\mathbf{M}$, we can obtain complete representations for tensor functions of $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$ and $\mathbf{T}_\lambda$ under $C_{2\nu}$, as shown in Tables 2 and 3.

3.6. Representations for $C_2(\mathbf{M}, \boldsymbol{\varepsilon})$

Instead of (2.5) and (2.6) we have

$$\begin{aligned}
(3.9)\quad &\mathbf{v}\cdot\mathbf{M}\mathbf{v},\ \mathbf{v}\cdot\mathbf{M}\boldsymbol{\varepsilon}\mathbf{v},\ \mathbf{v}\cdot\mathbf{v}_\varrho,\ \mathbf{v}\cdot\boldsymbol{\varepsilon}\mathbf{v}_\varrho,\ \mathrm{tr}\,\mathbf{A}_\alpha,\ \mathrm{tr}\,\mathbf{M}\mathbf{A}_\alpha,\ \mathrm{tr}\,\mathbf{M}\mathbf{A}_\alpha\boldsymbol{\varepsilon},\ \mathbf{v}\cdot\mathbf{t}^{\mathbf{M}}{}_\lambda,\\
&\qquad \mathbf{v}\cdot\boldsymbol{\varepsilon}\mathbf{t}^{\mathbf{M}}{}_\lambda,\ \mathrm{tr}\,\boldsymbol{\varepsilon}\mathbf{W}_\xi;\ \ \mathbf{v},\ \boldsymbol{\varepsilon}\mathbf{v};\ \ \mathbf{1},\ \mathbf{M},\ \mathbf{M}\boldsymbol{\varepsilon};\ \ \boldsymbol{\varepsilon};\\
&\qquad \langle\mathbf{v}\otimes\mathbf{M}\rangle,\ \langle\boldsymbol{\varepsilon}\mathbf{v}\otimes\mathbf{M}\rangle,\ \{\mathbf{v}\otimes\mathbf{1}\},\ \{\boldsymbol{\varepsilon}\mathbf{v}\otimes\mathbf{1}\},
\end{aligned}$$

where the redundant invariant $\mathbf{v}\cdot\mathbf{v}$ is removed due to the identity $(\mathbf{v}\cdot\mathbf{v})^2 = (\mathbf{v}\cdot\mathbf{M}\mathbf{v})^2 + (\mathbf{v}\cdot\mathbf{M}\boldsymbol{\varepsilon}\mathbf{v})^2$. Setting $\mathbf{A} = \mathbf{M}$ and $\mathbf{W} = \boldsymbol{\varepsilon}$ in (2.10)–(2.12) together with (3.9), we immediately obtain the complete representations under C_2, as presented in Tables 2 and 3.

3.7. Representations for $C_{1\nu}(\mathbf{a})$

Setting $\mathbf{v} = \mathbf{a}$ in (2.5) – (2.9) yields immediately the complete isotropic tensor function representations of $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$, $\mathbf{T}_\lambda$ and $\mathbf{a}$; namely, the complete tensor function representations of $\mathbf{A}_\alpha$, $\mathbf{W}_\xi$, $\mathbf{v}_\varrho$ and $\mathbf{T}_\lambda$ under $C_{1\nu}$ as given in Tables 2 and 3.

3.8. Representations for $C_1(\mathbf{a}, \boldsymbol{\varepsilon})$

Settting $\mathbf{v} = \mathbf{a}$ and $\mathbf{W} = \boldsymbol{\varepsilon}$ in (2.5) and (2.6) yields immediately the complete representations with respect to C_1, as given in Tables 2 and 3, where $\boldsymbol{\varepsilon}\mathbf{a}$ is replaced by $\mathbf{b}$ because of $\boldsymbol{\varepsilon}\,\mathbf{a} = \pm\mathbf{b}$.

4. Proof of the irreducibility of the derived representations

To verify the irreducibility of the representations established above in Sec. 3, we employ the technique developed by PENNISI and TROVATO [5]. It is obvious that the representations in Tables 2 and 3 with respect to C_1, C_2 and C_3 are irreducible. With respect to $C_{\infty\nu}$, C_∞, $C_{3\nu}$, $C_{2\nu}$ and $C_{1\nu}$, the irreducibility of all invariants, vector form-invariants, and second-order tensor form-invariants in Tables 2 and 3 when $\mathbf{T}_\lambda = \mathbf{0}$ has ben proved by ZHENG [8]; the irreducibility of all additional invariants and form-invariants when non-zero tensors exist among $\mathbf{T}_\lambda$, is verified in Tables 4 and 5, respectively; and the irreducibility of all third-order tensor form invariants is confirmed in Table 6.

Table 4. Irreducibility of the function bases.

variables	invariant	variables	invariant
	$C_{\infty\nu}$		
$\mathbf{T} = \mathbf{0}$ and $\mathbf{P}$	$\mathbf{T} \vdots \mathbf{T}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{v} = \mathbf{u}$, $\mathbf{W} = \pm\boldsymbol{\varepsilon}$	$\mathbf{v} \cdot \mathbf{W}\,\mathbf{t}^\mathbf{v}$
$\mathbf{T} = \mathbf{P}$, $\mathbf{A} = \pm\mathbf{M}$	$\mathbf{t}^\mathbf{A} \cdot \mathbf{A}\,\mathbf{t}^\mathbf{A}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{v} = \mathbf{b}$, $\mathbf{u} = \mathbf{b} \pm \sqrt{3}\mathbf{a}$	$\mathbf{u} \cdot \mathbf{t}^\mathbf{v}$
$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$, $\mathbf{B} = \mathbf{M} \pm \sqrt{3}\mathbf{M}\boldsymbol{\varepsilon}$	$\mathbf{t}^\mathbf{A} \cdot \mathbf{B}\,\mathbf{t}^\mathbf{A}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{S} = \pm\mathbf{P}$	$\mathbf{T} \vdots \mathbf{S}$
$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$, $\mathbf{W} = \pm\boldsymbol{\varepsilon}$	$\mathbf{t}^\mathbf{A} \cdot \mathbf{A}\mathbf{W}\,\mathbf{t}^\mathbf{A}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{S} = \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \pm\mathbf{M}\boldsymbol{\varepsilon}$	$\mathbf{t}^\mathbf{A} \cdot \mathbf{A}\,\mathbf{s}^\mathbf{A}$
$\mathbf{T} = \mathbf{P}$, $\mathbf{v} = \pm\mathbf{a}$	$\mathbf{v} \cdot \mathbf{t}^\mathbf{v}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{S} = \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$, $\mathbf{B} = \mathbf{M} \pm \sqrt{3}\mathbf{M}\boldsymbol{\varepsilon}$	$\mathrm{tr}\,\mathbf{T} : \mathbf{S})\mathbf{A}\mathbf{B}$
$\mathbf{T} = \mathbf{P}$, $\mathbf{A} = \mathbf{M}$, $\mathbf{v} = \pm\mathbf{a}$	$\mathbf{v} \cdot \mathbf{t}^\mathbf{A}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{S} = \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{W} = \pm\boldsymbol{\varepsilon}$	$\mathrm{tr}\,(\mathbf{T} : \mathbf{S})\mathbf{W}$
	C_∞		
$\mathbf{T} = \mathbf{0}$ and $\mathbf{P}$	$\mathbf{T} \vdots \mathbf{T}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{v} = \pm\mathbf{b}$	$\mathbf{v} \cdot \boldsymbol{\varepsilon}\,\mathbf{t}^\mathbf{v}$
$\mathbf{T} = \mathbf{P}$, $\mathbf{A} = \pm\mathbf{M}$	$\mathbf{t}^\mathbf{A} \cdot \mathbf{A}\,\mathbf{t}^\mathbf{A}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{S} = \pm\mathbf{P}$	$\mathbf{T} \vdots \mathbf{S}$
$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$	$\mathbf{t}^\mathbf{A} \cdot \mathbf{A}\boldsymbol{\varepsilon}\,\mathbf{t}^\mathbf{A}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{S} = \pm\mathbf{P}\boldsymbol{\varepsilon}$	$\mathrm{tr}\,(\mathbf{T} : \mathbf{S})\boldsymbol{\varepsilon}$
$\mathbf{T} = \mathbf{P}$, $\mathbf{v} = \pm\mathbf{a}$	$\mathbf{v} \cdot \mathbf{t}^\mathbf{v}$		

Table 4 [cont.]

variables	invariant	variables	invariant
		$C_{3\nu}$	
$\mathbf{T}=0$ and $\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{T}\vdots\mathbf{T}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{v}=\pm\mathbf{b}$	$\mathbf{v}\cdot\mathbf{t}^{\mathbf{v}}$
$\mathbf{T}=\pm\mathbf{P}$	$\mathbf{P}\vdots\mathbf{T}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\mathbf{M}\pm\sqrt{3}\mathbf{M}\boldsymbol{\varepsilon},\ \mathbf{v}=\mathbf{a}$	$\mathbf{v}\cdot\mathbf{t}^{\mathbf{A}}$
$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\pm\mathbf{M}\boldsymbol{\varepsilon},$	$\mathbf{p}^{\mathbf{A}}\cdot\mathbf{A}\,\mathbf{t}^{\mathbf{A}}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{v}=\mathbf{a},\ \mathbf{u}=\mathbf{a}\pm\sqrt{3}\mathbf{b}$	$\mathbf{u}\cdot(\mathbf{P}:\mathbf{T})\mathbf{v}$
$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{W}=\pm\boldsymbol{\varepsilon}$	$\mathrm{tr}\,(\mathbf{P}:\mathbf{T})\mathbf{W}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{S}=\pm\mathbf{P}\boldsymbol{\varepsilon},$	$\mathbf{T}\vdots\mathbf{S}$
$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\mathbf{M},\ \mathbf{B}=\mathbf{M}\pm\sqrt{3}\mathbf{M}\boldsymbol{\varepsilon}$	$\mathrm{tr}\,(\mathbf{P}:\mathbf{T})\mathbf{A}\mathbf{B}$		
		$C_{2\nu}$	
$\mathbf{T}=0$ and $\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{T}\vdots\mathbf{T}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\pm\mathbf{M}\boldsymbol{\varepsilon},\ \mathbf{v}=\mathbf{a}$	$\mathbf{v}\cdot\mathbf{t}^{\mathbf{A}}$
$\mathbf{T}=\mathbf{P}$ and $\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{t}^{\mathbf{M}}\cdot\mathbf{M}\,\mathbf{t}^{\mathbf{M}}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{W}=\pm\boldsymbol{\varepsilon},\ \mathbf{v}=\mathbf{a}$	$\mathbf{v}\cdot\mathbf{W}\,\mathbf{t}^{\mathbf{M}}$
$\mathbf{T}=\mathbf{R}+\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\pm\mathbf{M}$	$\mathbf{t}^{\mathbf{M}}\cdot\mathbf{A}\,\mathbf{t}^{\mathbf{M}}$	$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{S}=\pm(\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon})$	$\mathbf{T}\vdots\mathbf{S}$
$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{W}=\pm\boldsymbol{\varepsilon}$	$\mathbf{t}^{\mathbf{M}}\cdot\mathbf{M}\mathbf{W}\,\mathbf{t}^{\mathbf{M}}$	$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{S}=\pm(\mathbf{P}-\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{t}^{\mathbf{M}}\cdot\mathbf{M}\,\mathbf{s}^{\mathbf{M}}$
$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{v}=\pm(\mathbf{a}-\mathbf{b})$	$\mathbf{v}\cdot\mathbf{t}^{\mathbf{v}}$	$\mathbf{T}=\mathbf{P},\ \mathbf{S}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\pm\mathbf{M}\boldsymbol{\varepsilon}$	$\mathrm{tr}\,(\mathbf{T}:\mathbf{S})\mathbf{M}\mathbf{A}$
$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{v}=\pm(\mathbf{a}+\mathbf{b})$	$\mathbf{v}\cdot\mathbf{t}^{\mathbf{M}}$	$\mathbf{T}=\mathbf{P},\ \mathbf{S}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{W}=\pm\boldsymbol{\varepsilon}$	$\mathrm{tr}\,(\mathbf{T}:\mathbf{S})\mathbf{W}$
		$C_{1\nu}$	
$\mathbf{T}=0$ and $\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{T}\vdots\mathbf{T}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{W}=\pm\boldsymbol{\varepsilon}$	$\mathbf{a}\cdot\mathbf{W}\,\mathbf{t}^{\mathbf{a}}$
$\mathbf{T}=\pm\mathbf{P}$	$\mathbf{a}\cdot\mathbf{t}^{\mathbf{a}}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{v}=\pm\mathbf{b}$	$\mathbf{v}\cdot\mathbf{t}^{\mathbf{a}}$
$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\pm\mathbf{M}$	$\mathbf{a}\cdot\mathbf{t}^{\mathbf{A}}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{S}=\pm\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{T}\vdots\mathbf{S}$

Table 5. Irreducibility of vector- and second-order tensor-valued function representations.

	variables	form-invariant	variables	form-invariant
		vector-valued		
$C_{\infty\nu}$	$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\mathbf{M}$	$\mathbf{t}^{\mathbf{A}}$	$\mathbf{T}=\mathbf{P},\ \mathbf{A}=\mathbf{M},\ \mathbf{W}=\boldsymbol{\varepsilon}$	$\mathbf{W}\,\mathbf{t}^{\mathbf{A}}$
	$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\mathbf{M}$	$\mathbf{A}\,\mathbf{t}^{\mathbf{A}}$	$\mathbf{T}=\mathbf{P},\ \mathbf{v}=\mathbf{b}$	$\mathbf{t}^{\mathbf{x}}$
C_{∞}	$\mathbf{T}=\mathbf{P},\ \mathbf{A}=\mathbf{M}$	$\mathbf{t}^{\mathbf{A}}$	$\mathbf{T}=\mathbf{P},\ \mathbf{A}=\mathbf{M}$	$\boldsymbol{\varepsilon}\,\mathbf{t}^{\mathbf{A}}$
$C_{3\nu}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{v}=\mathbf{a}$	$\mathbf{t}^{\mathbf{v}}$		
$C_{2\nu}$	$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{t}^{\mathbf{M}}$	$\mathbf{T}=\mathbf{P},\ \mathbf{A}=\mathbf{M}\boldsymbol{\varepsilon}$	$\mathbf{t}^{\mathbf{A}}$
	$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{M}\,\mathbf{t}^{\mathbf{M}}$	$\mathbf{T}=\mathbf{P},\ \mathbf{W}=\boldsymbol{\varepsilon}$	$\mathbf{W}\,\mathbf{t}^{\mathbf{M}}$
$C_{1\nu}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{a},\ \mathbf{t}^{\mathbf{a}}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{t}^{\mathbf{a}}$
		second-order symmetric tensor-valued		
$C_{\infty\nu}$	$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\mathbf{M}$	$\mathbf{t}^{\mathbf{A}}\otimes\mathbf{t}^{\mathbf{A}}$	$\mathbf{T}=\mathbf{P},\ \mathbf{S}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\mathbf{M}$	$\mathbf{A}(\mathbf{T}:\mathbf{S})-(\mathbf{S}:\mathbf{T})\mathbf{A}$
	$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{v}=\mathbf{a}$	$\mathbf{T}^{\mathbf{v}}$		
$C_{3\nu}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{v}=\mathbf{a}$	$\mathbf{T}^{\mathbf{v}}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{A}=\mathbf{M}$	$\mathbf{A}(\mathbf{P}:\mathbf{T})-(\mathbf{T}:\mathbf{P})\mathbf{A}$
$C_{2\nu}$	$\mathbf{T}=\mathbf{P}+\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{t}^{\mathbf{M}}\otimes\mathbf{t}^{\mathbf{M}}$	$\mathbf{T}=\mathbf{P},\ \mathbf{S}=\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{M}(\mathbf{T}:\mathbf{S})-(\mathbf{S}:\mathbf{T})\mathbf{M}$
	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon},\ \mathbf{v}=\mathbf{a}$	$\mathbf{T}^{\mathbf{v}}$		
$C_{1\nu}$	$\mathbf{T}=\mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{T}^{\mathbf{a}}$		

Table 5 [cont.]

	variables	form-invariant	variables	form-invariant
second-order skew-symmetric tensor-valued				
$C_{\infty\nu}$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$	$\mathbf{t}^{\mathbf{A}} \otimes \mathbf{A}\mathbf{t}^{\mathbf{A}} - \mathbf{A}\mathbf{t}^{\mathbf{A}} \otimes \mathbf{t}^{\mathbf{A}}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{S} = \mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{T} : \mathbf{S} - \mathbf{S} : \mathbf{T}$
	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\mathbf{v} \otimes \mathbf{t}^{\mathbf{v}} - \mathbf{t}^{\mathbf{v}} \otimes \mathbf{v}$		
$C_{3\nu}$	$\mathbf{T} = \mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{P} : \mathbf{T} - \mathbf{T} : \mathbf{P}$		
$C_{2\nu}$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{t}^{\mathbf{M}} \otimes \mathbf{M}\mathbf{t}^{\mathbf{M}} - \mathbf{M}\mathbf{t}^{\mathbf{M}} \otimes \mathbf{t}^{\mathbf{M}}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{S} = \mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{T} : \mathbf{S} - \mathbf{S} : \mathbf{T}$
	$\mathbf{T} = \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\mathbf{v} \otimes \mathbf{t}^{\mathbf{M}} - \mathbf{t}^{\mathbf{M}} \otimes \mathbf{v}$		
$C_{1\nu}$	$\mathbf{T} = \mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{a} \otimes \mathbf{t}^{\mathbf{a}} - \mathbf{t}^{\mathbf{a}} \otimes \mathbf{a}$		

Table 6. Irreducibility of third-order tensor-valued function representations.

	variables	form-invariant	variables	form-invariant
$C_{\infty\nu}$	$\mathbf{v} = \mathbf{a}$	$\mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v}$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$	$\mathbf{T}$
	$\mathbf{v} = \mathbf{a}$	$\{\mathbf{v} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$	$\langle \mathbf{A}\mathbf{t}^{\mathbf{A}} \otimes \mathbf{A} \rangle$
	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\langle \mathbf{v} \otimes \mathbf{A} \rangle$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$	$\{\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\}$
	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\{\mathbf{A}\mathbf{v} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$	$\{\mathbf{A}\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\}$
	$\mathbf{W} = \boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\langle \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{W}\mathbf{v} \rangle$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$, $\mathbf{B} = \mathbf{M} + \sqrt{3}\mathbf{M}\boldsymbol{\varepsilon}$	$\mathbf{T}(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A})$
	$\mathbf{W} = \boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\{\mathbf{W}\mathbf{v} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{W} = \boldsymbol{\varepsilon}$	$\mathbf{T}\mathbf{W}$
	$\mathbf{v} = \mathbf{a}$, $\mathbf{u} = \mathbf{a} + \sqrt{3}\mathbf{b}$	$\langle \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{u} \rangle$	$\mathbf{T} = \mathbf{P}$, $\mathbf{A} = \mathbf{M}$, $\mathbf{W} = \boldsymbol{\varepsilon}$	$\{\mathbf{W}\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\}$
			$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\{\mathbf{t}^{\mathbf{v}} \otimes \mathbf{1}\}$
C_{∞}	$\mathbf{v} = \mathbf{a}$	$\mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{A} = \mathbf{M}$	$\mathbf{T}$
	$\mathbf{v} = \mathbf{a}$	$\langle \mathbf{v} \otimes \mathbf{v} \otimes \boldsymbol{\varepsilon}\mathbf{v} \rangle$	$\mathbf{T} = \mathbf{P}$, $\mathbf{A} = \mathbf{M}$	$\mathbf{T}\boldsymbol{\varepsilon}$
	$\mathbf{v} = \mathbf{a}$	$\{\mathbf{v} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{A} = \mathbf{M}$	$\{\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\}$
	$\mathbf{v} = \mathbf{a}$	$\{\boldsymbol{\varepsilon}\mathbf{v} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{A} = \mathbf{M}$	$\{\boldsymbol{\varepsilon}\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\}$
$C_{3\nu}$	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$	$\mathbf{P}$	$\mathbf{A} = \mathbf{M}$, $\mathbf{B} = \mathbf{M} + \sqrt{3}\mathbf{M}\boldsymbol{\varepsilon}$	$\mathbf{P}(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A})$
	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$	$\langle \mathbf{A}\mathbf{p}^{\mathbf{A}} \otimes \mathbf{A} \rangle$	$\mathbf{A} = \mathbf{M}$, $\mathbf{W} = \boldsymbol{\varepsilon}$	$\{\mathbf{W}\mathbf{p}^{\mathbf{A}} \otimes \mathbf{1}\}$
	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$	$\{\mathbf{p}^{\mathbf{A}} \otimes \mathbf{1}\}$	$\mathbf{A} = \mathbf{M} + \sqrt{3}\mathbf{M}\boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\langle \mathbf{v} \otimes \mathbf{A} \rangle$
	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$	$\{\mathbf{A}\mathbf{p}^{\mathbf{A}} \otimes \mathbf{1}\}$	$\mathbf{W} = \boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\{\mathbf{W}\mathbf{v} \otimes \mathbf{1}\}$
	$\mathbf{W} = \boldsymbol{\varepsilon}$	$\mathbf{P}\mathbf{W}$	$\mathbf{v} = \mathbf{a}$, $\mathbf{u} = \mathbf{a} + \sqrt{3}\mathbf{b}$	$\mathbf{P}(\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v})$
	$\mathbf{v} = \mathbf{b}$	$\mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v}$	$\mathbf{T} = \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$	$\mathbf{T}$
	$\mathbf{v} = \mathbf{b}$	$\{\mathbf{v} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{A} = \mathbf{M}$	$\{\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\}$
	$\mathbf{v} = \mathbf{b}$	$\{\mathbf{p}^{\mathbf{v}} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P}\boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\{\mathbf{t}^{\mathbf{v}} \otimes \mathbf{1}\}$
$C_{2\nu}$	$\mathbf{v} = \mathbf{a} + \mathbf{b}$	$\mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v}$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{T}$
	$\mathbf{v} = \mathbf{a} + \mathbf{b}$	$\langle \mathbf{v} \otimes \mathbf{M} \rangle$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$	$\langle \mathbf{M}\mathbf{t}^{\mathbf{M}} \otimes \mathbf{M} \rangle$
	$\mathbf{v} = \mathbf{a} + \mathbf{b}$	$\{\mathbf{v} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$	$\{\mathbf{t}^{\mathbf{M}} \otimes \mathbf{1}\}$
	$\mathbf{v} = \mathbf{a} + \mathbf{b}$	$\{\mathbf{M}\mathbf{v} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P} + \mathbf{P}\boldsymbol{\varepsilon}$	$\{\mathbf{M}\mathbf{T}^{\mathbf{M}} \otimes \mathbf{1}\}$
	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\langle \mathbf{v} \otimes \mathbf{A} \rangle$	$\mathbf{T} = \mathbf{P}$, $\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$	$\mathbf{T}(\mathbf{M}\mathbf{A} - \mathbf{A}\mathbf{M})$
	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\{\mathbf{A}\mathbf{v} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$	$\{\mathbf{t}^{\mathbf{A}} \otimes \mathbf{1}\}$
	$\mathbf{W} = \boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\langle \mathbf{W}\mathbf{v} \otimes \mathbf{M} \rangle$	$\mathbf{T} = \mathbf{P}$, $\mathbf{W} = \boldsymbol{\varepsilon}$	$\mathbf{T}\mathbf{W}$
	$\mathbf{W} = \boldsymbol{\varepsilon}$, $\mathbf{v} = \mathbf{a}$	$\{\mathbf{W}\mathbf{v} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P}$, $\mathbf{W} = \boldsymbol{\varepsilon}$	$\{\mathbf{W}\mathbf{t}^{\mathbf{M}} \otimes \mathbf{1}\}$
$C_{1\nu}$	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$	$\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a}$	$\mathbf{W} = \boldsymbol{\varepsilon}$	$\{\mathbf{W}\mathbf{a} \otimes \mathbf{1}\}$
	$\mathbf{A} + \mathbf{M}\boldsymbol{\varepsilon}$	$\langle \mathbf{a} \otimes \mathbf{A} \rangle$	$\mathbf{v} = \mathbf{b}$	$\langle \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{v} \rangle$
	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$	$\{\mathbf{a} \otimes \mathbf{1}\}$	$\mathbf{v} = \mathbf{b}$	$\{\mathbf{v} \otimes \mathbf{1}\}$
	$\mathbf{A} = \mathbf{M}\boldsymbol{\varepsilon}$	$\{\mathbf{A}\mathbf{a} \otimes \mathbf{1}\}$	$\mathbf{T} = \mathbf{P}\boldsymbol{\varepsilon}$	$\mathbf{T}$
	$\mathbf{W} = \boldsymbol{\varepsilon}$	$\langle \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{W}\mathbf{a} \rangle$	$\mathbf{T} = \mathbf{P}\boldsymbol{\varepsilon}$	$\{\mathbf{t}^{\mathbf{v}} \otimes \mathbf{1}\}$

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References

1. J. BETTEN and W. HELISH, *Simuleneous invariants in system of 2nd-order tensors*, ZAMM, **75**, 753–759, 1995.
2. J.P. BOEHLER, *A simple derivation of representations for non-polynominal constitutive equations in some cases of anisotropy*, ZAMM, **59**, 157–167, 1979.
3. K.C. HANNABUSS, *The irreducible components of homogeneous functions and symmetric tensors*, J. Inst. Math. Applics., **14**, 83–88, 1974.
4. S. PENNISI, *On third order tensor-valued isotropic functions*, Int. J. Engng. Sci., **30**, 679–692, 1992.
5. S. PENNISI and M. TROVATO, *On the irreducibility of Professor G.F. Smith's representations for isotropic functions*, Int. J. Engng. Sci., **25**, 1059–1065, 1987.
6. A.J.M. SPENCER, *A note on the decomposition of tensors into traceless symmetric tensors*, Int. J. Engng. Sci., **8**, 489–505, 1970.
7. Q.-S. ZHENG, *Two-dimensional tensor function representation for all kinds of material symmetry*, Proc. R. Soc. Lond., **A443**, 127–138, 1993.
8. Q.-S. ZHENG, *On the representations for isotropic vector-valued, symmetric tensor-valued and skew-symmetric tensor-valued functions*, Int. J. Engng. Sci., **31**, 1013–1024, 1993.
9. Q.-S. ZHENG, *A note on representation for isotropic functions of a 4th-order tensor in 2-dimensional space*, ZAMM, **75**, 357–359, 1994.
10. Q.-S. ZHENG and J. BETTEN, *On the tensor function representations of 2nd-order and 4th-order tensors*, I, ZAMM, **75**, 269–281, 1994.
11. Q.-S. ZHENG and J.P. BOEHLER, *The description, classification, and reality of material and physical symmetries*, Acta Mech., **102**, 73–89, 1994.
12. Q.-S. ZHENG and A.J.M. SPENCER, *On the canonical representations for Kronecker papers of orthogonal tensors with application to material symmetry problems*, Int. J. Engng. Sci., **31**, 617–635, 1993.

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