Porous media at finite strains
The new model with the balance equation for porosity

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THE PURPOSE of this work is the presentation of the governing equations describing the two-component porous material as the mixture with the additional field of the porosity. The additional field equation for this field is proposed. The governing equations are formulated in the new Lagrangian description. The constitutive relations under arbitrary elastic deformations of the skeleton are proposed. Various simplified models and their basic properties such as the propagation of sound waves are discussed. The work should be of interest for scientists working on continuum mechanics (problems with the free boundary), on numerical methods in continuum mechanics and on the wave propagation as the method of diagnosis of media with microstructure.

1. Introduction

THE THEORIES of porous materials have been developed primarily within the frame of soil mechanics. For granular soils (e.g. sand), clays and rocks, various engineering models were proposed to describe the flow of water or other fluids through the pores. The extensive literature concerning this subject as well as the introduction into the nomenclature of porous media can be found, for instance, in the excellent classical book of J. BEAR [1]. The connection of continuous models of porous materials with the modern theory of mixtures is explained in the review article of R.M. BOWEN [2].

R. DE BOER [3] presents in his major historical paper not only many details concerning the pioneering works of Terzaghi, Fillunger and some other engineers, who have contributed to the practical soil mechanics but he discusses also some new tendencies in the theories of porous media. Another practical aspect of these theories stems from combustion problems of granular materials which describe the behaviour of solid fuels. The review article on this subject has been written by S.L. PASSMAN, J.W. NUNZIATO, E.K. WALSH [4]. Much less has been done on the subject of multicomponent continua with large deformations of solids. Large elastic deformations which appear, for instance, in foams damping the sound waves or some filters in the chemical industry, were investigated experimentally but very little has been done from the continuum-mechanical point of view. Large plastic deformations, which accompany almost any loading of sands, are still described by means of the one-component models and, for instance, the influence of the changes of porosity is usually entirely neglected. Even the problems of large static deformations with the small dynamical disturbance (e.g. diagnosis of soils by propagating sound waves) are understood much better from the experimental standpoint than through some theoretical description. As an example, let us

mention a very competent book of T. BOURBIE, O. COUSSY, B. ZINSZNER, [5] who give the account on the wave experiments on porous materials but describe them theoretically by means of the old model of Biot in which the multicomponent character of the medium is accounted in a very poor and deficient manner.

The purpose of the present work is the presentation of the mechanical two-component model of the porous materials in which the skeleton may undergo arbitrary large elastic deformations, the fluid is inviscid but it may interact with the skeleton in an almost arbitrary way, and the porosity can change according to its own field equation. The irreversibility of processes in such a model follows from the diffusion and from the pore relaxation.

In the next section we present the necessity of the formulation of additional equations in the theory of porous materials when compared with the usual theory of mixtures of the same number of components. The third section is devoted to the brief presentation of the new consistent way of description of porous materials when the reference configuration of the skeleton is chosen as the reference for all other components as well. Apart from the advantages of this Lagrangian description in cases of large deformations, it is also a very convenient starting point for the numerical investigations of the model of porous media. In the fourth section we present the family of fields and field equations for this Lagrangian description of the two-component porous medium with the elastic skeleton and an ideal fluid component. The fifth section is devoted to the thermodynamic restrictions imposed on the constitutive relations assumed in the section four. The sixth section limits further the constitutive relations by the assumption of isotropy. One of the most important and rather surprising, very restrictive results follows in this section for the flux in the balance equation of porosity. In the seventh section we discuss some possibilities of further restrictions of constitutive relations by simplifying the way in which the components interact with each other. These simplifications are motivated by experimental results for rocks and granular materials.

The presentation of the model is supplemented in the eighth section with the discussion of the dynamic compatibility conditions and their connection with the boundary conditions for the porous medium. The most important part of this section concerns the conditions for the case of the free outstreaming fluid which yields the necessity of the additional scalar boundary condition describing the free boundary.

As an example of applications of the model we present in the ninth section the analysis of the propagation conditions for the sound waves. It is shown that the model indeed describes all these waves which are observed in reality. We present as well some possibilities of the application of this model to the diagnosis of porous media. The tenth section contains one of the possible linear models following from the general formulation. It is shown that quasi-static solutions of some boundary value problems for such a linear model are identical with the corresponding solutions of soil mechanics.

2. Closure problem: constitutive relations vs. differential equations for volume fractions

The main difference between the classical theory of mixtures of fluid-like components (miscible components) and the theory of porous materials (immiscible components) is connected with the existence of additional fields – a sort of internal variables – for the porous material, which describe the volume fraction of each component in the total microscopic control volume. For the A different components these volume fractions must satisfy the obvious normalisation condition

$$(2.1) \qquad \sum_{\alpha=1}^{A} n^{\alpha} = 1,$$

where n^{α} denotes the volume fraction of the α-component, $1 \leq \alpha \leq A$.

This relation is sometimes called the saturation condition. This name stems from the soil mechanics in which the porous materials with pores partially filled with water are frequently considered. In such cases the air is not accounted for as the third component and the medium is considered to be not fully saturated. The sum of volume fractions of the solid and of the water is smaller than one. It is quite obvious that it is not necessery to do so in the construction of the model. Particularly in processes of phase transitions such as the evaporation (drying processes, cavitation) the role of the gaseous phase is important. This gaseous component cannot be left out of the model even if its kinematics is identical with this of the fluid component.

It is easy to see that, in contrast to the classical theory of mixtures of miscible components, a theory of porous materials requires additional field equations. The continuum models of miscible components have been constructed by means of the partial balance equations of mass, momentum and energy for each component. In the Eulerian description these laws together with apppropriate constitutive relations were sufficient to yield the field equations for the partial mass densities ϱ^{α}, the partial velocities $\mathbf{v}^{\alpha}$ and the partial temperatures Θ^{α}. These balance laws are also used in models of the immiscible components but we have to supplement the theory with relations for the volume fractions.

A few solutions of this problem have been proposed. They can be divided into two classes:

1) additional constitutive relations are introduced,

2) additional differential equations in the form of either evolution equations or balance equations are proposed.

The simplest example of the model of the first class is the model proposed by R.M. Bowen [6]. Its prototype can be found in the papers of J.J. van Deemter and E.R. van der Laan [7] as well as of J.O. Hinze [8]. Also the work of R.S. Sampaio and W.O. Williams [9] is based on the similar notions. In this model

it is assumed that the volume fractions are proportional to the corresponding partial mass densities

(2.2) $$ n^\alpha = \frac{\varrho^\alpha}{\varrho^{\alpha R}}, \qquad 1 \le \alpha \le A, $$

where $\varrho^{\alpha R}$ are constants. These constants are called *true mass densities* and the corresponding components are called *incompressible*. In the Bowen's model this notion of incompressibility has nothing to do with the usual incompressibility of one-component continua. The classical incompressibility is the constraint requiring the sustaining reaction forces (e.g. reaction pressure). Such reaction forces do not appear in the Bowen's model. There is however a reaction force due to the saturation condition. Namely the relations (2.2) specify all volume fractions in terms of partial mass densities but they cannot be arbitrary due to the constraint of the saturation condition (2.1). This model has been extensively applied. However the recent results concerning in particular the boundary value problems for dynamic processes and the relaxation properties seem to indicate that the model has many very serious physical flaws.

Another model of the same class has been introduced by J.L.W. MORLAND [10]. He has assumed the constitutive relations describing the volume fractions. The model presented in the paper [11], concerning the two-component porous material belongs as well to this class. In the latter paper the saturation condition reduces the number of independent volume fractions to one. The additional constitutive relation has been proposed in a quite general form

(2.3) $$ \pi(\mathcal{C}) = 0, $$

where π denotes the arbitrary scalar function and $\mathcal{C}$ denotes the collection of all constitutive variables of the model. The thermodynamic considerations as well as the construction of the boundary value problems for such a model have been presented in the above mentioned paper. No practical applications have been made as yet.

Recently the much more sophisticated version of such a model is being investigated by J. BLUHM and R. DE BOER (see: [3, 12]). It is based on the semi-microscopic considerations referring to the "true" components. The local configuration of each component is assumed to be described by the so-called *realistic deformation gradient* $\mathbf{F}^{\alpha R}$ which is mapping the material vectors of the α-component from the reference configuration to the current configuration. These gradients are *not* assumed to be integrable. However one assumes that there exists the supplementary gradient $\mathbf{F}^{\alpha N}$ which combines with the realistic deformation gradient into the integrable partial deformation gradient $\mathbf{F}^\alpha$ of the α-component

(2.4) $$ \mathbf{F}^\alpha = \mathbf{F}^{\alpha N} \mathbf{F}^{\alpha R}. $$

The constitutive relations are assumed to hold for the objective combination of the realistic deformation gradients

$$(2.5) \qquad \mathbf{C}^{\alpha R} = \mathbf{C}^{\alpha R}(\mathcal{C}), \qquad \mathbf{C}^{\alpha R} \equiv \mathbf{F}^{\alpha R T} \mathbf{F}^{\alpha R}.$$

In particular these relations define the constitutive relations for volume fractions and the saturation condition becomes again the constraint. It has been shown that in some particular cases this model describes the phenomena which have been observed in the experimental soil mechanics. Moreover the model seems to be an appropriate starting point for the description of anisotropic structure of pores. Nothing has been done yet in this direction.

It should be mentioned that the models of this class do not describe the pore relaxation processes because the volume fractions are controlled by other macroscopic deformation variables.

Within the second class of the models, the most commonly used one seems to be that started by the M.A. GOODMAN and S.C. COWIN [13] who have proposed an additional balance equation for a scalar quantity with a rather obscure physical interpretation. This equation is called the *balance of equilibrated forces* and in various versions it has been extensively used to describe the two-component granular materials (e.g. see: J.W. NUNZIATO, E.K. WALSH [14], D.S. DRUMHELLER, A. BEDFORD [15], A. BEDFORD, D.S. DRUMHELLER [16], S.L. PASSMAN [17], S.L. PASSMAN, J.W. NUNZIATO, E.K. WALSH [4]). In particular the results for the combustion problems (solid fuels) indicate that such a model is quite reasonable in spite of its rather unclear microscopic foundations.

The same sort of the model has been investigated by J. BLUHM, R. DE BOER and K. WILMAŃSKI [18]. They have considered the model with balance equations for true mass densities $\varrho^{\alpha R}$. These were not assumed to be constant any more as it was the case for the "incompressible" model of Bowen. The purpose of this work was however solely to show that the incompressibilities in the Bowen's model, if considered in the same way as in the classical continuum mechanics, yield the structure of the partial stress tensors which eliminates some flaws of the original Bowen's model. The local properties of this model have been investigated in order to check the appearance of sound waves. It has been proved [19] that the so-called P1- and P2- longitudinal waves may appear as required by experimental observations if very specific constitutive restrictions on fluxes are satisfied.

Another type of the model in this class has been introduced by R.M. BOWEN [20] who postulates the *evolution equation* for each volume fraction. This procedure is quite common in thermodynamic theories with internal variables (e.g. macroscopic theories of mixtures with chemical reactions). It yields the spontaneous pore relaxation.

It should be mentioned that most of the above models admit large deformations of the skeleton. Although these have not been investigated in the above quoted papers, the problem has been recognized rather early. Some of its aspects

were mentioned, for instance, in the early papers of J.E. Adkins [21] and A.E. Green and J.E. Adkins [22]. These works do not contain however any propositions concerning the changes of volume fractions. An extension of these works under the Bowen's "incompressibility" assumption has been proposed by J. Kubik [23]. His work contains also many references connected with the problem of large deformations.

In the present work we shall discuss in some details a new version of the two-component model with the balance equation for porosity. It will be shown that the model easily admits large deformations of the skeleton (the solid component of the porous medium). Simultaneously it complies in the limit cases with the early engineering models of soils and rocks. The semi-microscopic motivation and thermodynamic details can be found in the paper [24]. A brief presentation of these arguments is contained in the Appendix to this paper.

3. Lagrangian description

The continuous theory of mixture with fluid components relies usually on the Eulerian description of the motion of components, similarly to the classical fluid mechanics of the single component. In the case of one solid component such as the skeleton of the porous medium this method is also possible but not very convenient. Namely, to describe the large deformations of the skeleton in the Eulerian way we have to introduce the deformation gradient F^S of the skeleton as the field in the space of actual configurations and then use the integrability condition for this gradient as the additional tensorial field equation (e.g. see: [25]). The attempts to use the mixed description – the Eulerian one for the fluid components and the Lagrangian one for the solid components (see: R.M. Bowen [2]) – does not seem to be appropriate either. It yields certain basic technical difficulties in the evaluation of the second law of thermodynamics and, most important of all, it is not suitable for the analysis of the boundary value problems. In the latter case, the field equations must be first transformed to the same independent variables – either Eulerian or Lagrangian and this transformation leads again to the technical difficulties apart from the fact that the problem can be formulated in the uniform description from the very beginning. In addition, the numerical analysis based on the finite element methods is simplified considerably when we use the same reference configuration for all components to define the spatial (Lagrangian) independent variables.

The most natural choice of such a reference configuration is the configuration of the skeleton for which its deformation gradient is the identity. Then the description of the deformation and of the kinematics is Lagrangian as in the nonlinear mechanics of solids. It remains to clear the question how to describe the fluid components in such a reference. This question has been answered in [11] (see also [26] for many details) where the two-component porous material

has been considered. We present here briefly these results limiting the further considerations of this work to the two-component porous materials as well. The extension to the cases of larger number of components is straightforward.

Let us begin with the motion of the skeleton. In the Lagrangian description it is given by the *function of motion*

$$(3.1) \qquad \mathbf{x} = \boldsymbol{\chi}^S(\mathbf{X}, t), \qquad \mathbf{x} \in E^3, \qquad \mathbf{X} \in \mathcal{B},$$

where $\mathbf{x}$ denotes the current position of the material point $\mathbf{X}$ of the skeleton, E^3 is the three-dimensional Euclidean space of motion and $\mathcal{B}$ denotes the reference configuration of the skeleton which, for the purpose of this work, can be identified for instance with the real configuration of the skeleton at the instant of time $t = t_0$. Then the deformation gradient and the velocity of the skeleton are defined as follows

$$(3.2) \qquad \mathbf{F}^S(\mathbf{X}, t) = \operatorname{Grad} \boldsymbol{\chi}^S(\mathbf{X}, t), \qquad \mathbf{x}'^S(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}^S}{\partial t}(\mathbf{X}, t).$$

In the case of the fluid component described in the Eulerian way, the kinematics is given by the velocity field defined on the current configuration

$$(3.3) \qquad \mathbf{v}^F = \mathbf{v}^F(\mathbf{x}, t), \qquad \mathbf{x} \in \boldsymbol{\chi}^S(\mathcal{B}, t).$$

It is rather obvious that the kinematics of the fluid is defined solely within the domain of the current configuration of the solid $\boldsymbol{\chi}^S(\mathcal{B}, t)$. We are not interested in the motion beyond this domain except for the phenomena appearing on the boundary of the skeleton. This problem shall be discussed in the sequel. We proceed to transform the relation (3.3) into the Lagrangian description of the skeleton. Let us concentrate the attention on the material point of the fluid which occupies the position $\mathbf{x}$ at the instant of time t. For the small time increment Δt the position of this material point is given by the relation

$$(3.4) \qquad \mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \mathbf{v}^F(\mathbf{x}(t), t)\Delta t \equiv \mathbf{x}(t) + \mathbf{F}^S(\mathbf{X}, t)\Delta\mathbf{X} + \mathbf{x}'^S(\mathbf{X}, t)\Delta t,$$

where

$$(3.5) \qquad \begin{aligned} \mathbf{X} &\equiv \mathbf{X}(t) = \boldsymbol{\chi}^{S-1}(\mathbf{x}(t), t), \\ \Delta\mathbf{X} &= \boldsymbol{\chi}^{S-1}(\mathbf{x}(t + \Delta t), t + \Delta t) - \boldsymbol{\chi}^{S-1}(\mathbf{x}(t), t). \end{aligned}$$

The second part of the relation (3.4) follows certainly from the fact that the material point of the fluid has changed the material point of the skeleton $\mathbf{X}$ with which it had shared the position at the instant of time t into the material point of the skeleton $\mathbf{X} + \Delta\mathbf{X}$, as indicated in the relation (3.5) (diffusion!). Consequently, after easy manipulations in (3.4) we see that the *image* of the material point of the

fluid in the reference configuration of the skeleton $\mathcal{B}$ moves with the following velocity

(3.6)
$$\mathbf{X}^{\prime F}(\mathbf{X}, t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{X}}{\Delta t} = \mathbf{F}^{S-1}(\mathbf{x}^{\prime F} - \mathbf{x}^{\prime S}),$$
$$\mathbf{x}^{\prime F} \equiv \mathbf{v}^{F}(\boldsymbol{\chi}^{S}(\mathbf{X}, t), t).$$

We call this velocity the *Lagrangian velocity of the fluid component*. It is obvious that this velocity together with the velocity of the skeleton and with the deformation gradient of the skeleton, determines uniquely the usual Eulerian velocity of the fluid component $\mathbf{v}^{F}$. Hence both ways of the description of kinematics of the fluid component are equivalent. However the Lagrangian way has the advantage that all fields are defined in the same domain $\mathcal{B}$.

4. Field equations

We proceed to specify the basic fields of the two-component model and the appropriate field equations. We limit our attention solely to *isothermal processes*. Then the processes in the skeleton are described by the initial mass density ϱ^{S} which is assumed to be constant (independent of the position in $\mathcal{B}$ – homogeneous material) and by the function of motion $\boldsymbol{\chi}^{S}(\cdot, \cdot)$. In addition to this vector field for the skeleton, the process in the porous medium is described by the vector field of the Lagrangian velocity $\mathbf{X}^{\prime F}(., .)$ as well as the mass density of the fluid component and the volume fraction of the fluid. We have to find the Lagrangian representation for the last two fields.

The usual current mass density of the fluid component $\varrho_{t}^{F}(\mathbf{x}, t)$ satisfies the following mass conservation law

(4.1)
$$\forall \mathcal{P}_{t} \subset \boldsymbol{\chi}^{S}(\mathcal{B}, t) : \frac{d}{dt} \int_{\mathcal{P}_{t}} \varrho_{t}^{F} \, dv = 0,$$

where $\mathcal{P}_{t}$ is *material* with respect to the motion of the fluid. It has been assumed that there are no mass sources which could appear in the case of the exchange of mass between components. The above relation can be easily written in the image on the reference configuration $\mathcal{B}$ of the skeleton. Namely

(4.2)
$$\forall \mathcal{P} \subset \mathcal{B} : \frac{d}{dt} \int_{\mathcal{P}} \varrho^{F} \, dv \equiv \int_{\mathcal{P}} \frac{\partial \varrho^{F}}{\partial t} dv + \oint_{\partial \mathcal{P}} \varrho^{F} \mathbf{X}^{\prime F} \cdot \mathbf{N} ds = 0,$$
$$dv = J^{S-1} \, d\mathbf{v},$$

where

(4.3)
$$\varrho^{F} = J^{S} \varrho_{t}^{F}, \qquad J^{S} \equiv \det \mathbf{F}^{S}, \qquad \mathcal{P} = \boldsymbol{\chi}^{S-1}(\mathcal{P}_{t}, t),$$

and $\partial \mathcal{P}$ denotes the boundary of the set $\mathcal{P}$, *material* with respect to the fluid component. The presence of the surface integral is certainly connected with the fact that the image of fluid on $\mathcal{B}$ changes in time according to the field of the Lagrangian velocity of the fluid component.

It remains to introduce the representation for the volume fractions. It can be done, for instance, by the consideration of the true mass densities defined by the relations (2.2). If these are going to have the meaning of the mass densities then they have to transform in the same way as ϱ^F in the relation (4.3), i.e.

$$(4.4) \qquad \varrho^{FR} = J^S \varrho_t^{FR} \;\Rightarrow\; n^F = n_t^F,$$

where ϱ^{FR} and ϱ_t^{FR} denote the reference value and the current value of the true mass density of the fluid component, respectively. The implication in the relation (4.4) follows, certainly, from (4.3). Consequently, we have the following relation for the volume fraction of the skeleton

$$(4.5) \qquad n^S = 1 - n = n_t^S, \qquad n \equiv n^F.$$

In the above relation the saturation condition for the two-component porous medium has been used. The volume fraction of the fluid component n^F is frequently called the *porosity* of such a medium and it is denoted by n, as indicated in (4.5). According to the above choice of the transformation rules preserving the geometrical meaning of the volume fractions, the porosity in the Lagrangian description is identical with that in the Eulerian description.

The above considerations yield the following set of fields which must be determined by the mechanical model of the two-component porous medium

$$(4.6) \qquad (\mathbf{X}, t) \mapsto \left\{ \varrho^F, n, \boldsymbol{\chi}^S, \mathbf{X}'^F \right\} \in V^8, \qquad \mathbf{X} \in \mathcal{B},$$

where V^8 is the eight-dimensional vector space of values of the fields.

For these fields we have to formulate the field equations. As usual we shall make use of the conservation laws. Obviously, the conservation of mass of the solid component is identically satisfied in the Lagrangian description. The local conservation of mass of the fluid component follows easily from the equation (4.2). We obtain

$$(4.7) \qquad \frac{\partial \varrho^F}{\partial t} + \mathrm{Div}\, \varrho^F \mathbf{X}'^F = 0.$$

The balance laws of momentum for both components are not conservation laws due to the interaction of components in the relative motion (*diffusive force*). We write first the integral form of these laws. Namely

$$\frac{d}{dt} \int_{\mathcal{P}} \varrho^S \mathbf{x}'^S \, dv = \oint_{\partial \mathcal{P}} \mathbf{P}^S \mathbf{N} \, ds + \int_{\mathcal{P}} (\mathbf{p}^* + \varrho^S \mathbf{b}^S) \, dv$$

for $\forall \mathcal{P} \subset \mathcal{B}$ – material with respect to skeleton,

$$(4.8) \qquad \frac{d}{dt} \int_{\mathcal{P}} \varrho^F \mathbf{x}'^F \, dv = \oint_{\partial \mathcal{P}} \mathbf{P}^F \mathbf{N} \, ds + \int_{\mathcal{P}} (-\mathbf{p}^* + \varrho^F \mathbf{b}^F) \, dv$$

for $\forall \mathcal{P} \subset \mathcal{B}$ – material with respect to fluid,
where $\mathbf{P}^S$ and $\mathbf{P}^F$ denote the *partial Piola-Kirchhoff stress tensors* related to the reference configuration of the skeleton. They are related to the Cauchy stress tensors of the current configuration by the relations

$$(4.9) \qquad \mathbf{P}^S = J^S \mathbf{T}^S \mathbf{F}^{S-T}, \qquad \mathbf{P}^F = J^S \mathbf{T}^F \mathbf{F}^{S-T},$$

$\mathbf{T}^S$ and $\mathbf{T}^F$ being the partial Cauchy stresses in the skeleton and in the fluid component, respectively.

The vector $\mathbf{p}^*$ denotes the *momentum source (diffusive force)* resulting from different velocity fields of the components. These, in reality, two sources for two momentum balance equations differ solely in sign as required by the continuum theory of mixtures.

The vector $\mathbf{N}$ is the unit vector orthogonal to the boundary $\partial \mathcal{P}$ and oriented outwards.

In any regular point of the domain $\mathcal{B}$, the above balance laws yield the following local equations

$$\varrho^S \frac{\partial \mathbf{x}'^S}{\partial t} - \mathrm{Div} \mathbf{P}^S = \mathbf{p}^* + \varrho^S \mathbf{b}^S,$$

(4.10)

$$\frac{\partial}{\partial t}(\varrho^F \mathbf{x}'^F) + \mathrm{Div}(\varrho^F \mathbf{x}'^F \otimes \mathbf{X}'^F - \mathbf{P}^F) = -\mathbf{p}^* + \varrho^F \mathbf{b}^F.$$

These equations and the mass balance for the fluid component (4.7) form the basis for the formulation of field equations if supplemented with constitutive laws. However we are still missing one equation for the eight fields (4.6). This is the closure problem which we have presented in Sec. 2. As indicated already we solve it by adding the *balance equation* for the porosity n. The semi-microscopic motivation of this equation can be found in the paper [24] and in the Appendix. In the present work this equation can be considered on the purely phenomenological footing (see as well: [27]). Namely we assume

$$(4.11) \qquad \frac{\partial n}{\partial t} + \mathrm{Div} \mathbf{J} = \nu,$$

and call $\mathbf{J}$ the *flux of porosity* and ν the *source of porosity*. Their physical meaning shall be presented in the sequel (see, also: [27, 28]).

In order to formulate the field equations we have to introduce the constitutive relations for the following *constitutive quantities*

$$(4.12) \qquad \mathcal{Z} = \left\{ \mathbf{J}, \nu, \mathbf{F}^{S-1} \mathbf{P}^S, \mathbf{F}^{S-1} \mathbf{P}^F, \mathbf{F}^{ST} \mathbf{p}^* \right\},$$

where the Piola–Kirchhoff stress tensors were multiplied by the deformation gradient for the *objectivity* reasons. We do not need to discuss this problem in the present work because it does not differ from the same problem of the nonlinear continuum mechanics of single-component media. However it is worth noticing that the vector **J** is also assumed to be independent of the observer which can be easily done in the Lagrangian description as we see further in this work. It is connected with the fact that the Lagrangian velocity is independent of the observer being defined by means of the relative velocity (see: (3.6)).

Further in this work we consider the simplest possible two-component porous medium for which it is assumed that the skeleton is *elastic* and the fluid is *ideal*. This certainly does not mean the reversibility of processes which are influenced by the diffusion and the sources of porosity, both these factors yielding dissipation. In terms of our fields the collection of *constitutive variables* in such a case is as follows

(4.13) $$\mathcal{C} = \left\{ \varrho^F, n, \mathbf{C}^S, \mathbf{X}'^F \right\},$$

where $\mathbf{C}^S$ denotes the right Cauchy–Green deformation tensor of the skeleton.

Finally we have the following *constitutive relations*

(4.14) $$\mathcal{Z} = \mathcal{Z}(\mathcal{C}),$$

all these functions being assumed to be twice continuously differentiable with respect to all arguments.

Equations (4.7), (4.10) and (4.11) together with the constitutive relations (4.14) form the closed set of eight field equations for the eight fields (4.6). It remains to formulate the boundary and initial conditions to obtain the initial-boundary value problem for the set of differential equations. We shall discuss the boundary conditions after the presentation of some thermodynamic admissibility conditions for the constitutive relations (4.14) which are as yet almost arbitrary except for the above mentioned mathematical regularity conditions.

5. Thermodynamic restrictions

We proceed to present the restrictions of the above described constitutive relations following from the assumption that the processes must satisfy the second law of thermodynamics.

Any solution of the field equations is called the *thermodynamic process*. According to the *second law of thermodynamics*, the thermodynamic process is *thermodynamically admissible* if the following inequality

(5.1) $$\varrho^S \frac{\partial \Psi^S}{\partial t} + \varrho^F \left(\frac{\partial \Psi^F}{\partial t} + \mathbf{X}'^F \cdot \operatorname{Grad} \Psi^F \right) - \mathbf{P}^S \cdot \frac{\partial \mathbf{F}^S}{\partial t}$$
$$- \mathbf{P}^F \cdot \operatorname{Grad} \mathbf{x}'^F - \mathbf{F}^{ST} \mathbf{p}^* \cdot \mathbf{X}'^F \leq 0$$

is identically satisfied. In the above inequality Ψ^S, Ψ^F denote the partial Helmholtz free energies of components. These are assumed to be the constitutive quantities, i.e.

$$(5.2) \qquad \Psi^S = \Psi^S(\mathcal{C}), \qquad \Psi^F = \Psi^F(\mathcal{C}).$$

The simple derivation of the inequality (5.1) from the entropy balance equations and the entropy inequality for isothermal conditions can be found, for instance, in the work [11].

In the standard way we eliminate now the constraint on solutions of the inequality (5.1) that it should hold solely for the thermodynamic processes. Namely we introduce the Lagrange multipliers for the field equations and require that the inequality

$$
(5.3) \qquad \varrho^S \frac{\partial \Psi^S}{\partial t} + \varrho^F \left(\frac{\partial \Psi^F}{\partial t} + \mathbf{X}'^F \cdot \operatorname{Grad} \Psi^F \right) - \mathbf{P}^S \cdot \frac{\partial \mathbf{F}^S}{\partial t}
$$
$$
- \mathbf{P}^F \cdot \operatorname{Grad} \mathbf{x}'^F - \mathbf{F}^{ST} \mathbf{p}^* \cdot \mathbf{X}'^F
$$
$$
- \Lambda^\varrho \left(\frac{\partial \varrho^F}{\partial t} + \operatorname{Div} \varrho^F \mathbf{X}'^F \right) - \Lambda^n \left(\frac{\partial n}{\partial t} + \operatorname{Div} \mathbf{J} - \nu \right)
$$
$$
- \mathbf{L}^S \cdot \left(\varrho^S \frac{\partial \mathbf{x}'^S}{\partial t} - \operatorname{Div} \mathbf{P}^S - \mathbf{p}^* - \varrho^S \mathbf{b}^S \right)
$$
$$
- \mathbf{L}^F \cdot \left(\frac{\partial}{\partial t}(\varrho^F \mathbf{x}'^F) + \operatorname{Div}(\varrho^F \mathbf{x}'^F \otimes \mathbf{X}'^F - \mathbf{P}^F) + \mathbf{p}^* - \varrho^F \mathbf{b}^F \right) \leq 0
$$

should hold for *arbitrary fields*. The multipliers are functions of the same constitutive variables as all other constitutive functions, i.e.

$$
(5.4) \qquad
\begin{aligned}
\Lambda^\varrho &= \Lambda^\varrho(\mathcal{C}), & \Lambda^n &= \Lambda^n(\mathcal{C}), \\
\mathbf{L}^S &= \mathbf{F}^S \mathbf{L}_0^S(\mathcal{C}), & \mathbf{L}^F &= \mathbf{F}^S \mathbf{L}_0^F(\mathcal{C}).
\end{aligned}
$$

The solutions of the above inequality are constructed in two different ways. In early 60-ies B.D. COLEMAN has proposed the method in which it was assumed that the class of volume forces was large enough to accomodate arbitrary changes of the other terms in the momentum balance equations. This means that these equations do not constrain the class of solutions of the entropy inequality. In such a case

$$(5.5) \qquad \mathbf{L}^S \equiv 0, \qquad \mathbf{L}^F \equiv 0.$$

However, if the class of volume forces is not large enough (e.g. if $\mathbf{b}^S = \mathbf{b}^F$ as it is the case for the gravitational forces), the inequality must be exploited by the absence of these forces. This has been investigated for the first time by

I. MÜLLER in 70-ies. It can be easily shown that the second way is less restrictive for the multicomponent media and both methods yield the same results for the single-component continua.

For the purpose of this work we rely on the COLEMAN's method. Consequently the results remain on the safe side as far as the thermodynamic restrictions are concerned.

Bearing in mind the constitutive relations (4.14) and (5.2) and making use of the chain rule of differentiation in (5.3) we obtain the inequality which is linear with respect to the following derivatives

$$(5.6) \qquad \left\{ \frac{\partial \varrho^F}{\partial t}, \frac{\partial n}{\partial t}, \text{Grad}\, \varrho^F, \text{Grad}\, n, \frac{\partial \mathbf{X}'^F}{\partial t}, \text{Grad}\, \mathbf{F}^S, \text{Grad}\, \mathbf{X}'^F \right\}.$$

Consequently the inequality can hold for arbitrary fields solely in the case when the coefficients of these derivatives vanish. We arrive at the set of the following identities

$$\Lambda^\varrho = \varrho^S \frac{\partial \Psi^S}{\partial \varrho^F} + \varrho^F \frac{\partial \Psi^F}{\partial \varrho^F}, \qquad \Lambda^n = \varrho^S \frac{\partial \Psi^S}{\partial n} + \varrho^F \frac{\partial \Psi^F}{\partial n},$$

$$(5.7) \qquad \varrho^S \frac{\partial \Psi^S}{\partial \varrho^F} \mathbf{X}'^F + \Lambda^n \frac{\partial \mathbf{J}}{\partial \varrho^F} = 0, \qquad \varrho^F \frac{\partial \Psi^F}{\partial n} \mathbf{X}'^F + \Lambda^n \frac{\partial \mathbf{J}}{\partial n} = 0,$$

$$\varrho^S \frac{\partial \Psi^S}{\partial \mathbf{X}'^F} + \varrho^F \frac{\partial \Psi^F}{\partial \mathbf{X}'^F} = 0,$$

$$\mathbf{P}^S + \mathbf{P}^F = 2\mathbf{F}^S \left(\varrho^S \frac{\partial \Psi^S}{\partial \mathbf{C}^S} + \varrho^F \frac{\partial \Psi^F}{\partial \mathbf{C}^S} \right),$$

$$(5.8) \qquad \mathbf{F}^{ST} \mathbf{P}^F = -\varrho^F \Lambda^\varrho \mathbf{1} + \varrho^F \frac{\partial \Psi^F}{\partial \mathbf{X}'^F} \otimes \mathbf{X}'^F - \Lambda^n \left(\frac{\partial \mathbf{J}}{\partial \mathbf{X}'^F} \right)^T,$$

$$\text{sym}^{23} \mathbf{P}^F \otimes \mathbf{X}'^F = \text{sym}^{23} \left[2\mathbf{F}^S \left(\varrho^F \frac{\partial \Psi^F}{\partial \mathbf{C}^S} \otimes \mathbf{X}'^F - \Lambda^n \left(\frac{\partial \mathbf{J}}{\partial \mathbf{C}^S} \right)^{T13} \right) \right].$$

There remains the *residual inequality* which defines the *dissipation* $\mathcal{D}$ of the process

$$(5.9) \qquad \mathcal{D} \equiv \mathbf{F}^{ST} \mathbf{p}^* \cdot \mathbf{X}'^F - \Lambda^n \nu \geq 0.$$

The above relations determine the Lagrange multipliers, relate partial stress tensors to the partial Helmholtz free energies and to the flux $\mathbf{J}$ and introduce certain additional restrictions on the constitutive relations. We do not try to exploit these results in their full generality and restrict our attention to the particular case of the *isotropic porous media*. This is the subject of the next section.

6. Isotropy

The assumption of the isotropy does not seem to limit the applicability of the present model very considerably because we have already assumed the porosity to be described by the volume fraction. Such an assumption eliminates any influence of the geometrical anisotropy of the pore structure from the model. In this respect the full isotropy assumption concerns solely the mechanical responses of the skeleton and reactions to the relative motion.

The constitutive relations for scalar functions of the isotropic medium must be invariant with respect to an arbitrary orthogonal transformation of the reference configuration. In our model there are three scalar functions (see: (4.12) and (5.2))

$$(6.1) \qquad \left\{ \nu, \Psi^S, \Psi^F \right\},$$

and these functions of constitutive variables (4.13) satisfy the above requirement if they depend on these variables solely through their invariants

$$(6.2) \qquad \mathcal{C}_{\mathrm{iso}} = \left\{ \varrho^F, n^F, \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI} \right\},$$

where

$$(6.3) \qquad \begin{aligned} \mathrm{I} &= \mathbf{1} \cdot \mathbf{C}^S, & \mathrm{II} &= \frac{1}{2}(\mathrm{I}^2 - \mathbf{1} \cdot \mathbf{C}^{S2}), \\ \mathrm{III} &= \det \mathbf{C}^S \equiv J^{S2}, & \mathrm{IV} &= \mathbf{X}'^F \cdot \mathbf{X}'^F, \\ \mathrm{V} &= \mathbf{C}^S \cdot (\mathbf{X}'^F \otimes \mathbf{X}'^F), & \mathrm{VI} &= \mathbf{C}^{S2} \cdot (\mathbf{X}'^F \otimes \mathbf{X}'^F). \end{aligned}$$

Simultaneously the model contains two vector constitutive functions for which the general isotropic representation is of the following form

$$(6.4) \qquad \begin{aligned} \mathbf{J} &= (\Phi_0 \mathbf{1} + \Phi_1 \mathbf{C}^S + \Phi_2 \mathbf{C}^{S2}) \mathbf{X}'^F, \\ \mathbf{F}^{ST} \mathbf{p}^* &= (\pi_0 \mathbf{1} + \pi_1 \mathbf{C}^S + \pi_2 \mathbf{C}^{S2}) \mathbf{X}'^F. \end{aligned}$$

In the above relations the coefficients are arbitrary isotropic scalar functions, i.e.

$$(6.5) \qquad \Phi_a = \Phi_a(\mathcal{C}_{\mathrm{iso}}), \qquad \pi_a = \pi_a(\mathcal{C}_{\mathrm{iso}}), \qquad a = 0, 1, 2.$$

Further we do not need the isotropic constitutive relations for the partial stress tensors because these follow from the identities (5.8) whose right-hand sides are determined by the isotropic scalar and vector functions.

Bearing in mind the thermodynamic relations (5.6) and (5.7) as well as the symmetries of the partial Cauchy stress tensors $\mathbf{T}^S$, $\mathbf{T}^F$ we obtain the following results.

The flux in the balance equation for the porosity must be parallel to the Lagrangian velocity $\mathbf{X}'^F$

$$(6.6) \qquad\qquad \mathbf{J} = \Phi_0 \mathbf{X}'^F, \qquad \Phi_1 = \Phi_2 \equiv 0.$$

The dependence of the Helmholtz free energies and of the coefficient Φ_0 on the invariants is restricted by the relations

$$(6.7)\quad
\begin{aligned}
&\varrho^S \frac{\partial \Psi^S}{\partial \varrho^F} + \Lambda^n \frac{\partial \Phi_0}{\partial \varrho^F} = 0,\\[4pt]
&\varrho^F \frac{\partial \Psi^F}{\partial \mathcal{A}_1} - \Lambda^n \frac{\partial \Phi_0}{\partial \mathcal{A}_1} = 0, \qquad \mathcal{A}_1 = n, \mathrm{I}, \mathrm{II}, \mathrm{IV}, \mathrm{VI},\\[4pt]
&\varrho^F \left(\varrho^F \frac{\partial \Psi^F}{\partial \varrho^F} + 2\mathrm{III} \frac{\partial \Psi^F}{\partial \mathrm{III}} \right)\\[4pt]
&\qquad\qquad - \Lambda^n \sqrt{\mathrm{III}} \left[\varrho^F \frac{\partial}{\partial \varrho^F} \left(\frac{\Phi_0}{\sqrt{\mathrm{III}}} \right) + 2\mathrm{III} \frac{\partial}{\partial \mathrm{III}} \left(\frac{\Phi_0}{\sqrt{\mathrm{III}}} \right) \right] = 0,
\end{aligned}$$

where the multiplier Λ^n is given by the relation $(5.7)_2$. Simultaneously

$$(6.8) \qquad\qquad \varrho^S \frac{\partial \Psi^S}{\partial \mathcal{A}_2} + \varrho^F \frac{\partial \Psi^F}{\partial \mathcal{A}_2} = 0, \qquad \mathcal{A}_2 = \mathrm{IV}, \mathrm{V}, \mathrm{VI}.$$

The Piola–Kirchhoff partial stress tensors have the following form

$$(6.9)_1 \quad
\begin{aligned}
\mathbf{P}^F = &-\left[\varrho^F \left(\varrho^F \frac{\partial \Psi^F}{\partial \varrho^F} + \varrho^S \frac{\partial \Psi^S}{\partial \varrho^F} \right) + \Lambda^n \Phi_0 \right] \mathbf{F}^{S-T}\\[4pt]
&+ 2\left[\varrho^F \frac{\partial \Psi^F}{\partial \mathbf{V}} + \Lambda^n \frac{\partial \Phi_0}{\partial \mathbf{V}} \right] \mathbf{F}^S (\mathbf{X}'^F \otimes \mathbf{X}'^F),
\end{aligned}$$

$$(6.9)_2 \quad
\begin{aligned}
\mathbf{P}^S = 2\mathbf{F}^{S-T} \Bigg\{ &\varrho^S \left[\frac{\partial \Psi^S}{\partial \mathrm{I}} \mathbf{C}^S + \left(\mathrm{II} \frac{\partial \Psi^S}{\partial \mathrm{II}} + \mathrm{III} \frac{\partial \Psi^S}{\partial \mathrm{III}} \right) \mathbf{1} \right.\\[4pt]
&\qquad\qquad \left. - \mathrm{III} \frac{\partial \Psi^S}{\partial \mathrm{II}} \mathbf{C}^{S-1} + 2\frac{\partial \Psi^S}{\partial \mathbf{V}} \mathbf{C}^S \mathbf{X}'^F \otimes \mathbf{X}'^F \right]\\[4pt]
&+ \Lambda^n \left[\frac{\partial \Phi_0}{\partial \mathrm{I}} \mathbf{C}^S + \left(\mathrm{II} \frac{\partial \Phi_0}{\partial \mathrm{II}} + \mathrm{III} \frac{\partial \Phi_0}{\partial \mathrm{III}} \right) \mathbf{1} - \mathrm{III} \frac{\partial \Phi_0}{\partial \mathrm{II}} \mathbf{C}^{S-1} \right.\\[4pt]
&\qquad\qquad\qquad \left. + 2\frac{\partial \Psi_0}{\partial \mathbf{V}} \mathbf{C}^S \mathbf{X}'^F \otimes \mathbf{C}^S \mathbf{X}'^F \right] \Bigg\}.
\end{aligned}$$

The proofs of these relations are rather technical; they are based on the spectral representation of the deformation tensor and of the Lagrangian velocity. They shall not be quoted in the present paper. The details can be found in the work [25].

In spite of the complexity of the above relations, some important properties of the isotropic model are immediately seen.

First of all the relations (6.6) yield the considerable simplification of the additional field equation (4.11) of the model. The collinearity of the flux of porosity and of the relative Lagrangian velocity of components couples the diffusion processes with this surface mechanism of changes of the porosity which is absent in the models based on the evolution equations for porosity. This property simplifies as well the problem of an additional boundary condition which is necessary for this field equation in the fully nonlinear case of the present model. The latter problem shall not be discussed in this work.

Simultaneously the scalar coefficient Φ_0 in the relation for the flux $\mathbf{J}$ plays the crucial role in the "static" coupling between the components. This "static" coupling is understood as the description of the interactions between components reflected by the dependence of the free energy of the fluid Ψ^F on the deformation of the skeleton through the invariants I, II, III of the Cauchy–Green deformation tensor, as well as the dependence of the free energy of the skeleton Ψ^S on the mass density of the fluid ϱ^F. The former is easily seen in the relations $(6.7)_{2,3}$ and the latter in the relations $(6.7)_1$. The additional most important "static" coupling is reflected by the dependence of both partial free energies on the current values of the porosity n. The dependence on n of at least one of these energies is necessery for the non-triviality of the relation $(5.7)_2$ for the multiplier Λ^n. This multiplier is solely responsible for the additional static interaction terms in all relations quoted above. For instance in the case of lack of diffusion, the vanishing multiplier Λ^n would yield the classical relation for the stress tensor in the one-component ideal fluid and the classical relation for the stress tensor in the one-component nonlinear elastic solid. In addition, all these interactions of components are described by the model independently of the fact whether the particular process is connected with the relative motion of components or not.

The above relations for stresses show also a rather complicated influence of the relative velocity on the mechanical responses of the two-component medium. Quite clearly this influence is at least quadratic. This means that the small diffusion velocity yields primarily the explicit linear dependence of the diffusion forces (momentum source) p^* and of the porosity flux $\mathbf{J}$ on this velocity, and the partial stresses contain solely the influence of the static interactions of components. In such a case the partial Cauchy stress tensor for the fluid component is reduced to the spherical form (pressure!).

Let us finally mention that the residual inequality (5.9) is in the isotropic case of the following form

(6.10)

Obviously the first term of this dissipation inequality describes the dissipation

due to the diffusion, and the second one – due the changes of porosity caused by the source ν in the field equation of porosity. In the thermodynamical equilibrium the relative velocity as well as the porosity source must vanish. These are the two mechanisms of the thermodynamical relaxation in the present model.

Let us briefly review the above results for the general case. The thermodynamic admissibility and the isotropy reduce the constitutive problem of the model to the following scalar constitutive functions

$$(6.11) \qquad \left\{ \Psi^S, \Psi^F, \Phi_0, \pi_0, \pi_1, \pi_2, \mathcal{N} \right\},$$

which, in general, may depend on the constitutive variables (6.2) and are subject to the conditions (6.7), (6.8) as well as (6.10). The vector fluxes and the stress tensors are then determined by these functions through the appropriate differentiation. Further in this paper we discuss some possibilities of the effective construction of these functions for certain real porous materials.

7. Simplified nonlinear models

The purpose of this section is the construction of some simplified models based on the general considerations of the sixth section. We shall not discuss all important particular cases because the research on this subject is still in progress. We want solely to illustrate the connection of the general mechanical model of large deformations of the porous two-component medium with some other models whose range of applicability is more restricted and with observations of some real materials.

We begin with the assumption that processes deviate not too far from the *thermodynamical equilibrium*. The latter is defined as the state with the vanishing dissipation. According to the inequality (6.10) we have in such a state

$$(7.1) \qquad \mathbf{X}'^F|_E = 0, \qquad \nu|_E = 0, \qquad \to n|_E = n_0 = \text{const}.$$

The above assumption means then that the relative velocity of components is small and the deviation of the porosity from the homogeneous initial state n_0 is small as well. In this approximation

$$\nu = -\frac{\mathcal{N}}{\varrho^F} \left(\varrho^S \frac{\partial \Psi^S}{\partial n} + \varrho^F \frac{\partial \Psi^F}{\partial n} \right),$$

$$(7.2)$$

$$\mathcal{N} = \mathcal{N}(n_0, \mathrm{I}, \mathrm{II}, \mathrm{III}, \varrho^F) \geq 0,$$

and the functions π_0, π_1, π_2 must be dependent on the same variables as $\mathcal{N}$.

Simultaneously the state of the thermodynamical equilibrium is the state in which the dissipation reaches its minimum. Consequently

$$(7.3) \qquad \frac{\partial}{\partial n}(\varrho^S \Psi^S + \varrho^F \Psi^F)|_{n=n_0} = 0, \qquad \frac{\partial^2}{\partial n^2}(\varrho^S \Psi^S + \varrho^F \Psi^F)|_{n=n_0} > 0.$$

Bearing in mind the identities (6.7) in the first approximation of the deviation from the state of equilibrium, we obtain after easy calculations

$$\Psi^S = \Psi_0^S + \frac{1}{2}\Psi_2^S(n - n_0)^2,$$

(7.4)
$$\Psi^F = \Psi_0^F + \frac{1}{2}\Psi_2^F(n - n_0)^2,$$

$$\Phi_0 = \left[\gamma(n_0) + \Phi_0^1(n - n_0)\right]\sqrt{\text{III}},$$

where

(7.5)
$$\Psi_0^S = \Psi_0^S(n_0, \text{I}, \text{II}, \text{III}), \qquad \Psi_0^F = \Psi_0^F(n_0, \varrho_t^F),$$

$$\varrho_t^F \equiv \varrho^F \text{III}^{-1/2},$$

and

(7.6) $$\nu = -\frac{n - n_0}{\tau}, \qquad \tau \equiv \frac{\varrho^F}{\mathcal{N}}(\varrho^S\Psi_2^S + \varrho^F\Psi_2^F)^{-1}, \qquad \Lambda^n = \frac{\varrho^F}{\tau\mathcal{N}}(n - n_0).$$

The material parameter τ has the interpretation of the *relaxation time* of the porosity and, according to the condition $(7.3)_2$ of the *stability of the thermodynamic equilibrium*, it must be positive. It creates the damping of the acoustic waves in addition to the damping connected with the diffusion.

Simultaneously

(7.7)
$$\frac{1}{2}\varrho^S\frac{\partial\Psi_2^S}{\partial\varrho_t^F} + \frac{\varrho^F}{\tau\mathcal{N}}\sqrt{\text{III}}\frac{\partial\Phi_0^1}{\partial\varrho_t^F} = 0,$$

$$\frac{1}{2}\frac{\partial\Psi_2^F}{\partial\mathcal{A}_3} - \frac{1}{\tau\mathcal{N}}\sqrt{\text{III}}\frac{\partial\Phi_0^1}{\partial\mathcal{A}_3} = 0, \qquad \mathcal{A}_3 = \text{I}, \text{II}, \text{III}.$$

It is convenient to use further the spatial representation of the constitutive variables and functions. In order to do so we introduce the current mass density ϱ_t^S, the left Cauchy–Green deformation tensor $\mathbf{B}^S$ and the real relative velocity $\mathbf{w}$

(7.7)′ $$\varrho_t^S = \varrho^S J^{S-1}, \qquad \varrho_t^F = \varrho^F J^{S-1}, \qquad \mathbf{B}^S = \mathbf{F}^S\mathbf{F}^{ST}, \qquad \mathbf{w} = (\mathbf{x}'^F - \mathbf{x}'^S).$$

The invariants I, II, III are certainly identical for the tensor $\mathbf{B}^S$ with those of the tensor $\mathbf{C}^S$, and the invariants IV, V, VI are immaterial under the present simplifying assumption of the small deviation from the thermodynamical equilibrium.

The partial Cauchy stress tensors, related to the Piola–Kirchhoff stress tensors by the relations (4.9), are in this case given by the following relations

(7.8)
$$\mathbf{T}^S = \mathcal{I}_1\mathbf{B}^S + \mathcal{I}_0\mathbf{1} + \mathcal{I}_{-1}\mathbf{B}^{-1},$$

$$\mathbf{T}^F = -p^F\mathbf{1},$$

where

(7.9)

and

(7.10)
$$p^F = (\varrho_t^F)^2 \frac{\partial \Psi_t^F}{\partial \varrho_t^F} + \gamma \varrho^F \frac{n - n_0}{\tau \mathcal{N}}.$$

We have used the relations (7.5) and neglected terms quadratic in the deviation of the porosity n from its equilibrium value n_0. The latter causes the symmetry of interactions in the partial stress tensors.

The similarity of the relations (7.8) to the classical relations for nonlinear elastic materials and for the ideal fluids is, certainly, only apparent. The *response coefficient* $\mathcal{I}_0$ depends in the present case not only on the deformation invariants I, II, III, as it is the case in the classical one-component model but also on the mass density ϱ_t^F and on the porosity n. Simultaneously the *partial pressure* in the fluid p^F depends not only on the current mass density ϱ_t^F but also on the invariants I, II, III and on the porosity n. Crucial for this coupling of components is the presence of the constant γ which is the part of the flux of porosity as well as the presence of the two additional material parameters τ and $\mathcal{N}$, both of them connected with the changes of porosity.

Let us finally mention two other simplified models which may have the practical bearing on the soil mechanics. In both models we assume the linearity with respect to the diffusion velocity.

The first one follows from the assumption that the balance equation for the porosity (4.11) reduces to the evolution equation which describes the changes of the porosity along trajectories of the fluid. Then

(7.11) $\Phi_0 \approx n,$ i.e. $\gamma(n_0) = n_0,$ $\Phi_0^1 = 1,$ III $\approx 1.$

In such a case the identities (6.7) yield

(7.12)
$$\Psi^S = \Psi^S(n_0, \mathrm{I}, \mathrm{II}, \mathrm{III}),$$
$$\Psi^F = \Psi^F(n_0, \varrho_t^F, \kappa), \qquad \kappa \equiv nJ^{S-1},$$
$$\Lambda^n = \varrho_t^F \frac{\partial \Psi^F}{\partial \kappa},$$

and the partial Cauchy stress tensors have the form

(7.13)
$$\mathbf{T}^S = 2\varrho_t^S \frac{\partial \Psi^S}{\partial \mathrm{I}} \mathbf{B}^S + 2\varrho_t^S \left(\mathrm{II} \frac{\partial \Psi^S}{\partial \mathrm{II}} + \mathrm{III} \frac{\partial \Psi^S}{\partial \mathrm{III}} \right) \mathbf{1} - 2\mathrm{III}\varrho_t^S \frac{\partial \Psi^S}{\partial \mathrm{II}} \mathbf{B}^{S-1},$$

$$\mathbf{T}^F = - \left[(\varrho_t^F)^2 \frac{\partial \Psi^F}{\partial \varrho_t^F} + \varrho_t^F \frac{\partial \Psi^F}{\partial \kappa} n \right] \mathbf{1}.$$

Hence the interaction of components is not symmetric in this case. The changes of porosity influence the stresses in the fluid but not in the skeleton.

The second simplified model follows from the assumption that the evolution equation of porosity is carried by the skeleton, i.e.

(7.14) $$\Phi_0 \approx 0.$$

According to the identities (6.7) we obtain then

(7.15)
$$\Psi^S = \Psi^S(n, \mathrm{I}, \mathrm{II}, \mathrm{III}),$$
$$\Psi^F = \Psi^F(n_0, \varrho_t^F),$$
$$\Lambda^n = \varrho^S \frac{\partial \Psi^S}{\partial n},$$

and the partial Cauchy stresses are

(7.16)
$$\mathbf{T}^S = 2\varrho_t^S \frac{\partial \Psi^S}{\partial \mathrm{I}} \mathbf{B}^S + 2\varrho_t^S \left(\mathrm{II} \frac{\partial \Psi^S}{\partial \mathrm{II}} + \mathrm{III} \frac{\partial \Psi^S}{\partial \mathrm{III}} \right) \mathbf{1} - 2\mathrm{III}\varrho_t^S \frac{\partial \Psi^S}{\partial \mathrm{II}} \mathbf{B}^{S-1},$$

$$\mathbf{T}^F = - \left[(\varrho_t^F)^2 \frac{\partial \Psi^F}{\partial \varrho_t^F} \right] \mathbf{1}.$$

Consequently the interaction of components is again non-symmetric. The changes of porosity influence solely stresses in the skeleton through the dependence of the free energy Ψ^S on the porosity.

We rest here as far as the discussion of the construction of nonlinear models is concerned. In the next section we present briefly the boundary conditions which are necessary to pose the mathematical problem for the field equations. Some physical properties of various models will be discussed in connection with the wave propagation.

8. Boundary value problems, permeable boundary of the skeleton

The set of field equations for the fields (4.6) requires – similarly to the mixture theory – *two vector conditions* on the boundary, connected with the vector

equations following from the momentum balance laws and, in general, *one scalar condition* for the scalar balance equation of porosity. The latter may not appear in particular cases when the coefficient of the flux of porosity Φ_0 is identical with n itself. It is easy to show that it may appear at least in two cases. The first one concerns the skeleton whose interactions with the fluid vanish entirely from the Helmholtz free energy Ψ^S as discussed in the previous section. This seems to appear for some rocks in the range of moderate porosities. The second one follows from the relation $(7.4)_3$ as the approximation of the small volume changes of the skeleton: $III \cong 1$, $\gamma(n_0) = n_0$ and $\Phi_0^1 = 1$. In both cases the stress tensor in the skeleton does not contain contributions from the fluid – it is indeed purely elastic. We skip here the details justifying these assumptions in some practical applications whose main purpose is to estimate the order of magnitude of the new material parameters. We shall accept them however in examples to be considered further in this paper. The general case has not been considered as yet.

In addition to the above boundary conditions one has to describe the motion of the *free surface* if the fluid flows out of the porous skeleton and the boundary is identified with the boundary of the skeleton. We proceed to present some elements of these problems.

Let us begin with the so-called *dynamical compatibility conditions*. These are the jump conditions for fields and their functions which follow from the general balance equations in the limit on singular surface. The derivation is standard and we shall not present here any details.

In order to simplify the considerations let us assume that the surface is *material with respect to the skeleton*. This means that its velocity is identically zero in the Lagrangian image used in the work. The general case has been considered in the paper [11]. Then the mass balance for the skeleton does not yield any non-trivial conditions. The mass balance of the fluid (4.2) leads to the following relation

(8.1)
$$m^F \equiv (\varrho^F \mathbf{X}'^F)^- \cdot \mathbf{N} = (\varrho^F \mathbf{X}'^F)^+ \cdot \mathbf{N}, \qquad \text{i.e.}$$

$$\left[\left[\varrho^F \mathbf{X}'^F\right]\right] \cdot \mathbf{N} = 0, \qquad [[\ldots]] \equiv (\ldots)^+ - (\ldots)^-,$$

where $(\ldots)^-$ is the limit of the expression in parenthesis from the negative side of the surface (this is the internal side of the surface if the surface is the boundary), and $(\ldots)^+$ is the limit from the positive side (the exterior for the boundary) for the other quantity. The quantity m^F describes the amount of the mass of the fluid which flows through the unit surface in the unit time. According to the above condition, the mass is neither produced nor does it sink on the surface. Such surfaces are called *ideal*.

The momentum balance equations (4.8) yield the following conditions

(8.2)
$$\left[\left[\mathbf{P}^S\right]\right] \cdot \mathbf{N} = 0,$$

$$\left[\left[\mathbf{P}^F\right]\right] \mathbf{N} = m^F \left[\left[\mathbf{x}'^F\right]\right],$$

where the first condition does not differ from the classical *Poisson's condition* of continuity of the stress vector in the skeleton. The presence of the right-hand side in the relation for the fluid means that due to the non-material character of the surface, it is not the ideal surface for the fluid with respect to the convective transport of momentum.

The surface balance for porosity is determined by the equation (4.11) which holds for an arbitrary regular point but can be easily written in the integral form and then extended to hold also in the limit on the singular surface. The corresponding jump condition is then of the form

$$(8.3) \qquad \left[\left[\Phi_0 \mathbf{X}'^F\right]\right] \cdot \mathbf{N} = 0.$$

We shall not discuss this problem any further in this work.

We proceed now to formulate the boundary conditions for the boundary of the skeleton on which the external load is given and the boundary is permeable for the fluid. Many details concerning this problem as well as its applications in the weak formulation and numerical codes for the two-component porous media can be found in the work of W. KEMPA [30].

The first vector boundary condition follows from the assumption that the external load, say $\mathbf{t}_{\mathrm{ext}}$, is given on the boundary of the skeleton $\partial \mathcal{B}$. We assume that this load is taken over by the resultant stress vector of both components on the positive side of the boundary, i.e.

$$(8.4) \qquad \left\{ (\mathbf{P}^S \mathbf{N})^- + (\mathbf{P}^F \mathbf{N})^- + m^F \left[\left[\mathbf{x}'^F\right]\right] \right\} |_{\partial \mathcal{B}} = \mathbf{t}_{\mathrm{ext}},$$

where the sum of the dynamic compatibility conditions (8.2) has been used. Apart from the limits of fields from the interior, this relation contains as well the unspecified quantity $(\mathbf{x}'^F)^+$. We relate this vector to a scalar quantity in the sequel (Eq. $(8.6)_1$).

In order to expose the most essential feature we consider the second vector condition under the additional assumption that the Cauchy stress tensor in the fluid is spherical, i.e. we neglect the higher order contributions of the relative velocity. In such a case we can assume that the tangential component of the relative velocity is continuous on the boundary of the skeleton and the fluid does not flow tangentially to the skeleton in the exterior. In the Lagrangian description we have then

$$(8.5) \qquad (\mathbf{X}'^F - (\mathbf{X}'^F \cdot \mathbf{N})\mathbf{N})^- |_{\partial \mathcal{B}} = 0.$$

Solely two components of this vector are independent. For this reason we need in addition one scalar condition. We formulate this condition assuming that the flow of the fluid m^F on the boundary of the skeleton is controlled by the pressure

difference between the fluid inside of the porous material $(p^F)^-$ and the pressure of the surrounding p_{ext}. Consequently

(8.6)
$$\left[\left[x'^F\right]\right] = (C^{S-1} \cdot (N \otimes N))^{1/2} m^F \left[\left[\frac{1}{\varrho^F}\right]\right] n,$$
$$m^{F2} \left[\left[\frac{1}{\varrho^F}\right]\right] = -J^S C^{S-1} \cdot (N \otimes N) \left[\left[p^F\right]\right],$$
$$m^F = -\alpha(p_{ext} - p^{F-}),$$

where the relation (8.4) has been accounted for, as well as the following relations for the unit vector n normal to the boundary in the current configuration [11] and for the pressure in the fluid have been used

(8.7)
$$n = (C^{S-1} \cdot (N \otimes N))^{-1/2} F^{S-T} N, \qquad p^F = -\frac{1}{3} J^S (P^F F^{ST}) \cdot 1,$$

and the parameter α is constitutive. If this parameter as well as the mass density $(\varrho^F)^+$ and the pressure p_{ext} were known, the relations (8.6) would complete the formulation of the boundary value problem. We shall not go into any further details referring an interested reader to the work [30]. Let us solely notice that the *constitutive relation* for the boundary $(8.6)_3$ does not contain any influence of the pressure gradient projected on the normal to the boundary. Sometimes it seems to be necessery to have this type of condition. The linear combination of the jump of pressure and such a normal derivative would yield the boundary condition similar to that appearing in the heat conduction problems with the boundary characterised by its own thermal conductivity.

9. Acceleration waves in two-component media

The model constructed above in this paper contains a number of constitutive quantities which must be measured in experiments. In the case of porous materials such experiments are usually of the two different types. Either the measurements are done by means of devices which are in contact with real components or they are delivering the mean quantities in which the contribution of separate real components is not clearly specified. To the first type belong, for instance, the measurements of true mass densities of components separated from each other or the measurements of the real pore water pressure. The most important class of measurements of the second class are the measurements of speeds of propagation and the attenuation of acoustic waves in porous materials. The waves propagate in the multicomponent porous media and they deliver an information on the whole system rather than on separate real components. Many examples of such measurements can be found in the book of T. Bourbie, O. Coussy, B. Zinszner [5].

In this section we present the most fundamental properties of acoustic waves described by the present two-component model. We follow here the papers [19, 27, 28]) where also the extensive discussion and the comparison with the experimental data can be found.

Within the continuum mechanics the *acoustic wave* is defined as the so-called *weak discontinuity wave* in which the motion and the velocities are continuous and the accelerations suffer the jump on a singular surface. This surface is assumed to be orientable and it is called the *wave front*. It moves through the material with the *speed of propagation of the wave*.

According to the above definition we assume in the case of the two-component medium the following relations to hold on the wave front of the acoustic wave

$$(9.1) \qquad \left[\!\left[\boldsymbol{\chi}^S\right]\!\right] = 0, \qquad \left[\!\left[\mathbf{x}'^{S}\right]\!\right] = 0, \qquad \left[\!\left[\mathbf{x}'^{F}\right]\!\right] = 0, \qquad [[n]] = 0.$$

Under these conditions the so-called *iterated geometrical* and *kinematical compatibility conditions* yield

$$
\begin{aligned}
&\left[\!\left[\mathbf{F}^S\right]\!\right] = 0, \qquad \left[\!\left[\mathbf{X}'^{S}\right]\!\right] = 0, \qquad \left[\!\left[\varrho^F\right]\!\right] = 0, \\[2mm]
&\left[\!\left[\mathbf{P}^S\right]\!\right]\mathbf{N} = 0, \qquad \left[\!\left[\mathbf{P}^F\right]\!\right]\mathbf{N} = 0, \\[2mm]
&\left[\!\left[\frac{\partial \mathbf{x}'^{S}}{\partial t}\right]\!\right] = \mathbf{a}^S U^2, \qquad \left[\!\left[\mathrm{Grad}\,\mathbf{F}^S\right]\!\right] = \mathbf{a}^S \otimes \mathbf{N} \otimes \mathbf{N}, \\[2mm]
&\left[\!\left[\frac{\partial \mathbf{F}^S}{\partial t}\right]\!\right] = -U\mathbf{a}^S \otimes \mathbf{N}, \\[2mm]
(9.2)\qquad &\left[\!\left[\frac{\partial \mathbf{x}'^{F}}{\partial t}\right]\!\right] = \mathbf{a}^F U^2, \qquad \left[\!\left[\mathrm{Grad}\,\mathbf{x}'^{F}\right]\!\right] = -U\mathbf{a}^F \otimes \mathbf{N}, \\[2mm]
&\left[\!\left[\mathrm{Grad}\,\mathbf{X}'^{F}\right]\!\right] = (U - \mathbf{X}'^{F}\cdot\mathbf{N})\mathbf{F}^{S-1}\mathbf{a}^S \otimes \mathbf{N} - U\mathbf{F}^{S-1}\mathbf{a}^F \otimes \mathbf{N}, \\[2mm]
&\left[\!\left[\frac{\partial \varrho^F}{\partial t}\right]\!\right] = -Ur, \qquad \left[\!\left[\mathrm{Grad}\,\varrho^F\right]\!\right] = r\mathbf{N}, \qquad \left[\!\left[\frac{\partial n}{\partial t}\right]\!\right] = -Un, \\[2mm]
&[[\mathrm{Grad}\,n]] = n\mathbf{N},
\end{aligned}
$$

where $\mathbf{N}$ denotes the unit normal vector to the wave front and $\mathbf{a}^S$, $\mathbf{a}^F$, r and n denote the so-called *amplitudes* of discontinuity of the acceleration in the skeleton, the acceleration in the fluid, the fluid mass density gradient and the porosity gradient, respectively. The *speed of propagation* of the wave front is denoted by U.

In order to find the speed of propagation U and the relation between the direction of the amplitude and the direction of propagation, it is now sufficient to evaluate the limits of field equations on both sides of the wave front. This evaluation for the mass balance in the fluid (4.7) and for the balance equation of

porosity (4.11) yield

$$rU\left(1 - \frac{X_N'^F}{U}\right) - \varrho^F U\left(1 - \frac{X_N'^F}{U}\right)\mathbf{F}^{S-T}\cdot(\mathbf{a}^S \otimes \mathbf{N})$$

$$+ \varrho^F U \mathbf{F}^{S-T}\cdot(\mathbf{a}^F \otimes \mathbf{N}) = 0,$$

(9.3) $\quad nU\left(1 - \frac{\partial\Phi_0}{\partial n}\frac{X_N'^F}{U}\right) - \frac{\partial\Phi_0}{\partial\varrho^F}rX_N'^F - 2X_N'^F\frac{\partial\Phi_0}{\partial\mathbf{C}^S}\cdot(\mathbf{F}^{ST}\mathbf{a}^S \otimes \mathbf{N})$

$$-\Phi_0 U\left(1 - \frac{X_N'^F}{U}\right)\mathbf{F}^{S-T}\cdot(\mathbf{a}^S \otimes \mathbf{N}) + \Phi_0 U \mathbf{F}^{S-T}\cdot(\mathbf{a}^F \otimes \mathbf{N}) = 0,$$

$$X_N'^F = \mathbf{X}'^F\cdot\mathbf{N}.$$

In most cases of the practical bearing the relative velocity of components is much smaller than the smallest speed of propagation of the acoustic wave. For this reason we can make the simplifying assumption

(9.4) $$\left|\frac{X_N'^F}{U}\right| \ll 1;$$

the usual order of magnitude of the left-hand side is 10^{-4}. If all other terms in the relations (9.3) are of the same order of magnitude then we have approximately

(9.5) $\quad r = \varrho^F \mathbf{F}^{S-T}\cdot(\mathbf{a}^S - \mathbf{a}^F)\otimes\mathbf{N}, \qquad n = \Phi_0 \mathbf{F}^{S-T}\cdot(\mathbf{a}^S - \mathbf{a}^F)\otimes\mathbf{N}.$

Hence the amplitudes of the mass density gradient in the fluid and the amplitude of the porosity gradient are determined by the amplitudes of the acceleration. They do not yield their own waves and are carried by the other sorts of waves. This would not be the case if we did not make the simplifying assumption (9.4). A rather unusual type of waves appears if we make a better approximation (see: [28]) but there is no experimental evidence that such waves do indeed exist.

We proceed now to investigate the momentum balance equations (4.10) from both sides of the wave front. We limit the attention to the case of small relative velocities for which the Cauchy stress tensor in the fluid is spherical (see: (7.2)). Then bearing in mind the simplification (9.4) and the remaining constitutive assumptions we obtain easily

$$\varrho^S \mathbf{a}^S U^2 = J^S(\mathbf{F}^{S-T}\cdot(\mathbf{a}^S - \mathbf{a}^F)\otimes\mathbf{N})\left\{\varrho^F\frac{\partial\mathbf{T}^S}{\partial\varrho^F} + \Phi_0\frac{\partial\mathbf{T}^S}{\partial n}\right\}\cdot(\mathbf{F}^{S-T}\mathbf{N}) + \mathbf{Q}^S\mathbf{a}^S,$$

(9.6) $\quad \varrho^F \mathbf{a}^F U^2 = -J^S\left\{(\mathbf{F}^{S-T}\cdot(\mathbf{a}^S - \mathbf{a}^F)\otimes\mathbf{N})\left(\varrho^F\frac{\partial p^F}{\partial\varrho^F} + \Phi_0\frac{\partial p^F}{\partial n}\right)\right.$

$$\left. +2\frac{\partial p^F}{\partial\mathbf{B}^S}\cdot(\mathbf{a}^S \otimes \mathbf{F}^S\mathbf{N})\right\}\mathbf{F}^{S-T}\mathbf{N},$$

where

$$(9.7) \qquad \mathbf{Q}^S \equiv 2J^S \left(\frac{\partial \mathbf{T}^S}{\partial \mathbf{B}^S} \right)^{T23} \cdot (\mathbf{F}^{S-T}\mathbf{N} \otimes \mathbf{F}^S\mathbf{N}).$$

This tensor of the second order is called the *acoustic tensor* in the classical theory of acoustic waves in single-component nonlinear elastic materials. Its eigenvalues determine the wave speeds, and its eigenvectors – the relation of the directions of amplitude to the directions of propagation in this classical case. It is not so in the case under considerations.

Let us notice that the second relation (9.6) implies that the amplitude $\mathbf{a}^F$ must be parallel to the vector **n** which is given in the current configuration by the relation (8.7) and which is perpendicular to the wave front. Consequently the waves carrying the discontinuity of the acceleration in the fluid must be *longitudinal*.

It is also easy to check that the amplitude $\mathbf{a}^S$ can have an arbitrary direction. As pointed out in the work [27], these solutions of the set of algebraic equations (9.6) determine three types of acoustic waves: *two longitudinal* so-called P1- and P2-*waves* and *one transversal* S-*wave*.

We shall discuss some properties of these waves for the linear model in the next section. However it is important to stress that all three waves are observed in porous materials. The fastest one is the P1-wave. It propagates, for instance, in soils with the speed $3-5$ km/s. The second fastest is the transversal wave carried primarily by the skeleton. The slow P2-wave (Biot's wave) has, for instance, in soils the speed $0.5-1.5$ km/s. These speeds as well as other properties of the waves (for instance – attenuation) are dependent on the deformation of both components and on the current porosity. This delivers the *in situ* methods of diagnosis of porous materials by propagating acoustic waves and measuring the arrival time and amplitudes of various sorts of waves. To a certain extent such methods are already used, for instance, in geology. The difficulties are connected with the analysis of the available data for which the old models of porous materials were not adequate.

10. Linear models, some simple analytical considerations

For the purpose of illustration we close this work with a few remarks concerning the linear version of the model. It is obvious that the construction of any analytical solution of the fully nonlinear boundary value problem shall be almost impossible. We can expect, however, that the numerical codes shall be developed. The work on this subject is already in progress. For this reason it is convenient to have some simple hints from the linear and simplified problems in which we do not have to eliminate the artefacts connected with the numerical approximations. We consider now a few examples of such problems.

Let us consider the case in which the following assumptions are satisfied

$$\mathbf{E}^S \equiv \frac{1}{2}(\mathbf{C}^S - 1), \qquad \|\mathbf{E}^S\| = \sup_{\mathbf{n},|\mathbf{n}|=1} |\mathbf{E}^S \cdot (\mathbf{n} \otimes \mathbf{n})|,$$

(10.1)

$$\sup_{x,t} \|\mathbf{E}^S\| \ll 1,$$

$$\sup_{x,t} \left| \frac{\varrho^F - \varrho_0^F}{\varrho_0^F} \right| \ll 1,$$

$$\sup_{x,t} \left| \frac{\Delta}{n_0} \right| \ll 1, \qquad \Delta \equiv n - n_0,$$

where ϱ_0^F and n_0 denote the constant initial values of the mass density of the fluid and of the porosity, respectively.

Under these assumptions the constitutive relations for the source of porosity (7.11) and for the partial stresses (7.2) become

$$\nu = -\frac{\Delta}{\tau}, \qquad \tau = \tau(n_0),$$

$$\mathbf{T}^S = \lambda^S(\mathbf{E}^S \cdot 1)1 + 2\mu^S \mathbf{E}^S, \qquad \lambda^S = \lambda^S(n_0), \qquad \mu^S = \mu^S(n_0),$$

(10.2)

$$\mathbf{T}^F = -p^F 1, \qquad p^F = K^F \varrho^F + \frac{n_0 \varrho_0^F}{\tau \mathcal{N}} \Delta,$$

$$K^F = K^F(n_0), \qquad \mathcal{N} = \mathcal{N}(n_0).$$

In the above relations we have used the assumption mentioned in the section on the boundary conditions and concerning the form of the flux Φ_0. Namely it has been assumed to be equal to the porosity n itself. In the linear model this assumption yields the constant flux of the value n_0. The coupling of stresses is then one-sided: the stress in the skeleton is independent of the presence and properties of the fluid in pores.

The fields in this case

(10.3) $$\left\{ \varrho^F, \Delta, \mathbf{u}^S, \mathbf{v}^F \right\}, \qquad \mathbf{u}^S \equiv \chi^S(\mathbf{X}, t) - \mathbf{X},$$

where $\mathbf{u}^S$ is the *displacement* of the skeleton, are described by the following fully linearized set of field equations

(10.4)
$$\frac{\partial \varrho^F}{\partial t} + \varrho_0^F \operatorname{Div}(\mathbf{v}^F) = 0,$$

$$\frac{\partial \Delta}{\partial t} + n_0 \operatorname{Div}(\mathbf{v}^F) = -\frac{\Delta}{\tau},$$

$$\varrho^S \frac{\partial^2 \mathbf{u}^S}{\partial t^2} = (\lambda^S + \mu^S) \operatorname{Grad} \operatorname{Div}(\mathbf{u}^S) + \mu^S \operatorname{Div} \operatorname{Grad}(\mathbf{u}^S) + \pi_3 \mathbf{w} + \varrho^S \mathbf{b}^S,$$

(10.4)
[cont.]
$$\varrho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = -\operatorname{Grad}\left(K^F \varrho^F + \frac{n_0 \varrho_0^F}{\tau \mathcal{N}} \Delta\right) - \pi_3 \mathbf{w} + \varrho^F \mathbf{b}^F,$$

$$\mathbf{w} \equiv \mathbf{v}^F - \frac{\partial \mathbf{u}^S}{\partial t},$$

where

(10.5)
$$\pi_3 = \pi_0 + \pi_1 + \pi_2 = \pi_3(n_0).$$

We can now make the analysis of the propagation condition of acoustic waves completely explicit. We obtain the following equations for the amplitudes

(10.6)
$$r + \varrho_0^F \mathbf{a}^F \cdot \mathbf{n} = 0,$$
$$\mathbf{n} + n_0 \mathbf{a}^F \cdot \mathbf{n} = 0,$$
$$\varrho^S U^2 \mathbf{a}^S = (\lambda^S + \mu^S)(\mathbf{a}^S \cdot \mathbf{n})\mathbf{n} + \mu^S \mathbf{a}^S,$$
$$\varrho_0^F U^2 \mathbf{a}^F = \left\{-K^F r - \frac{n_0 \varrho_0^F}{\tau \mathcal{N}} \mathbf{n}\right\} \mathbf{n}.$$

Consequently the amplitudes of the mass density gradient r and the amplitude of the porosity gradient n are not connected with their own waves – as it was already the case in the nonlinear problem. The amplitude of the acceleration wave in the fluid possesses solely the normal component and the speeds of propagations are given by the following relations

(10.7)
$$U_L^S = \sqrt{\frac{\lambda^S + 2\mu^S}{\varrho^S}} \qquad \text{longitudinal P1-wave,}$$

$$U_T^S = \sqrt{\frac{\mu^S}{\varrho^S}} \qquad \text{transversal S-wave,}$$

$$U_L^F = \sqrt{K^F + \frac{n_0^2}{\tau \mathcal{N}}} \qquad \text{longitudinal P2-wave.}$$

Hence the measurements of these three speeds of propagation deliver immediately three relations for the material parameters in function of the porosity n_0. These data are easily available and we show further a numerical example.

In order to analyze the attenuation of waves it is easier to consider a one-dimensional example of the monochromatic wave. Let us denote by v^F the x-component of the velocity of the fluid, by v^S – the x-component of the velocity of the skeleton and by ε^S – the extension of the skeleton in the x-direction. These three quantities together with ϱ^F and Δ fully describe the one-dimensional process. We look for the solution of the set of field equations in the following

form

$$\begin{aligned}
\varrho^F &= \varrho_0^F + \varepsilon R^F \exp(i(\omega t - k^* x)), \\
v^F &= \varepsilon V^F \exp(i(\omega t - k^* x)), \\
v^S &= \varepsilon V^S \exp(i(\omega t - k^* x)), \\
\Delta &= \varepsilon D \exp(i(\omega t - k^* x)), \\
\varepsilon^S &= \varepsilon E^S \exp(i(\omega t - k^* x)),
\end{aligned}$$

(10.8)

where $\varrho_0^F, R^F, V^F, V^S, D, E^S$ are constants and

(10.9) $$0 < \varepsilon \ll 1.$$

In the above relations the frequency ω denotes the real frequency of the monochromatic wave which is considered to be given. The wave number k^* is assumed to be complex. Namely

(10.10) $$k^* = k + i\alpha,$$

where k is the inverse of the *wavelength* and α denotes the *attenuation* of the wave.

Substitution of the relations (10.8) in the field equations yields the following *dispersion relation for the monochromatic waves*

(10.11) $$\left\{ \omega^2 - U_L^{F2} k^{*2} + \frac{n_0^2}{\tau \mathcal{N}} \frac{i1/\tau}{\omega + i1/\tau} k^{*2} - i \frac{\pi_3}{\varrho_0^F} \omega \right\}$$
$$\cdot \left\{ \omega^2 - U_L^{S2} k^{*2} - i \frac{\pi_3}{\varrho^S} \omega \right\} + \left(\frac{\pi_3}{\varrho^S} \right) \left(\frac{\pi_3}{\varrho_0^F} \right) \omega^2 = 0,$$

which is the equation for k^* as a function of ω. It is easy to check that the limit case of almost empty pores for which we can neglect the influence of diffusion yields the frequency-dependent speeds of propagation of two different types of waves corresponding to the two longitudinal waves discussed above. Moreover the limit $\omega \to \infty$ yields the same speeds of propagation as before.

As far as the attenuation coefficient α is concerned we obtain the following relation

(10.12) $$\frac{1}{\tau} = \frac{\omega}{2Q} \left\{ \left(Q^2 - \frac{1}{4} \right) \frac{n_0^2}{\tau \mathcal{N} K^F} \right.$$
$$\left. + \sqrt{ \left(Q^2 - \frac{1}{4} \right)^2 \left(\frac{n_0^2}{\tau \mathcal{N} K^F} \right)^2 - 4Q^2 \left(1 + \left(\frac{n_0^2}{\tau \mathcal{N} K^F} \right) \right) } \right\},$$

where

(10.13) $$Q = \frac{k}{2\alpha},$$

is the so-called the *quality factor* of the monochromatic wave (see: Sec. 3.3.3.
of [5]). Hence the relaxation time τ for the porosity is indeed one of the two
parameters describing the attenuation of waves. The second one is the classical
diffusion coefficient π_3. The quality factor is also easily attainable to the mea-
surements. This yields the possibility of measuring the additional parameter τ of
the model discussed in this section.

In order to illustrate the above considerations we present the numerical results
for the *Massillon sandstone*. For the porosity $n_o = 23\%$ and the water saturation
$S_w = 0.1\%$ we have the following experimental data [5] and the results of the
wave analysis

Measurements:	$U_L^S \cong 3.1 \times 10^3$ m/s	$U_L^F \cong 0.9 \times 10^3$ m/s	$\frac{1000}{Q} = 40$ for $\omega = 2 \times 10^3$ Hz		
	$U_T^S \cong 1.6 \times 10^3$ m/s	$U_{air} \cong 0.3 \times 10^3$ m/s	$\varrho^S \cong 2.4 \times 10^3$ kg/m³		

Results (the wave analysis):	$\lambda^S = 10.776$ $\times 10^3$ MPa	$\mu^S = 6.144$ $\times 10^3$ MPa	$K^F = 0.9$ $\times 10^5$ m²/s	$\tau \mathcal{N} = 7.347$ $\times 10^{-8}$ s²/m²	$\tau = 3.699$ $\times 10^{-6}$ s

These values check well with the available experimental results obtained by
the standard methods of measuring the material parameters.

In addition, the above simple examples justify to a certain extent the assump-
tions made in the nonlinear model. For instance the measurements of the speeds
of the P1-wave in many rocks show that they are almost independent of the wa-
ter saturation in pores. It means that these speeds do not react to the art of the
substance in the pores – they are independent of ϱ^F and Δ. This justifies for such
materials the assumption of independence of the free energy of the skeleton of
the mass density of the fluid and of the changes of the porosity which we have
mentioned in the section on the boundary conditions.

We complete this section with another standard example stemming from the
soil mechanics (see: [29] for further details). First of all let us notice that the
equations $(10.4)_{1,2}$ can be combined in the following way

(10.14) $$\frac{\partial \Delta}{\partial t} + \frac{\Delta}{\tau} = \frac{n_0}{\varrho_0^F} \frac{\partial \varrho^F}{\partial t}.$$

If the mass density ϱ^F were known, we could find the changes of porosity from
this equation. Consequently the *formal solution* can be written in the form

(10.15) $$\Delta = \frac{n_0}{\varrho_0^F} \left\{ \varrho^F - \varrho_0^F e^{-t/\tau} - \frac{1}{\tau} \int_0^t \varrho^F(\mathbf{x}, \eta) e^{-(t-\eta)\tau} \, d\eta \right\}.$$

As expected, the equation for porosity yields a sort of *memory effect* which in the linear theory is described by the Boltzmann integral. It means that the present value of the porosity depends not only on the present value of the mass density but also on its past history. The influence of the past history is, however, modified by the exponential function. Hence, in the first approximation, we can neglect these effects entirely. We obtain

(10.16)
$$\Delta \approx \frac{n_0}{\varrho_0^F}(\varrho^F - \varrho_0^F).$$

Substitution of this relation in Eq. (10.14) shows immediately that this equation is satisfied solely in the case of the infinite relaxation time. In such a case there is no dissipation due to the changes of porosity. The porosity changes according to the change of the mass density of the fluid. The similar property appears under the assumption of the incompressibility of real materials of components which has been discussed by R. BOWEN [6]. However in contrast to the work of Bowen, in our case it is only the approximation which does not lead to any reaction forces on constraints.

Bearing in mind the above approximation we solve now the one-dimensional quasi-static consolidation problem which has been solved for the first time by Fröhlich in 1938 within the frame of the Terzaghi model of consolidation. Namely we consider the compression of the semi-infinite prism of the porous material filled with water with the free flow of the water through the boundary $x = 0$. The external pressure p_a is atmospheric and the loading is given as the body force on the skeleton

(10.17)
$$\varrho^S b^S = q H(t)\delta(x),$$
$$\varrho^F b^F = 0,$$

where $H(\cdot)$ is the Heaviside distribution and $\delta(\cdot)$ is the Dirac distribution. The constant q is the load in the direction of the x-axis.

Simple manipulations of the field equations yield the following set of equations for the pressure p^F and the normal component of the stress σ^S in the direction of the x-axis

(10.18)
$$\frac{\partial}{\partial t}\left(\frac{\partial p^F}{\partial x}\right) - D\frac{\partial^2}{\partial x^2}\left(\frac{\partial p^F}{\partial x}\right) = Mq\delta(t)\delta(x),$$
$$\frac{\partial \sigma^S}{\partial x} - \frac{\partial p^F}{\partial x} = -qH(t)\delta(x),$$

where

$$M \equiv \frac{\varrho_0^F \left(K^F + \frac{n_0^2}{\tau \mathcal{N}} \right)}{E^S + \varrho_0^F \left(K^F + \frac{n_0^2}{\tau \mathcal{N}} \right)}, \qquad E^S \equiv \lambda^S + 2\mu^S,$$

(10.19)

$$D \equiv \frac{\varrho_0^F}{\pi_3} \frac{\varrho_0^F \left(K^F + \frac{n_0^2}{\tau \mathcal{N}} \right) E^S}{E^S + \varrho_0^F \left(K^F + \frac{n_0^2}{\tau \mathcal{N}} \right)}.$$

The equations (10.18) can be easily solved. For instance, we obtain the following result for the so-called *hydraulic gradient* i

(10.20)

$$\left(\frac{2\sqrt{\pi} H}{M q} \right) i = \frac{1}{\sqrt{t'}} \exp \left(-\frac{x'^2}{4t'} \right),$$

$$i \equiv \frac{\partial p^F}{\partial x}, \qquad t' \equiv t \frac{D}{H^2}, \qquad x' \equiv \frac{x}{H},$$

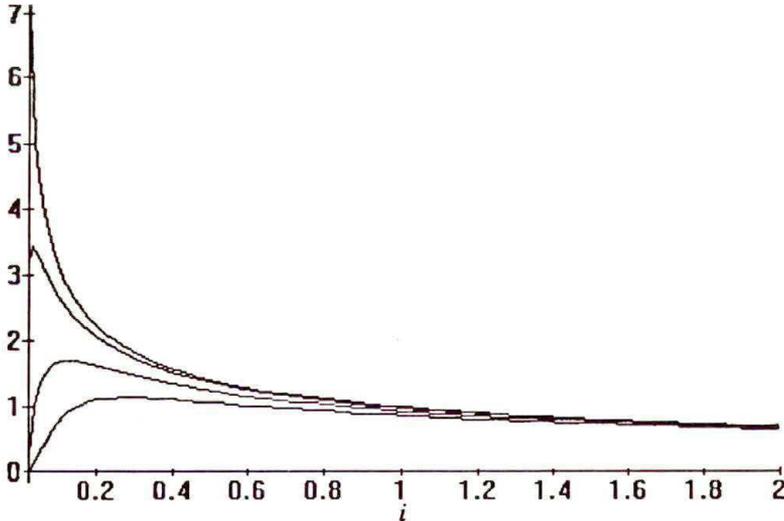and H is a constant with the dimension of length.

FIG. 1. Time changes of the hydraulic gradient i for $x/H = 0$, 0.25 and 0.75.

This solution is shown in the Fig. 1 for various values of the depth. The result complies quantitatively with the results obtained for the model of Terzaghi for times shorter than app. 1.5.

For large times the decay in the present model is much slower even though both solutions approach zero for the infinite time. This is most likely the result of approximations applied by Fröhlich.

The above results allow also to find the last material constant of the linear model – the coefficient of diffusion π_3. Consequently the model can be used in the practical applications to describe processes of small deformations and small changes of porosity. Little is known about the constitutive functions for nonlinear cases. This is however also the deficiency of the experiments which are available at the present time.

11. Final remarks

The simple examples of the last section have demonstrated how strong must be the simplifying assumptions to lead to the classical results of the theory of porous materials. Almost nothing has been done yet as far as the solutions for large deformations are concerned. At the present stage of research there seems to be a good chance for obtaining the first numerical results in the case of purely mechanical processes in materials with the elastic skeleton and the ideal fluid. However even in this case there are no mathematical results available and the free boundary may yield difficulties connected with the existence of classical solutions.

Even less developed are the models combining the large deformations with non-mechanical variables. Particularly important are here the non-isothermal problems. There exist already the first attempts to incorporate these effects, particularly in connection with the phase transformations (e.g. drying processes in ceramics). The situation is, however, not very satisfactory. The thermal variables connected with the problem of free boundaries yield difficulties with the construction of the model which would contain physically measurable quantities (e.g. see [11]).

On the other hand there seems to be no doubt that the modern continuum theory of mixtures of immiscible components is the only possibility to obtain the mathematical models of porous materials. The purely structural theories may deliver some important hints concerning, for instance, transport coefficients but they are hardly in the position to be applicable in numerous engineering problems of geology, chemistry, acoustics etc. independently of the capacity of future computers. The new chance for the continuum theories is certainly connected with the unified Lagrangian description of all components. Its application in this work has shown that the relatively complex model can be handled without many technical difficulties and the first experience with this description in numerical methods also indicates considerable simplifications.

Appendix: Motivation of the equation for porosity (4.11)

In this Appendix we present the brief semimicroscopical motivation of the balance equation for the porosity (4.11). Mathematical details of the derivation of this equation are rather involved due to the lack of smoothness. We discuss them elsewhere [24].

It is assumed that the skeleton, the solid component of the porous medium, is a continuum on the semimicroscopical level of observation. This means that each point X of the macroscopical manifold $\mathcal{B}$ is connected with a certain time-dependent microstructure $\mathcal{M}_X$ which is schematically shown in Fig. 2.
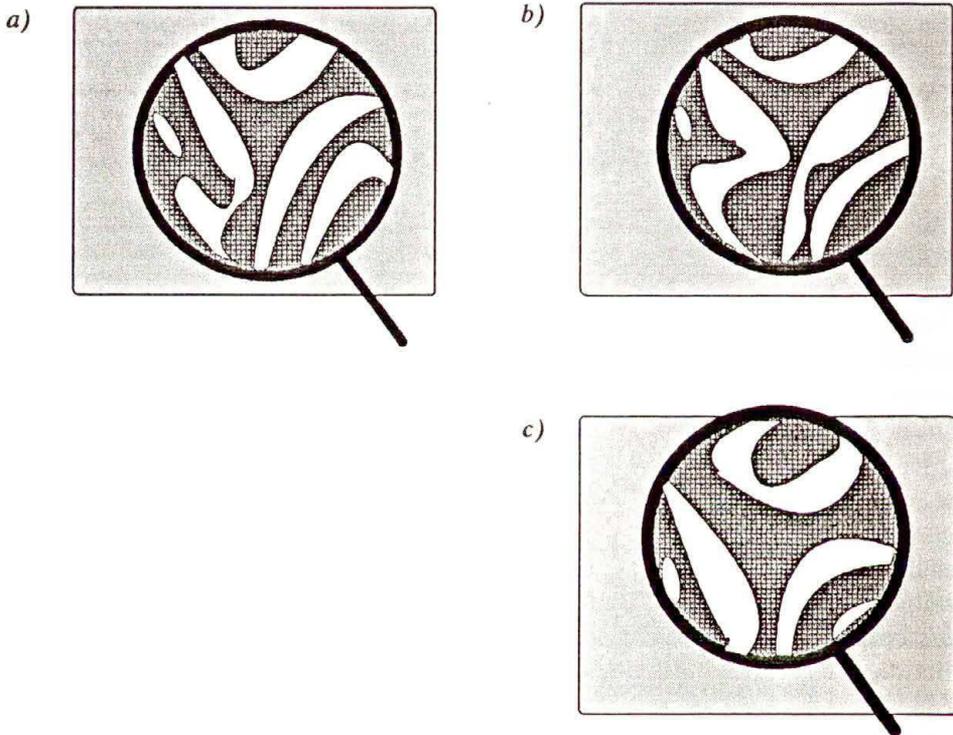

FIG. 2. The semimicroscopical mechanisms yielding the changes of porosity. The centre of the magnifying glass is located in all three cases at the same point X; a) initial microstructure, b) changes of microstructure due to the pore relaxation (micromotion and microsources), c) changes of microstructure due to the macroscopical flux (motion of microstructure relative to the macroscopical skeleton).

The instantaneous geometry of this microstructure is established by the real solid body $\mathcal{B}_{real}$ embedded for each instant of time in the three-dimensional Euclidean configuration space $\mathcal{R}^3$. The hull which is identical with the closed boundary surface of the geometrical three-dimensional figure $\mathcal{M}_X$ (the frame of the magnifying glass in Fig. 2) is now shifted over the configuration space and the

average properties of the part of the real body contained in the interior of the hull are prescribed to the point of the space $\mathcal{R}^3$ coinciding with a chosen internal point of $\mathcal{M}_X$ (the centre of the magnifying glass in the simple example of Fig. 2) and occupied at the same instant of time by the material point $\mathbf{X}$ of the skeleton. For simplicity one assumes that the shape of the hull does not change in time.

This type of the volume averages are used quite commonly in the theories of bodies with microstructure. For instance the volume averages of material properties of composites are calculated in this way. In the theory of porous materials with diffusion processes there are also numerous attempts in this direction (e.g. F. DOBRAN [31], W.A. GRAY, S.M. HASSANIZADEH [32], J. BEAR, Y. BACHMAT [33]). None of them seems to be yet effective and reliable enough to yield the macroscopical model without any need for additional macroscopical constitutive relations. For this reason we use the above described construction solely to motivate the equation for the porosity.

Instead of constructing averages in the configuration space $\mathcal{R}^3$ we use the procedure on the reference configuration $\mathcal{B}$ of the macroscopical skeleton. This corresponds with our Lagrangian approach.

We seek the equation describing the volume changes of the part of the real skeleton which at a given point $\mathbf{X} \in \mathcal{B}$ and at a given instant of time t lies inside the hull of the figure $\mathcal{M}_X$. The arbitrary point $\mathbf{Y}$ from $\mathcal{M}_X$ can be described by the location vector

$$(A.1) \qquad \mathbf{Y} = \mathbf{X} + \varepsilon \mathbf{Z}, \qquad \mathbf{X} \in \mathcal{B}, \qquad \mathbf{Y} \in \mathcal{M}_X,$$

where ε is the small parameter of the order of the cubic root of the ratio of the volume of microstructure to the characteristic macroscopic volume. If we denote by $\mathcal{H}(., t)$ the characteristic function of the real skeleton contained in $\mathcal{M}_X$

$$(A.2) \qquad \mathcal{H}(\mathbf{Y}, t) = \begin{cases} 1 & \text{for } \mathbf{Y} \text{ belonging to the domain of the real skeleton,} \\ 0 & \text{otherwise,} \end{cases}$$

then the porosity is defined by the relation

$$(A.3) \qquad 1 - n(\mathbf{X}, t) \equiv \frac{1}{V_C} \int_{\mathcal{M}_X} \mathcal{H}(\mathbf{Y}, t) \, dV, \qquad V_C \equiv \int_{\mathcal{M}_X} dV = \text{const},$$

where V_c is sometimes called the control volume of the averaging.

We want to find the time changes of the porosity. The changes of the microscopic geometry of the real skeleton are due to the two factors:

• The redistribution of the real solid material in the domain $\mathcal{M}_X$ due to its microscopic deformation. This may follow from the compressibility of the real material and/or from the microscopic motion of the skeleton inside of this domain which shifts the solid material to the parts of the pore space. Such processes are

not controllable on the macroscopical level and yield the *pore relaxation processes*. They are schematically shown in Fig. 2b. The material of the real skeleton in the microscopic configurations of Fig. 2a and 2b (the interior of the magnifying glass) is the same but its distribution within $\mathcal{M}_X$ has changed due to the above described mechanisms;

• The flux of the real material through the hull of the microstructure into the neighbouring regions of the real body. This is demonstrated in Fig. 2c by the shift of the real material relative to the magnifying glass whose centre still lies in **X**.

The balance equation for the whole microstructure describing these changes of the geometry has the form

$$
(A.4) \qquad \frac{\partial}{\partial t} \int_{\mathcal{M}_X} \mathcal{H}(\mathbf{Y}, t)\, dV = \oint_{\mathcal{M}_X} \mathcal{H}(\mathbf{Y}, t) \mathbf{v}_{\text{real}}^S \cdot \mathbf{n}\, dA + \int_{\mathcal{M}_X} \widehat{\mathcal{H}}(\mathbf{Y}, t)\, dV,
$$

where $\mathbf{v}_{\text{real}}^S$ denotes the velocity field for the points occupied in the microstructure by the real skeleton. This field is highly singular and usually cannot be integrated to describe any smooth trajectories (see: [24]). The operations performed on the above equation, which must be understood in the distributional sense, require certain additional smoothing procedures which we do not present in this Appendix. The vector **n** is the outward normal vector of the boundary of the microstructure $\partial \mathcal{M}_X$ and $\widehat{\mathcal{H}}(\mathbf{Y}, t)$ is the intensity of the source of the domain occupied by the real skeleton. The latter is due to the changes of the volume of the real skeleton in the microstructure caused by the changes of the real mass density of the skeleton (see: Fig. 2b).

The surface integral in (A.4) can be transformed into the volume integral under the above mentioned smoothing procedures. Subsequently we apply the *multiscaling* indicated by the relation (A.1) and obtain

$$
(A.5) \qquad \oint_{\mathcal{M}_X} \mathcal{H}(\mathbf{Y}, t) \mathbf{v}_{\text{real}}^S \cdot \mathbf{n}\, dA = \text{Div}_X \int_{\mathcal{M}_X} \mathcal{H}(\mathbf{X}, \mathbf{Z}, t) \mathbf{v}_{\text{real}}^S(\mathbf{X}, \mathbf{Z}, t)\, dV_Z
$$

$$
+ \varepsilon \int_{\mathcal{M}_X} \text{Div}_Z(\mathcal{H}(\mathbf{X}, \mathbf{Z}, t) \mathbf{v}_{\text{real}}^S(\mathbf{X}, \mathbf{Z}, t))\, dV_Z,
$$

where the differentiation and integration with respect to the microvariable **Z** has been separated from the differentiation with respect to the macroscopical Lagrange variable **X**.

The first term on the right-hand side of this relation describes the macroscopical flux of the porosity defined by the relation (A.3). Simultaneously the second term follows from the microscopical motions of the real skeleton within the microstructure and contributes to the pore relaxation processes – independently of the fact whether the real components are assumed to be compressible or incompressible.

Consequently, if we introduce the notation

$$- \mathbf{J}(\mathbf{X}, t) = \frac{1}{V_C} \left\{ \int_{\mathcal{M}_X} \mathcal{H}(\mathbf{X}, \mathbf{Z}, t) \mathbf{v}^S_{\text{real}}(\mathbf{X}, \mathbf{Z}, t) \, dV_Z \right\},$$

(A.6) $$- \hat{n}(\mathbf{X}, t) = \frac{1}{V_C} \left\{ \varepsilon \int_{\mathcal{M}_X} \text{Div}_Z \mathcal{H}(\mathbf{X}, \mathbf{Z}, t) \mathbf{v}^S_{\text{real}}(\mathbf{X}, \mathbf{Z}, t) \, dV_Z \right\}$$

$$+ \frac{1}{V_C} \left\{ \int_{\mathcal{M}_X} \hat{\mathcal{H}}(\mathbf{X}, \mathbf{Z}, t) \, dV_Z \right\},$$

we obtain from the equation (A.4) the balance equation of the porosity (4.11).

The above considerations must be considered solely as the clarification of certain microscopical mechanisms yielding the "internal" variables and the motivation of this balance equation and not as its derivation because, apart from the above mentioned smoothness problems, the relations of this Appendix are not effective if we do not have the full set of microscopical field equations. The problem must be still closed by constitutive relations and this is obviously simpler on the macroscopical level as we have done in the paper.

Acknowledgement

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