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Surface stress waves in a nonhomogeneous elastic half-space Part II. Existence of surface waves for an arbitrary variation of Poisson's ratio Approximate solution based on perturbation methods

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Two approaches to the solution of the nonlinear eigenvalue problem of propagation of surface waves in a nonhomogeneous isotropic elastic half-space are considered. In Sec. 1 the nonlinear eigenvalue problem is transformed to the equivalent integral equation, and the method of solving this equation is proposed. In Sec. 2 Friedrich's perturbation theory [6] is used to solve an eigenvalue problem describing the surface stress waves in a "weakly" nonhomogeneous isotropic elastic half-space. Two cases are discussed in detail: a) a half-space with a "weak" variation of density, b) a half-space with a "weak" variation of the shear modulus. In both cases an asymptotic solution is obtained and numerical results are given.

1. Effective form of amplitude of surface stress waves in a non-homogeneous isotropic elastic half-space

1.1. Formulation of the problem

It is shown in [1] that the problem of propagation of surface waves in a non-homogeneous isotropic elastic half-space can be reduced to the following eigenvalue problem: to find a nonvanishing pair $(\beta(x), c_R)$ satisfying the relations

$$(1.1)\qquad \left(\frac{1}{s^2}D\frac{1}{1-\Omega}D-1\right)\frac{1}{1-\kappa}\,\frac{\Omega}{2-\Omega}\left[D^2-s^2(1-\Omega\kappa)\right]\beta + 4\left[\frac{1}{2-\Omega}D^2-D\frac{1}{1-\Omega}D\frac{1-\Omega}{2-\Omega}\right]\beta=0 \qquad \text{for} \quad x\in(0,\infty),$$

and

$$(1.2)\qquad \begin{cases} \beta(0)=\beta(\infty)=0,\\ \dfrac{1}{s^2(2-\Omega)}D\left\{\dfrac{\Omega}{2-\Omega}\,\dfrac{1}{1-\kappa}\left[D^2-s^2(1-\Omega\kappa)\right]\beta-4s^2\dfrac{1-\Omega}{2-\Omega}\beta\right\}\Bigg|_{\substack{x_2=0\\x_2=\infty}}=0.\end{cases}$$

Here

$$(1.3)\qquad \frac{1-2\nu(x)}{2-2\nu(x)}=\kappa(x),\qquad \Omega(x)=\frac{C_R^2}{\mu(x)},\qquad D=\frac{d}{dx},$$

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in which $\nu(x)$ and $\mu(x)$ are the Poisson's ratio and shear modulus, respectively, while the symbol $c_R = p/s$, where $2\pi/p$ is the wave period and $2\pi/s$ is the wave length, denotes the velocity of surface wave. The eigenvalue c_R corresponding to the eigenfunction β is to be identified with the Rayleigh velocity.

Next we consider the case

$$(1.4) \qquad \begin{cases} \beta \in C^4[0,\infty), & \\ \kappa = \kappa(x) \in C^2[0,\infty), & 0 < \kappa_0 \le \kappa(x) \le \kappa_1 < 3/4, \\ \mu(x) \equiv \mu_0 = 1, & \Omega(x) \equiv \Omega_0 = C_R^2. \end{cases}$$

These hypotheses assure that the elastic energy of the half-space is stricly positive. The system (1.1) – (1.2) subjected to conditions (1.4) is equivalent to the following equations:

$$(1.5) \qquad \frac{1}{1-\kappa(x)}\left[D^2 - s^2(1-\Omega_0\kappa(x))\right]\beta = C_1 e^{-s\sqrt{1-\Omega_0}x} + C_2 e^{s\sqrt{1-\Omega_0}x},$$

$$(1.6) \qquad \beta(0) = \beta(\infty) = 0,$$

$$(1.7) \qquad \begin{aligned} D\left\{\frac{\Omega_0}{1-\kappa(x)}\left[D^2 - s^2(1-\Omega_0\kappa(x))\right]\beta - 4s^2(1-\Omega_0)\beta\right\}\Big|_{x=0} &= 0, \\ D\left\{\frac{\Omega_0}{1-\kappa(x)}\left[D^2 - s^2(1-\Omega_0\kappa(x))\right]\beta - 4s^2(1-\Omega_0)\beta\right\}\Big|_{x=\infty} &= 0, \end{aligned}$$

where C_1 is an arbitrary constant, $C_2 = 0$, $\beta \in C^4[0,\infty)$, $\beta(\infty) = 0$.

The aim of this section is to transform the problem (1.5) – (1.7) to an equivalent integral equation and to construct an iteration method of solving this equation. To this end consider the differential operator L associated with (1.5):

$$(1.8) \qquad L\beta \equiv -D^2\beta + s^2(1-\Omega_0\kappa(x))\beta,$$

$$(1.9) \qquad \beta(0) = \beta(\infty) = 0.$$

Let $g = g(x,t;\Omega_0,s)$; $(x,t;\Omega_0,s) \in [0,\infty)\times[0,\infty)\times(0,1)\times(0,\infty)$ be the Green function for the operator L with a "frozen" coefficient κ. In other words, the Green function g fulfills the relations:

$$(1.10) \qquad \frac{\partial^2 g}{\partial t^2} - s^2(1-\Omega_0\kappa(x))g = 0 \qquad \text{for} \quad t \neq x,$$

$$(1.11) \qquad g = 0 \qquad \text{for} \quad t = 0,$$

$$(1.12) \qquad \frac{\partial g}{\partial t}\Big|_{t=x+0} - \frac{\partial g}{\partial t}\Big|_{t=x-0} = -1.$$

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For the operator with a variable coefficient κ, the Green function $G = G(x, t; \Omega_0, s)$ satisfies the following equation (c.f. [2], 123–149):

$$(1.13)\qquad G(x, t; \Omega_0, s) = g(x, t; \Omega_0, s) - s^2\Omega_0 \int\limits_0^\infty g(x, \xi; \Omega_0, s)[\kappa(x) - \kappa(\xi)]G(\xi, t; \Omega_0, s)\, d\xi$$

for every $(x, t; \Omega_0, s) \in [0, \infty) \times [0, \infty) \times (0, 1) \times (0, \infty)$.

It is easy to show, that the function $g = g(x, t; \Omega_0, s)$ fulfilling the conditions (1.10)–(1.12) has the following form

$$(1.14)\qquad g(x, t; \Omega_0, s) = \begin{cases} \dfrac{1}{2s\sqrt{1-\Omega_0\kappa(x)}}\left[e^{-s\sqrt{1-\Omega_0\kappa(x)}(t-x)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)}\right] & \text{for} \quad x \le t < \infty, \\ \dfrac{1}{2s\sqrt{1-\Omega_0\kappa(x)}}\left[e^{-s\sqrt{1-\Omega_0\kappa(x)}(x-t)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)}\right] & \text{for} \quad 0 \le t \le x. \end{cases}$$

In the subsequent part, the properties of the Green functions $G = G(x, t; \Omega_0, s)$ will be investigated and the solution of eigenvalue problem (1.5)–(1.7) will be expressed using the function G.

1.2. Integral equation for Green function

Let us denote by X the Banach space of real functions $A(x, t)$, $(x, t) \in [0, \infty)\times[0, \infty)$ with norm $\|\cdot\|_X$ given by

$$(1.15)\qquad \|A(x, t)\|_X^2 = \int\limits_0^\infty \left\{ \int\limits_0^\infty |A(x, t)|^2\, dt \right\} dx < \infty.$$

Let N be the operator in X of the form:

$$(1.16)\qquad NA(x, t) = s^2\Omega_0 \int\limits_0^\infty g(x, \xi; \Omega_0, s)\,[\kappa(x) - \kappa(\xi)]\, A(\xi, t)\, d\xi\,,$$

where $g(x, \xi; \Omega_0, s)$ is defined by Eq. (1.14).

One can observe, that for every $(x, \xi) \in [0, \infty) \times [0, \infty)$ there exists such m, that

$$(1.17)\qquad |\kappa(x) - \kappa(\xi)| \le m|x - \xi|.$$

The existence of m follows from assumption (1.4) and from the fact, that $\kappa(x) \in C^2[0, \infty)$. It can be assumed that

$$m = \sup_{x\in[0,\infty)} \left|\frac{d\kappa}{dx}\right|.$$

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The following lemma is valid:

LEMMA 1. If the inequality

$$q \equiv \Omega_0 m(1 - \Omega_0\kappa_1)^{-3/2}s^{-1} < 1 \tag{1.18}$$

is satisfied, then operator N is a contraction in the space X, i.e.

$$\|NA\|_X \leq q\|A\|_X \,.$$

P r o o f. Due to (1.16) and (1.4) we obtain

$$\begin{aligned}
&NA(x,t) \equiv M(x,t,\Omega_0,s)\\
&= \frac{1}{2}s\Omega_0 \int\limits_x^\infty (1-\Omega_0\kappa)^{-1/2}\left[e^{-s\sqrt{1-\Omega_0\kappa}(\xi-x)} - e^{-s\sqrt{1-\Omega_0\kappa}(x+\xi)}\right][\kappa(x)-\kappa(\xi)]A(\xi,t)d\xi\\
&+\frac{1}{2}s\Omega_0 \int\limits_0^x (1-\Omega_0\kappa)^{-1/2}\left[e^{-s\sqrt{1-\Omega_0\kappa}(x-\xi)} - e^{-s\sqrt{1-\Omega_0\kappa}(x+\xi)}\right][\kappa(x)-\kappa(\xi)]A(\xi,t)d\xi\\
&\equiv a(x,t;\Omega_0,s) + b(x,t;\Omega_0,s).
\end{aligned}$$

Hence, the following estimate can be deduced

$$|M(x,t;\Omega_0,s)| \leq |a(x,t;\Omega_0,s)| + |b(x,t;\Omega_0,s)|. \tag{1.19}$$

Estimating from above the function a we get

$$\begin{aligned}
a^2(x,t;\Omega_0,s) \leq \frac{1}{4}s^2\Omega_0^2 \cdot \Bigg\{ \int\limits_x^\infty (1-\Omega_0\kappa_1)^{-1/2}\\
\left[e^{-s\sqrt{1-\Omega_0\kappa}(\xi-x)} - e^{-s\sqrt{1-\Omega_0\kappa}(x+\xi)}\right] \cdot [\kappa(x)-\kappa(\xi)] \cdot |A(\xi,t)|\,d\xi \Bigg\}^2.
\end{aligned} \tag{1.20}$$

From the inequalities (1.17) and (1.4) we have

$$\begin{aligned}
(1-\Omega_0\kappa)^{-1/2}\left[e^{-s\sqrt{1-\Omega_0\kappa}(\xi-x)} - e^{-s\sqrt{1-\Omega_0\kappa}(x+\xi)}\right] \cdot [\kappa(x)-\kappa(\xi)] \cdot |A(\xi,t)|\\
\leq (1-\Omega_0\kappa_1)^{-1/2}e^{-s\sqrt{1-\Omega_0\kappa_1}(\xi-x)}m(\xi-x) \cdot |A(\xi,t)|,
\end{aligned} \tag{1.21}$$

and finally

$$\begin{aligned}
a^2(x,t;\Omega_0,s) \leq \frac{1}{4}s^2m^2\Omega_0^2(1-\Omega_0\kappa_1)^{-1}\\
\cdot \Bigg\{ \int\limits_x^\infty (\xi-x)e^{-s\sqrt{1-\Omega_0\kappa}(\xi-x)}|A(\xi,t)|\,d\xi \Bigg\}^2.
\end{aligned} \tag{1.22}$$

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Integrating inequality (1.22) with respect to x on the interval $[0, \infty)$ and changing the variables we obtain

$$\int_0^\infty a^2(x,t;\Omega_0,s)\,dx \le \frac{1}{4}m^2\Omega_0^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\int_0^\infty |A(x,t)|^2\,dx. \tag{1.23}$$

Integrating the inequality (1.23) with respect to t on the interval $[0, \infty)$ we get

$$\|a(x,t;\Omega_0,s)\|_X \le \frac{1}{2}\Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1}\|A(x,t)\|_X\,. \tag{1.24}$$

Now we shall estimate the norm $\|b(x,t;\Omega_0,s)\|_X$. From the definition of the function $b(x,t;\Omega_0,s)$ we have:

$$b(x,t;\Omega_0,s) = \frac{1}{2}s\Omega_0\int_0^x (1-\Omega_0\kappa)^{-1/2} \left[e^{-s\sqrt{1-\Omega_0\kappa}(x-\xi)} - e^{-s\sqrt{1-\Omega_0\kappa}(x+\xi)}\right]\cdot[\kappa(x)-\kappa(\xi)]\cdot|A(\xi,t)|\,d\xi. \tag{1.25}$$

The inequalities (1.26) and (1.4) lead to

$$b^2(x,t;\Omega_0,s) \le \frac{1}{4}s^2\Omega_0 m^2(1-\Omega_0\kappa_1)^{-1} \left\{\int_0^x (x-\xi)e^{-s\sqrt{1-\Omega_0\kappa_1}(x-\xi)}|A(\xi,t)|\,d\xi\right\}^2. \tag{1.26}$$

Similarly to the case of inequality (1.22), from (1.26) we get the following estimate

$$\|b(x,t;\Omega_0,s)\|_X \le \frac{1}{2}\Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1}\|A(x,t)\|_X\,. \tag{1.27}$$

From the inequality (1.19), (1.24) and (1.27) it follows that the operator N is a contraction in the space X, if

$$q = \Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1} < 1 \tag{1.28}$$

which ends the proof of Lemma 1.

In the further analysis it will be convenient to introduce two other Banach spaces $X_2^{(1)}, X_1^{(-1/2)}$ with the following norms

$$\|A(x,y)\|^2_{X_2^{(1)}} = \sup_{y\in[0,\infty)}\int_0^\infty |A(x,y)|\,dx, \tag{1.29}$$

$$\|A(x,y)\|^2_{X_1^{(-1/2)}} = \sup_{y\in[0,\infty)}\int_0^\infty \frac{|A(x,y)|^2}{s^2[1-\Omega_0\kappa(x)]}\,dx. \tag{1.30}$$

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The following lemma are valid:

LEMMA 2. The operator N given by formula (1.16) is a contraction in $X_2^{(1)}$, i.e.

$$\|NA\|_{X_2^{(1)}} \le q_1 \|A\|_{X_2^{(1)}}, \qquad \text{if} \qquad q_1 = \sqrt{q} < 1. \tag{1.31}$$

Here q is defined by the formula (1.18).

LEMMA 3. The operator N given by formula (1.16) is a contraction in $X_1^{(-1/2)}$, i.e.

$$\|NA\|_{X_1^{(-1/2)}} \le q \|A\|_{X_1^{(-1/2)}}, \qquad \text{if} \qquad q < 1. \tag{1.32}$$

Here q is defined by the formula (1.18).

The proof of Lemma 2 and 3 is given in the Appendix I (I_e, I_a, I_d and I_f).

We need the following lemma:

LEMMA 4. For every $(\Omega_0, s) \in (0,1) \times (0,\infty)$ the functions $g(x,t;\Omega_0,s)$, $\dfrac{\partial g(x,t;\Omega_0,s)}{\partial t}$, $\dfrac{\partial^2 g(x,t;\Omega_0,s)}{\partial t^2}$ belong to $X_2^{(1)}$, $X_1^{(-1/2)}$.

P r o o f. First, we show that the function $\dfrac{\partial g(x,t;\Omega_0,s)}{\partial t}$ belongs to $X_2^{(1)}$. Indeed, differentiating the functions defined by (1.14) with respect to t we obtain

$$\frac{\partial g(x,t;\Omega_0,s)}{\partial t} = \begin{cases} \dfrac{1}{2}\left[e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(t-x)}\right] & \text{for} \quad x < t < \infty, \\ \dfrac{1}{2}\left[e^{-s\sqrt{1-\Omega_0\kappa(x)}(x-t)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)}\right] & \text{for} \quad 0 < t \le x, \end{cases} \tag{1.33}$$

and we obtain the estimate:

$$\begin{aligned} \int_0^\infty \left|\frac{\partial g(x,t;\Omega_0,s)}{\partial t}\right| dt &\le \frac{1}{2}\int_x^\infty \left|e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(t-x)}\right| dt \\ &\quad + \frac{1}{2}\int_0^x \left|e^{-s\sqrt{1-\Omega_0\kappa(x)}(x-t)} - e^{-s\sqrt{1-\Omega_0\kappa(x)}(x+t)}\right| dt \\ &\le \frac{1}{2}\left(\int_x^\infty e^{-s\sqrt{1-\Omega_0\kappa(x)}(t-x)}\, dt + \int_0^x e^{-s\sqrt{1-\Omega_0\kappa(x)}(x-t)}\, dt \right. \\ &\quad \left. + \int_0^x e^{-s\sqrt{1-\Omega_0\kappa(x)}(t+x)}\, dt\right) \le \frac{1}{2}\left(\int_x^\infty e^{-s\sqrt{1-\Omega_0\kappa_1}(t-x)}\, dt\right. \end{aligned} \tag{1.34}$$

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(1.34) [cont.]

$$+\int_0^x e^{-s\sqrt{1-\Omega_0\kappa_1}(x-t)}\,dt+\int_0^x e^{-s\sqrt{1-\Omega_0\kappa_1}(t+x)}\,dt\Bigg)$$

$$=\frac{1}{2}\left(\frac{-1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-s\sqrt{1-\Omega_0\kappa_1}(t-x)}\Big|_{t=x}^{t=\infty}+\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-s\sqrt{1-\Omega_0\kappa_1}(t-x)}\Big|_{t=0}^{t=x}\right.$$

$$\left.+\frac{-1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-s\sqrt{1-\Omega_0\kappa_1}(t+x)}\Big|_{t=0}^{t=x}\right)=\frac{1}{2}\left(\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}+\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}\right.$$

$$-\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{s\sqrt{1-\Omega_0\kappa_1}x}-\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-2s\sqrt{1-\Omega_0\kappa_1}x}$$

$$\left.+\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-s\sqrt{1-\Omega_0\kappa_1}x}\right)\leq\frac{1}{2}\left(\frac{3}{s\sqrt{1-\Omega_0\kappa_1}}\right),$$

due to

$$-\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{s\sqrt{1-\Omega_0\kappa_1}x}<0,\qquad -\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-2s\sqrt{1-\Omega_0\kappa_1}x}<0,$$

and

$$\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}e^{-s\sqrt{1-\Omega_0\kappa_1}x}\leq\frac{1}{s\sqrt{1-\Omega_0\kappa_1}}.$$

And finally

$$\sup_{x\in[0,\infty)}\int_0^\infty\left|\frac{\partial g}{\partial t}\right|dt\leq\frac{3}{2s\sqrt{1-\Omega_0\kappa_1}}<\infty. \tag{1.35}$$

This implies that $\dfrac{\partial g(s,t;\Omega_0,s)}{\partial t}\in X_2^{(1)}$.

For the other functions the proof is similar.

Using the formula (1.16), Eq. (1.13) can be written in the form

$$G(s,t;\Omega_0,s)=g(s,t;\Omega_0,s)-NG(s,t;\Omega_0,s) \tag{1.36}$$

and a solution to this equation can be obtained by the iteration procedure ([3] pp. 30–31)

$$g_{n+1}=-Ng_n+g_0, \tag{1.37}$$

where

$$g_0=g(s,t;\Omega_0,s). \tag{1.38}$$

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1.3. Time derivatives of the Green function G

From Eq. (1.13) by formal differentiation with respect to t, we obtain:

$$(1.39)\qquad \frac{\partial G(s,t;\Omega_0,s)}{\partial t} = \frac{\partial g(s,t;\Omega_0,s)}{\partial t} - s^2\Omega_0 \int_0^\infty g(s,t;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\frac{\partial}{\partial t}G(\xi,t;\Omega_0,s)\,d\xi.$$

From Lemma 2 and Lemma 4 it follows that the solution of Eq. (1.39) beongs to the $X_2^{(1)}$ space. It is easy to show:

THEOREM 1. *Function* $\frac{\partial G(s,t;\Omega_0,s)}{\partial t}$ *is continuous for every*

$$(x,t;\Omega_0,s) \in [0,\infty)\times[0,\infty)\times(0,1)\times(0,1)$$

such that $t \neq x$.

P r o o f. Equation (1.39) may be written in the form:

$$(1.40)\qquad \frac{\partial G(s,t;\Omega_0,s)}{\partial t} - \frac{\partial g(s,t;\Omega_0,s)}{\partial t} = -s^2\Omega_0 \int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\frac{\partial g(\xi,t;\Omega_0\,s)}{\partial t}d\xi$$
$$-s^2\Omega_0 \int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\left\{\frac{\partial G(\xi,t;\Omega_0,s)}{\partial t} - \frac{\partial g(\xi,t;\Omega_0,s}{\partial t}\right\}dt.$$

Applying the estimates similar to those used in the proof of Lemma 2 one can show, that the function

$$(1.41)\qquad l(x,t;\Omega_0,s) = -s^2\Omega_0 \int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\frac{\partial g(\xi,t;\Omega_0,s)}{\partial t}d$$

is continuous with respect to t, for every $x \in [0,\infty)$ and $(\Omega_0,s)\in(0,1)\times(1,\infty)$. Indeed, the integral is uniformly convergent with respect to t, due to the estinates used in the proof of Lemma 2.

Continuity of l with respect to t and Eq. (1.41) imply that $\frac{\partial G}{\partial t} - \frac{\partial g}{\partial t}$ beongs to X, $X_2^{(1)}$ or $X_1^{(-1/2)}$ if the condition (1.18) is fulfilled.

Applying the iterative procedure to Eq. (1.40) one can show, that the function

$$\frac{\partial}{\partial t}G(x,t;\Omega_0,s) - \frac{\partial}{\partial t}g(x,t;\Omega_0,s)$$

is continuous with respect to t, for $t \neq x$, which ends the proof.

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One can prove:

THEOREM 2. *The function* $\frac{\partial}{\partial t}G(x,t;\Omega_0,s)$ *has the same points of discontinuity as the function* $\frac{\partial}{\partial t}g(x,t;\Omega_0,s)$.

P r o o f. From formula (1.33) it follows that

$$\frac{\partial}{\partial t}g(x,t;\Omega_0,s)\Big|_{t=x+0}-\frac{\partial}{\partial t}g(x,t;\Omega_0,s)\Big|_{t=x-0}=-1. \tag{1.42}$$

Due to Theorem 1 the function $\frac{\partial G(x,t;\Omega_0,s)}{\partial t}$ is continuous with respect to t, except $t=x$, where the discontinuity of the first kind appears, i.e.

$$\frac{\partial G(x,t;\Omega_0,s)}{\partial t}\Big|_{t=x+0}-\frac{\partial G(x,t;\Omega_0,s)}{\partial t}\Big|_{t=x-0}=-1. \tag{1.43}$$

The type of discontinuity of function G follows from the definition of the Green function for the operator L. In order to establish the properties of the second derivative with respect to t, we shall transform Eq. (1.39) to the form

$$\mathcal{L}(x,t;\Omega_0,s)=\widehat{l}(x,t;\Omega_0,s)-s^2\Omega_0\int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\mathcal{L}(\xi,t;\Omega_0,s)\,d\xi, \tag{1.44}$$

where

$$\begin{aligned}\mathcal{L}(x,t;\Omega_0,s)&=\frac{\partial}{\partial t}G(x,t;\Omega_0,s)-\frac{\partial}{\partial t}g(x,t;\Omega_0,s),\\ \widehat{l}(x,t;\Omega_0,s)&=-s^2\Omega_0\int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\frac{\partial\mathcal{L}(\xi,t;\Omega_0,s)}{\partial t}d\xi.\end{aligned} \tag{1.45}$$

Taking the derivative with respect to t we obtain

$$\frac{\partial\mathcal{L}}{\partial t}=\frac{\partial\widehat{l}}{\partial t}-s^2\Omega_0\int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\cdot\frac{\partial\mathcal{L}(\xi,t;\Omega_0,s)}{\partial t}d\xi \qquad \text{for} \quad t\neq x. \tag{1.46}$$

Denote the first term on the R.H.S. of Eq. (1.46) by $\widetilde{m}(x,t;\Omega_0,s)$ and consider the equation

$$M(x,t;\Omega_0,s)=\widetilde{m}(x,t;\Omega_0,s)-s^2\Omega_0\int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]M(\xi,t;\Omega_0,s)\,d\xi \tag{1.47}$$

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in which $M(x,t;\Omega_0,s)$ is an unknown function. It can be shown that the function $\widetilde{m} \in X_1^{(-1/2)}$. The proof is analogous to that proposed by KOSTUČENKO [2].

From Lemma 3 it follows that if

$$q = \Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1} < 1,$$

then a solution of Eq. (1.47) belongs to $X_1^{(-1/2)}$. We are to show that this solution for $t \neq x$ is identical with the function

$$\frac{\partial^2}{\partial t^2}[G(x,t;\Omega_0,s) - g(x,t;\Omega_0,s)].$$

In order to do this we integrate (1.47) with respect to t over the interval $[0,t]$ and we get

$$(1.48)\qquad \int_0^t M(x,\hat{t};\Omega_0,s)\,d\hat{t} = \int_0^t \widetilde{m}(x,\hat{t};\Omega_0,s)\,d\hat{t}$$
$$-s^2\Omega_0\int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)]\left\{\int_0^t M(\xi,\hat{t};\Omega_0,s)\,d\hat{t}\right\}d\xi.$$

From (1.44) it follows that the equation

$$(1.49)\qquad \mathcal{L}(x,t;\Omega_0,s) - \mathcal{L}(x,0;\Omega_0,s) = \hat{l}(x,t;\Omega_0,s) - \hat{l}(x,0;\Omega_0,s)$$
$$-s^2\Omega_0\int_0^\infty g(x,\xi;\Omega_0,s)[\kappa(x)-\kappa(\xi)][\mathcal{L}(\xi,t;\Omega_0,s)-\mathcal{L}(x,0;\Omega_0,s)]\,d\xi,$$

and existence as well as uniqueness of the solution of Eq. (1.48) imply that

$$(1.50)\qquad \int_0^t M(x,\hat{t};\Omega_0,s)\,d\hat{t} = \mathcal{L}(x,t;\Omega_0,s) - \mathcal{L}(x,0;\Omega_0,s).$$

The last relation implies

$$(1.51)\qquad M(x,t;\Omega_0,s) = \frac{\partial}{\partial t}\mathcal{L}(x,t;\Omega_0,s)$$
$$= \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}G(x,t;\Omega_0,s) - \frac{\partial}{\partial t}g(x,t;\Omega_0,s)\right].$$

Because $\dfrac{\partial^2}{\partial t^2}g(x,t;\Omega_0,s)$ for $x \neq t$ belongs to $X_1^{(-1/2)}$, therefore $\dfrac{\partial^2 G}{\partial t^2}$ belongs to $X_1^{(-1/2)}$. From $\dfrac{\partial^2 G}{\partial t^2} \in X_1^{(-1/2)}$ and Eq. (1.51) we obtain:

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THEOREM 3. *If* $q = \Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1} < 1$, *then* $G(x,t;\Omega_0,s)$ *satisfies the equation*

$$(1.52)\qquad \begin{aligned} \frac{\partial^2 G(x,t;\Omega_0,s)}{\partial t^2} &= s^2(1-\Omega_0\kappa(t))G(x,t;\Omega_0,s),\\ G(x,t;\Omega_0,s) &= G(t,x;\Omega_0,s), \end{aligned}$$

and conditions

$$(1.53)\qquad \begin{aligned} G(x,t;\Omega_0,s)|_{t=0} &= 0,\\ \frac{\partial G(x,t;\Omega_0,s)}{\partial t}\Big|_{t=x+0} - \frac{\partial G(x,t;\Omega_0,s)}{\partial t}\Big|_{t=x-0} &= -1. \end{aligned}$$

In other words, $G(x,t;\Omega_0,s)$ is a Green function for the boundary value problem:

$$(1.54)\qquad \begin{aligned} L\beta(x) &= 0,\\ \beta(0) &= 0. \end{aligned}$$

Clearly, a solution to (1.5)–(1.6) expressed by G takes the form

$$(1.55)\qquad \beta(x;\Omega_0,s) = C_1\int_0^\infty G(x,t;\Omega_0,s)[1-\kappa(t)]e^{-s\sqrt{1-\Omega_0}t}\,dt \qquad (C_1 = \text{const}).$$

Since condition (1.7) can be written in the form

$$(1.56)\qquad \beta(0;\Omega_0,s) = -C_1\Omega_0\sqrt{1-\Omega_0}/4s(1-\Omega_0),$$

a solution to the eigen-problem (1.5)–(1.7) is defined by the pair $(\Omega_0,\beta(x))$ in which Ω_0 is a solution to the equation

$$(1.57)\qquad \int_0^\infty \left[\frac{\partial}{\partial t}G\right]_{x=0}[1-\kappa(t)]e^{-st\sqrt{1-\Omega_0}}\,dt + \Omega_0\sqrt{1-\Omega_0}/4s(1-\Omega_0) = 0,$$

and $\beta(x)$ is given by the formula (1.55).

Using the formulae (1.37)–(1.38), (1.55) and (1.57), we get a solution of the eigen-problem if $q < 1$, e.g.

$$(1.58)\qquad \Omega_0 m < s(1-\Omega_0\kappa_1)^{3/2}.$$

In general, the Eq. (1.57) has a finite number of solutions $\Omega_0 = \Omega_0(s)$, (cf. [4]).

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2. Surface waves in a weakly nonhomogeneous isotropic elastic half-space

The problem of propagation of a surface stress wave of the form

$$
\begin{aligned}
\tau_{11}(x_1,x_2,t) &= \alpha_{11}(x_2)\cos(sx_1 - t\sqrt{\lambda}),\\
\tau_{22}(x_1,x_2,t) &= \alpha_{22}(x_2)\cos(sx_1 - t\sqrt{\lambda}),\\
\tau_{12}(x_1,x_2,t) &= -\alpha_{12}(x_2)\sin(sx_1 - t\sqrt{\lambda}),
\end{aligned}
\tag{2.1}
$$

in a nonhomogeneous elastic half-space

$$X = \{(x_1,x_2):\quad x_2 \ge 0,\ \ |x_1| < \infty\}$$

reduces to the following eigenvalue problem [5]: find a real symmetric tensor field $\alpha_{ij} = \alpha_{ij}(x_2)$ ($\alpha_{ij} \in C^2[0,\infty)$; $i,j = 1,2$) and a real number λ ($\lambda > 0$) satisfying the system of equations

$$
\begin{aligned}
\varrho^{-1}(s^2\alpha_{11} + s\,\dot{\alpha}_{12}) - \lambda(2\mu)^{-1}(\alpha_{11} - \nu\alpha_{\gamma\gamma}) &= 0,\\
-[\varrho^{-1}(\dot{\alpha}_{22} + s\alpha_{12})]^{\cdot} - \lambda(2\mu)^{-1}(\alpha_{22} - \nu\alpha_{\gamma\gamma}) &= 0,\\
-[\varrho^{-1}(s^2\,\dot{\alpha}_{12} + s\alpha_{11})]^{\cdot} - s\varrho^{-1}(\dot{\alpha}_{22} - s\alpha_{12}) - \lambda(2\mu)^{-1}2\alpha_{12} &= 0
\end{aligned}
\tag{2.2}
$$

$$\text{for}\quad x_2 \in (0,\infty)\quad (\gamma = 1,2)$$

and the boundary conditions

$$\alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = 0, \tag{2.3}$$

s being the wave number ($s > 0$), and $\varrho = \varrho(x_2)$, $\mu = \mu(x_2)$, $\nu = \nu(x_2)$ denoting, respectively, the density of the medium, the shear modulus, and the Poisson ratio. The functions are assumed to be of the $C^2[0,\infty)$ class, and to satisfy the following inequalities

$$
\begin{aligned}
0 < \varrho_0 \le \varrho(x_2) \le \varrho_1 < \infty,\\
0 < \mu_0 \le \mu(x_2) \le \mu_1 < \infty,\\
-1 < \nu_0 \le \nu(x_2) \le \nu_1 < 1/2.
\end{aligned}
\tag{2.4}
$$

A dot over a symbol denotes differentiation with respect to the variable x_2; we will also use the symbol D to denote the derivative.

The aim of this paper is to give an approximate solution of the eigenvalue problem (2.2)–(2.3) in the following two cases:

1) density $\varrho = \varrho(x_2)$ is a "weakly" variable function, and μ and ν are constant;

2) shear modulus $\mu = \mu(x_2)$ is a "weakly" variable function, and ϱ and ν are constant.

In both cases we obtain the approximate solution by using the perturbation method proposed by FRIEDRICHS in [6].

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2.1. Analysis of the case $\dfrac{1}{\varrho(x_2)} = \dfrac{1}{\varrho_1} + \dfrac{\varepsilon}{\widehat{\varrho}(x_2)}$, $\mu(x_2) \equiv \mu_1$, $\nu(x_2) \equiv \nu_1$

Let us consider in the real Hilbert space $\mathcal{H}$ generated by the scalar product

$$(\alpha, \beta) = \int_0^\infty (\alpha_{11}\beta_{11} + \alpha_{22}\beta_{22} + \alpha_{12}\beta_{12})\, dx_2$$

and satisfying the condition

$$\|\alpha\|^2 = \int_0^\infty (\alpha_{11}^2 + \alpha_{22}^2 + \alpha_{12}^2)\, dx_2 < \infty.$$

Equation (2.2) written in operator form

$$A\alpha - \lambda B\alpha = 0, \tag{2.5}$$

where

$$\alpha = \begin{pmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{12} \end{pmatrix},$$

$$A \equiv A(s;p) \equiv \begin{bmatrix} \dfrac{s^2}{\varrho} & 0 & \dfrac{s}{\varrho} D \\ 0 & -D\dfrac{1}{\varrho}D & sD\dfrac{1}{\varrho} \\ -sD\dfrac{1}{\varrho} & -\dfrac{s}{\varrho}D & -D\dfrac{1}{\varrho}D + \dfrac{s^2}{\varrho} \end{bmatrix},$$

$$B \equiv B(\mu;\nu) \equiv \begin{bmatrix} \dfrac{1-\nu}{2\mu} & \dfrac{-\nu}{2\mu} & 0 \\ \dfrac{-\nu}{2\mu} & \dfrac{1-\nu}{2\mu} & 0 \\ 0 & 0 & \dfrac{1}{\mu} \end{bmatrix}.$$

The domains of operators A and B may be defined as

$$\begin{aligned} \mathcal{D}(A) &= \left\{\alpha\colon\ \alpha_{ij} \in C^2[0,\infty),\ \alpha_{12}(0) = \alpha_{22}(0) = \alpha_{12}(\infty) = \alpha_{22}(\infty) = 0\right\}, \\ \mathcal{D}(B) &= \left\{\alpha\colon\ \alpha_{ij} \in C^2[0,\infty)\right\}, \quad i,j = 1,2. \end{aligned} \tag{2.6}$$

The sets $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are dense in $\mathcal{H}$ since the set $C_0^\infty[0,\infty) \times C_0^\infty[0,\infty) \times C_0^\infty[0,\infty)$ is dense in $\mathcal{H}$ and is contained in $\mathcal{D}(A)$ and $\mathcal{D}(B)$.

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It can be demonstrated that operators A and B are symmetric. The symmetry of operator A results from the fact that operators on both sides of the principal diagonal are formally adjoint, e.g. $-sD\frac{1}{\varrho}$ and $\frac{s}{\varrho}D$, $sD\frac{1}{\varrho}$ and $\frac{-s}{\varrho}D$. For any $\alpha, \beta \in \mathcal{D}(A)$ we have

$$(A\alpha,\beta) = \int_0^\infty \Big\{\varrho^{-1}(s^2\alpha_{11} + s\,\dot{\alpha}_{12})\beta_{11} - \Big[\varrho^{-1}(\dot{\alpha}_{22} - s\alpha_{12})\Big]^{\cdot}\beta_{22}$$
$$- \Big[\varrho^{-1}(\dot{\alpha}_{12} + s\alpha_{11})\Big]^{\cdot}\beta_{12} - s\varrho^{-1}(s^2\dot{\alpha}_{12} - s\alpha_{22})\beta_{12}\Big\}dx_2.$$

Integration by parts with the use of boundary conditions shows that

$$(A\alpha,\beta) = (\alpha, A\beta).$$

The symmetry of operator B is obvious. Matrix B is positive definite and for every $\alpha \in \mathcal{D}(B) \subset \mathcal{H}$ we have (¹)

$$(2.7) \qquad (B\alpha,\alpha) \geq k(\alpha,\alpha),$$

where

$$k = \min_{x_2\in[0,\infty)}\left(\frac{1-2\nu}{2\mu}, \frac{1}{\mu}, \frac{1}{2\mu}\right).$$

If in Eq. (2.5) we put $\varrho \equiv \widetilde{\varrho} = \text{const}$, $\mu \equiv \widetilde{\mu} = \text{const}$, $\nu \equiv \widetilde{\nu} = \text{const}$ (homogeneous medium), the problem has precisely one solution $(\widetilde{\alpha}, \widetilde{\lambda})$ of the form

$$(2.8) \qquad \widetilde{\alpha}(\widetilde{\varrho},\widetilde{\mu},\widetilde{\nu}) = \begin{bmatrix} -\widetilde{\beta}_0\left[e^{-x_2\widetilde{h}_2} - \dfrac{2+\widetilde{\omega}(1-2\widetilde{\kappa})}{2-\widetilde{\omega}}e^{-x_2\widetilde{h}_1}\right] \\ \widetilde{\beta}_0\left[e^{-x_2\widetilde{h}_2} - e^{-x_2\widetilde{h}_1}\right] \\ -\dfrac{2}{s}\dfrac{\widetilde{\beta}_0}{2-\widetilde{\omega}}\widetilde{h}_1\left[e^{-x_2\widetilde{h}_2} - e^{-x_2\widetilde{h}_1}\right] \end{bmatrix},$$

where

$$\widetilde{\kappa} = \frac{1-2\widetilde{\nu}}{2-2\widetilde{\nu}}, \qquad \widetilde{h}_1 = s\sqrt{1-\widetilde{\omega}\widetilde{\kappa}}, \qquad \widetilde{h}_2 = s\sqrt{1-\widetilde{\omega}}$$

and $\widetilde{\omega}$ is a root of the equation

$$(2.9) \qquad (2-\widetilde{\omega})^2 = 4\sqrt{(1-\widetilde{\omega})(1-\widetilde{\omega}\widetilde{\kappa})}$$

(¹) The eigenvalues of matrix B are $\frac{1-2\nu}{2\mu}, \frac{1}{\mu}, \frac{1}{2\mu}$. From the theorem in [8] saying that a symmetric matrix B is positive definite iff all its eigenvalues are positive and $(B\alpha,\alpha) \geq \min_i \lambda_i(\alpha,\alpha)$, results the Eq. (27).

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such that $0 < \widetilde{\omega} < 1$; $\widetilde{\beta}_0$ is an arbitrary real number. The surface wave velocity in this homogeneous medium is given by

$$\widetilde{C}_R = \sqrt{\frac{\widetilde{\mu}}{\widetilde{\varrho}}} \cdot \sqrt{\widetilde{\omega}}.$$

The relation between $\widetilde{\lambda}$ and $\widetilde{C}_R$ is of the form

$$\sqrt{\widetilde{\lambda}} = s\widetilde{C}_R.$$

Let us consider the case when

$$\frac{1}{\varrho(x_2)} = \frac{1}{\varrho_1} + \frac{\varepsilon}{\widehat{\varrho}(x_2)}, \qquad \nu \equiv \nu_1, \qquad \mu \equiv \mu_1 \tag{2.10}$$

and ε is a sufficiently small positive real number. Moreover, ϱ_1 is a positive constant, and $\widehat{\varrho}(x_2)$ is a positive function (cf. (2.4)). After substituting (2.10) into (2.5) we get the equation

$$A_0\alpha + \varepsilon V\alpha = \lambda B\alpha\,, \tag{2.11}$$

where

$$\begin{aligned} A_0 &= A(s; \varrho_1), \\ V &= V(s; \widehat{\varrho}), \\ B &= B(\mu_1; \nu_1). \end{aligned}$$

The constraints on ϱ (cf. (2.4)) and (2.10) yield the constraints on $\widehat{\varrho}$ for $x_2 \in [0, \infty)$. Hence for every $\alpha \in \mathcal{D}(A) \equiv \mathcal{D}(V) \subset \mathcal{H}$

$$(V\alpha, \alpha) < \infty. \tag{2.12}$$

Moreover the operators A_0, V and B are symmetric in the space $\mathcal{H}$. From the fact that $\widetilde{\lambda}(\varrho_1, \mu_1, \nu_1)$ is a simple eigenvalue (the eigenspace is one-dimensional) it follows that $(A_0 - \widetilde{\lambda}B)^{-1}$ is defined in the subspace $\mathcal{H}$ orthogonal to the vector $\widetilde{\alpha}(\varrho_1, \mu_1, \nu_1)$ (²). Hence for sufficiently small ε in a neighbourhood of $(\widetilde{\lambda}(\varrho_1, \nu_1, \mu_1), \widetilde{\alpha}(\varrho_1, \nu_1, \mu_1))$ there exists a solution $(\lambda_\varepsilon, \alpha_\varepsilon)$ satisfying Eq. (2.11), analytical with respect to ε, of the form

$$\lambda_\varepsilon = \widetilde{\lambda} + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots\,, \tag{2.13}$$

$$\alpha_\varepsilon = \widetilde{\alpha} + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \ldots\,, \tag{2.14}$$

(²) FRIEDRICHS [6] formulates the following assumptions of a perturbation theory; the operator must be symmetric, it must allow for a spectral decomposition, it must have a simple eigenvalue $\widetilde{\lambda}$ with the corresponding eigenvector $\widetilde{\alpha}$. Hence the equation $[A_0 - \widetilde{\lambda}B]\alpha = \Psi$ has a solution for any right-hand side of Ψ orthogonal to $\widetilde{\alpha}$ in the space $\mathcal{H}$. It is easily seen that Friedrich's assumptions are satisfied for the problem (2.5) – (2.6).

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where

$$\alpha_i = \begin{pmatrix} \alpha_{11}^{(i)} \\ \alpha_{22}^{(i)} \\ \alpha_{12}^{(i)} \end{pmatrix}, \qquad i = 1, 2, 3 \ldots .$$

Substituting (2.13) and (2.14) into (2.11) and comparing the expressions appearing at suitable powers of ε we get

$$\begin{aligned}
&(A_0 - \widetilde{\lambda} B)\widetilde{\alpha} = 0, \\
&(A_0 - \widetilde{\lambda} B)\alpha_1 = -(V - \lambda_1 B)\widetilde{\alpha}, \\
&(A_0 - \widetilde{\lambda} B)\alpha_2 = -(V - \lambda_1 B)\alpha_1 + \lambda_2 B\widetilde{\alpha}, \\
&(A_0 - \widetilde{\lambda} B)\alpha_3 = -(V - \lambda_1 B)\alpha_2 + \lambda_2 B\alpha_1 + \lambda_3 B\widetilde{\alpha}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .
\end{aligned} \tag{2.15}$$

Multiplying the first equation by α_1 and the second by $\widetilde{\alpha}$, and subtracting we get

$$\lambda_1 = \frac{(V\widetilde{\alpha}, \widetilde{\alpha})}{(B\widetilde{\alpha}, \widetilde{\alpha})} . \tag{2.16}$$

Analogously, multiplying the first equation by α_2, and the third one by $\widetilde{\alpha}$ and subtracting we obtain

$$\lambda_2 = \frac{(V\alpha_1, \widetilde{\alpha}) - \lambda_1(B\alpha_1, \widetilde{\alpha})}{(B\widetilde{\alpha}, \widetilde{\alpha})} . \tag{2.17}$$

In general, we get λ_i ($i \geq 3$) by multiplying the first equation by α_i, multiplying the $(i+1)$-th equation by $\widetilde{\alpha}$ and subtracting both sides of the relations.

Equations (2.13), (2.16) and (2.17) effectively determine the approximate eigenvalue in the problem with weakly variable density in the considered half-space.

We now proceed to construct the series α_i. It is easy to demonstrate that the right-hand sides of the system (2.15) are elements of a subspace $\mathcal{H}$ orthogonal to the vector $\widetilde{\alpha}$. The construction of the series α_i is thus reduced to finding an operator $[A_0 - \widetilde{\lambda}(\varrho_1, \mu_1, \nu_1)B(\mu_1, \nu_1)]^{-1}$ on a subspace orthogonal to $\widetilde{\alpha}(\varrho_1, \mu_1, \nu_1)$. To this end, let us consider the equation

$$A_0\widehat{\alpha} - \widetilde{\lambda}(\varrho_1, \mu_1, \nu_1)B\widehat{\alpha} = g, \tag{2.18}$$

where

$$\widehat{\alpha} = \begin{pmatrix} \widehat{\alpha}_{11} \\ \widehat{\alpha}_{22} \\ \widehat{\alpha}_{12} \end{pmatrix}, \qquad g = \begin{pmatrix} g_{11} \\ g_{22} \\ g_{12} \end{pmatrix},$$

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and g is a vector of the subspace $\mathcal{H}$ satisfying the condition

$$(g, \widetilde{\alpha}) = 0. \tag{2.19}$$

The vector α_ε given by (2.14) should belong to $\mathcal{D}(A)$. Thus to construct α_i it is enough to find $\widehat{\alpha}$ satisfying (2.18), (2.19) such that $\widehat{\alpha} \in \mathcal{D}(A)$. It can be shown that vector $\widehat{\alpha}$ is of the form

$$\widehat{\alpha} = \begin{bmatrix} \widehat{\alpha}_{11}(x_2) \\ \widehat{\alpha}_{22}(x_2) \\ \widehat{\alpha}_{12}(x_2) \end{bmatrix} = \begin{bmatrix} \int\limits_0^\infty K_1(x_2, t)F(t)dt \\ \int\limits_0^\infty K_2(x_2, t)F(t)dt \\ \int\limits_0^\infty K_3(x_2, t)F(t)dt \end{bmatrix} + \begin{bmatrix} G_{11}(x_2) \\ G_{22}(x_2) \\ G_{12}(x_2) \end{bmatrix}, \tag{2.20}$$

where

$$K_1(x_2, t) = \begin{cases} l_3 e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(t-x_2)} + l_4 e^{-s\sqrt{1-\widetilde{\omega}}(t-x_2)} \\ \qquad + \left[\dfrac{2(1-\widetilde{\omega}\kappa_1)+\widetilde{\omega}}{\widetilde{\omega}-2}\right] a_1 e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(t+x_2)} \\ \qquad + \left[\dfrac{2(1-\widetilde{\omega}\kappa_1)+\widetilde{\omega}}{\widetilde{\omega}-2}\right] a_2 e^{-s(\sqrt{1-\widetilde{\omega}}t+\sqrt{1-\widetilde{\omega}\kappa_1}x_2)} \\ -b_1 e^{-s(\sqrt{1-\widetilde{\omega}\kappa_1}t+\sqrt{1-\widetilde{\omega}}x_2)} - b_2 e^{-s\sqrt{1-\widetilde{\omega}}(t+x_2)} \quad \text{for } t \geq x_2, \\ l_3 e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(x_2-t)} + l_4 e^{-s\sqrt{1-\widetilde{\omega}}(x_2-t)} \\ \qquad + \left[\dfrac{2(1-\widetilde{\omega}\kappa_1)+\widetilde{\omega}}{\widetilde{\omega}-2}\right] a_1 e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(t+x_2)} \\ \qquad + \left[\dfrac{2(1-\widetilde{\omega}\kappa_1)+\widetilde{\omega}}{\widetilde{\omega}-2}\right] a_2 e^{-s(\sqrt{1-\widetilde{\omega}}t+\sqrt{1-\widetilde{\omega}\kappa_1}x_2)} \\ -b_1 e^{-s(\sqrt{1-\widetilde{\omega}\kappa_1}t+\sqrt{1-\widetilde{\omega}}x_2)} - b_2 e^{-s\sqrt{1-\widetilde{\omega}}(t+x_2)} \quad \text{for } x_2 \geq t, \end{cases} \tag{2.21}$$

$$K_2(x_2, t) = \begin{cases} l_1 e^{s\sqrt{1-\widetilde{\omega}\kappa_1}(x_2-t)} + l_2 e^{-s\sqrt{1-\widetilde{\omega}}(x_2-t)} + a_1 e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(t+x_2)} \\ \qquad + a_2 e^{-s(\sqrt{1-\widetilde{\omega}}t+\sqrt{1-\widetilde{\omega}\kappa_1}x_2)} + b_1 e^{-s(\sqrt{1-\widetilde{\omega}\kappa_1}t+\sqrt{1-\widetilde{\omega}}x_2)} \\ \qquad + b_2 e^{-s\sqrt{1-\widetilde{\omega}}(t+x_2)} \qquad \text{for } t \geq x_2, \\ l_1 e^{s\sqrt{1-\widetilde{\omega}\kappa_1}(x_2-t)} + l_2 e^{s\sqrt{1-\widetilde{\omega}}(x_2-t)} + a_1 e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(t+x_2)} \\ \qquad + a_2 e^{-s(\sqrt{1-\widetilde{\omega}}t+\sqrt{1-\widetilde{\omega}\kappa_1}x_2)} + b_1 e^{-s(\sqrt{1-\widetilde{\omega}\kappa_1}t+\sqrt{1-\widetilde{\omega}}x_2)} \\ \qquad + b_2 e^{-s\sqrt{1-\widetilde{\omega}}(t+x_2)} \qquad \text{for } x_2 \geq t, \end{cases} \tag{2.22}$$

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$$(2.23)\qquad K_3(x_2,t)=\begin{cases} l_5e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(t-x_2)}+l_6e^{-s\sqrt{1-\widetilde{\omega}}(t-x_2)}+a_1l_7e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(x_2+t)} \\ \quad +a_2l_7e^{-s(\sqrt{1-\widetilde{\omega}}t+\sqrt{1-\widetilde{\omega}\kappa_1}x_2)}+b_1l_8e^{-s(\sqrt{1-\widetilde{\omega}\kappa_1}t+\sqrt{1-\widetilde{\omega}}x_2)} \\ \quad +b_2l_8e^{-s\sqrt{1-\widetilde{\omega}}(x_2+t)} \qquad \text{for } t\ge x_2, \\ -l_5e^{s\sqrt{1-\widetilde{\omega}\kappa_1}(t-x_2)}-l_6e^{s\sqrt{1-\widetilde{\omega}}(t-x_2)}+a_1l_7e^{-s\sqrt{1-\widetilde{\omega}\kappa_1}(x_2+t)} \\ \quad +a_2l_7e^{-s(\sqrt{1-\widetilde{\omega}}t+\sqrt{1-\widetilde{\omega}\kappa_1}x_2)}+b_1l_8e^{-s(\sqrt{1-\widetilde{\omega}\kappa_1}t+\sqrt{1-\widetilde{\omega}}x_2)} \\ \quad +b_2l_8e^{-s\sqrt{1-\widetilde{\omega}}(t+x_2)} \qquad \text{for } x_2\ge t, \end{cases}$$

$$(2.24)\qquad F(t)=-\varrho_1\left[D^2-k_1^2\right]g_{22}(t)+\varrho_1\widetilde{\omega}^{-1}\left[2-2\widetilde{\omega}-2\kappa_1+\widetilde{\omega}\kappa_1\right]\left[D^2+k_2^2\right]g_{11}(t) +2\varrho_1 s(2-\widetilde{\omega})(1-\kappa_1)\widetilde{\omega}^{-1}Dg_{12}(t),$$

$$(2.25)\qquad G_{11}(x_2)=\frac{2\varrho_1}{s^2(\widetilde{\omega}-2)}(g_{22}-g_{11}),$$

$$(2.26)\qquad G_{22}(x_2)=0,$$

$$(2.27)\qquad G_{12}(x_2)=0.$$

The coefficients $l_1, l_2, ..., l_8$, k_1^2, k_2^2, a_1, a_2, b_1, b_2, appearing in Eqs. (2.21)–(2.27) are given in the Appendix II.

Using Eqs. (2.18)–(2.27) and the relations (2.16) and (2.17) we can find successively (λ_i, α_i).

Let us now analyse the eigenvalue λ_ε (cf. (2.13)) in the case when the function $\widehat{\varrho}=\widehat{\varrho}(x_2)$ (cf. (2.10)) is a monotonic function of the half-space depth coordinate.

Assume that

$$(2.28)\qquad \frac{1}{\varrho(x_2)}=\frac{1}{\varrho_1}+\frac{\varepsilon}{\widehat{\varrho}_\infty}(1-e^{-ax_2}), \qquad (a\ge 0), \qquad (\widehat{\varrho}_\infty>0).$$

Since $\dfrac{1}{\varrho_1}\le\dfrac{1}{\varrho(x_2)}\le\dfrac{1}{\varrho_0}$ we have on the one hand $\max\limits_{x_2\in[0,\infty)}\dfrac{1}{\varrho(x_2)}=\dfrac{1}{\varrho_0}$ and on the other hand, $\max\limits_{x_2\in[0,\infty)}\dfrac{1}{\varrho(x_2)}=\dfrac{1}{\varrho_1}+\dfrac{\varepsilon}{\widehat{\varrho}_\infty}$. Comparing these values we get

$$(2.29)\qquad \varepsilon=\widehat{\varrho}_\infty\left(\frac{1}{\varrho_0}-\frac{1}{\varrho_1}\right),$$

where $\varrho_1/\varrho_0\sim 1$.

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Substituting (2.28) and (2.29) into (2.16), taking into account the relations

$$\lambda_\varepsilon = (C_R^2)_\varepsilon s^2, \qquad \widetilde{\lambda} = \widetilde{C}_R^2 s^2, \qquad \lambda_1 = (C_R^2)_1 s^2, \qquad \kappa_1 = \frac{1-2\nu_1}{2-2\nu_1}$$

and limiting ourselves to two terms in Eq. (2.13) we get for the square of surface wave velocity the relation

$$(2.30) \qquad (C_R^2)_\varepsilon = \widetilde{C}_R^2 + \left(\frac{1}{\varrho_0} - \frac{1}{\varrho_1}\right)\frac{\mu_1 a}{2(1-\widetilde{\omega})} \times \left[\frac{P_0(\widetilde{\omega})}{\sqrt{1-\widetilde{\omega}\kappa_1}(a+2s\sqrt{1-\widetilde{\omega}\kappa_1})} + \frac{P_1(\widetilde{\omega})}{\sqrt{1-\widetilde{\omega}}(a+2s\sqrt{1-\widetilde{\omega}})} + \frac{P_2(\widetilde{\omega})}{(\sqrt{1-\widetilde{\omega}}+\sqrt{1-\widetilde{\omega}\kappa_1})(a+s\sqrt{1-\widetilde{\omega}}+s\sqrt{1-\widetilde{\omega}\kappa_1})}\right] \times \left[\frac{P_3(\widetilde{\omega},\kappa_1)}{\sqrt{1-\widetilde{\omega}}} + \frac{P_4(\widetilde{\omega},\kappa_1)}{\sqrt{1-\widetilde{\omega}\kappa_1}} + \frac{P_5(\widetilde{\omega},\kappa_1)}{\sqrt{1-\widetilde{\omega}}+\sqrt{1-\widetilde{\omega}\kappa_1}}\right]^{-1}$$

(the polynomials $P_0, P_1, P_2, P_3, P_4, P_5$ are given in the Appendix II). Introducing the following notation

$$\theta = \frac{\varrho_1}{\varrho_0}, \quad C_2^2 = \frac{\mu_1}{\varrho_1}, \qquad a = \frac{s}{2\pi}\widehat{a} \qquad (\widehat{a} \in [0,\infty)),$$

$$\widetilde{\omega} = \frac{\widetilde{C}_R^2}{C_2^2}, \qquad \omega = \frac{(C_R^2)}{C_2^2},$$

we rewrite the formula (2.30) in the form

$$(2.31) \qquad \omega = \widetilde{\omega} + (\theta-1)\frac{\widehat{a}}{2(1-\widetilde{\omega})}\left[\frac{P_0}{\sqrt{1-\widetilde{\omega}\kappa_1}(\widehat{a}+4\pi\sqrt{1-\widetilde{\omega}\kappa_1})} + \frac{P_1}{\sqrt{1-\widetilde{\omega}}(\widehat{a}+4\pi\sqrt{1-\widetilde{\omega}})} + \frac{P_2}{(\sqrt{1-\widetilde{\omega}}+\sqrt{1-\widetilde{\omega}\kappa_1})(\widehat{a}+2\pi\sqrt{1-\widetilde{\omega}}+2\pi\sqrt{1-\widetilde{\omega}\kappa_1})}\right] \cdot \left[\frac{P_3}{\sqrt{1-\widetilde{\omega}}} + \frac{P_4}{\sqrt{1-\widetilde{\omega}\kappa_1}} + \frac{P_5}{\sqrt{1-\widetilde{\omega}}+\sqrt{1-\widetilde{\omega}\kappa_1}}\right]^{-1}.$$

It is easy to demonstrate that the function $\omega = \omega(\widehat{a},\theta,\kappa_1)$ described by (2.31) is for every fixed θ and κ_1 an increasing function of the variable $\widehat{a}$. Figure 1 shows the function ω for: 1) $\kappa_1 = \frac{1}{2}$, $\theta = 1.1$; 2) $\kappa_1 = \frac{1}{2}$, $\theta = 1.01$; 3) $\kappa_1 = \frac{1}{2}$, $\theta = 1$.

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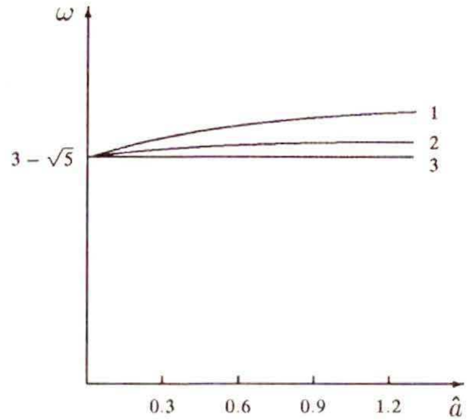

FIG. 1.

2.2. Surface wave in an elastic half-space with a weakly variable shear modulus ([3])

Assume now that

$$\frac{1}{\mu(x_2)} = \frac{1}{\mu_1} + \frac{\varepsilon}{\widehat{\mu}(x_2)}, \qquad \varrho \equiv \varrho_1, \qquad \nu \equiv \nu_1 . \tag{2.32}$$

Substituting (2.32) into (2.5) we get

$$A(s,\varrho_1)\alpha - \lambda[B(\mu_1,\nu_1) + \varepsilon B(\widehat{\mu},\nu_1)]\alpha = 0. \tag{2.33}$$

Operators $A(s,\varrho_1)$, $B(\mu_1,\nu_1)$, $B(\widehat{\mu},\nu_1)$ are symmetric in $\mathcal{H}$. Moreover, B is a positive definite operator. According to the perturbation theory, there exists a solution of the eigenproblem (2.33) determined in some neighbourhood of $(\widetilde{\lambda},\widetilde{\alpha})$ which is an analytical function of the parameter ε. The pair $(\lambda_\varepsilon,\alpha_\varepsilon)$ is given by (2.13) and (2.14), while $(\widetilde{\lambda},\widetilde{\alpha})$ is given by (2.8) and (2.9), where $\widetilde{\varrho} = \varrho_1$, $\widetilde{\mu} = \mu_1$, $\nu = \nu_1$. Substituting (2.13) and (2.14) into (2.33) and comparing the values at suitable powers of ε we obtain the following system

$$\begin{aligned}
&[A(s,\varrho_1) - \widetilde{\lambda}B(\mu_1,\nu_1)]\widetilde{\alpha} \;= 0,\\
&[A(s,\varrho_1) - \widetilde{\lambda}B(\mu_1,\nu_1)]\alpha_1 = [\widetilde{\lambda}B(\widehat{\mu},\nu_1) + \lambda_1 B(\mu_1,\nu_1)]\widetilde{\alpha},\\
&[A(s,\varrho_1) - \widetilde{\lambda}B(\mu_1,\nu_1)]\alpha_2 = \widetilde{\lambda}B(\widehat{\mu},\nu_1)\alpha_1 + \lambda_1[B(\mu_1,\nu_1)\alpha_1 + B(\widehat{\mu},\nu_1)\widetilde{\alpha}]\\
&\qquad\qquad\qquad\qquad\qquad + \lambda_2 B(\mu_1,\nu_1)\widetilde{\alpha},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .
\end{aligned} \tag{2.34}$$

Performing scalar multiplication of $(2.34)_1$ by α_1, of $(2.34)_2$ by $\widetilde{\alpha}$ and subtracting by sides, we get

$$\lambda_1 = \frac{-\widetilde{\lambda}[B(\widehat{\mu},\nu_1)\widetilde{\alpha},\widetilde{\alpha}]}{[B(\mu_1,\nu_1)\widetilde{\alpha},\widetilde{\alpha}]} . \tag{2.35}$$

([3]) This problem was also analysed in [7], using another approach.

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Proceeding similarly as in the derivation of the series (2.16) and (2.17), we get

$$\lambda_2 = \frac{-\widetilde{\lambda}[B(\widehat{\mu},\nu_1)\alpha_1,\widetilde{\alpha}] - \lambda_1\{[B(\mu_1,\nu_1)\alpha_1,\widetilde{\alpha}] + [B(\widehat{\mu},\nu_1)\widetilde{\alpha},\widetilde{\alpha}]\}}{[B(\mu_1,\nu_1)\widetilde{\alpha},\widetilde{\alpha}]} \tag{2.36}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .$$

The vectors α_i are defined by the equations

$$\begin{aligned}\alpha_1 &= [A(s,\varrho_1) - \widetilde{\lambda}B(\mu_1,\nu_1)]^{-1}\widetilde{\lambda}B(\widehat{\mu},\nu_1)\widetilde{\alpha} + \lambda_1 B(\mu_1,\nu_1)\widetilde{\alpha},\\ \alpha_2 &= [A(s,\varrho_1) - \widetilde{\lambda}B(\mu_1,\nu_1)]^{-1}\{\widetilde{\lambda}B(\widehat{\mu},\nu_1)\alpha_1 + \lambda_1[B(\mu_1,\nu_1)\alpha_1\\ &\qquad + B(\widehat{\mu},\nu_1)\widetilde{\alpha}] + \lambda_2 B(\mu_1,\nu_1)\widetilde{\alpha}\},\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .\end{aligned} \tag{2.37}$$

We continue similarly to the case of the half-space with "weakly variable" density and we assume

$$\frac{1}{\widehat{\mu}(x_2)} = \frac{1}{\widehat{\mu}_\infty}(1 - e^{-ax_2}) \qquad (a \geq 0), \quad \nu \equiv \nu_1, \quad \varrho \equiv \varrho_1,$$

where

$$\varepsilon = \widehat{\mu}_\infty\left(\frac{1}{\mu_0} - \frac{1}{\mu_1}\right), \qquad \frac{\mu_1}{\mu_0} \sim 1.$$

From relation (2.35) for the square of wave velocity we get

$$\begin{aligned}(C_R^2)_\varepsilon = \widetilde{C}_R^2 - \left(\frac{1}{\mu_0} - \frac{1}{\mu_1}\right)\widetilde{C}_R^2 a\mu_1 \Bigg[&\frac{P_3}{\sqrt{1-\widetilde{\omega}}(a + 2s\sqrt{1-\widetilde{\omega}})}\\ &+ \frac{P_4}{(\sqrt{1-\widetilde{\omega}} + \sqrt{1-\widetilde{\omega}\kappa_1})(a + s\sqrt{1-\widetilde{\omega}} + s\sqrt{1-\widetilde{\omega}\kappa_1})}\\ &+ \frac{P_5}{(\sqrt{1-\widetilde{\omega}\kappa_1}(a + 2s\sqrt{1-\widetilde{\omega}\kappa_1})}\Bigg]\\ &\cdot\left[\frac{P_3}{\sqrt{1-\widetilde{\omega}}} + \frac{P_5}{\sqrt{1-\widetilde{\omega}\kappa_1}} + \frac{P_4}{\sqrt{1-\widetilde{\omega}} + \sqrt{1-\widetilde{\omega}\kappa_1}}\right]^{-1}.\end{aligned} \tag{2.38}$$

Introducing the following notations

$$\frac{\mu_1}{\varrho_1} = C_2^2, \qquad \theta = \frac{\mu_1}{\mu_0}, \qquad a = \frac{s}{2\pi}\widehat{a},$$

$$\widetilde{\omega} = \frac{\widetilde{C}_R^2}{C_2^2}, \qquad \omega = \frac{(C_R^2)_\varepsilon}{C_2^2},$$

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we reduce (2.38) to the form

$$(2.39)\qquad \omega = \widetilde{\omega} - \widetilde{\omega}(\theta - 1)\widehat{a}\left[\frac{P_3}{\sqrt{1-\widetilde{\omega}}(\widehat{a} + 4\pi\sqrt{1-\widetilde{\omega}})} + \frac{P_4}{(\sqrt{1-\widetilde{\omega}} + \sqrt{1-\widetilde{\omega}\kappa_1})(\widehat{a} + 2\pi\sqrt{1-\widetilde{\omega}} + 2\pi\sqrt{1-\widetilde{\omega}\kappa_1})} + \frac{P_5}{(\sqrt{1-\widetilde{\omega}\kappa_1}(\widehat{a} + 4\pi\sqrt{1-\widetilde{\omega}\kappa_1})}\right] \times \left[\frac{P_3}{\sqrt{1-\widetilde{\omega}}} + \frac{P_5}{\sqrt{1-\widetilde{\omega}\kappa_1}} + \frac{P_4}{\sqrt{1-\widetilde{\omega}} + \sqrt{1-\widetilde{\omega}\kappa_1}}\right]^{-1}.$$

The function ω given by (2.39) for a fixed θ and κ_1 is a decreasing function of the argument $\widehat{\alpha}$. Figure 2 shows the diagrams of the function $\omega(\theta, \kappa_1, \widehat{\alpha})$ for 1) $\kappa_1 = 0.5$, $\theta = 1.1$; 2) $\kappa_1 = 0.5$, $\theta = 1.01$; 3) $\kappa_1 = 0.5$, $\theta = 1$.

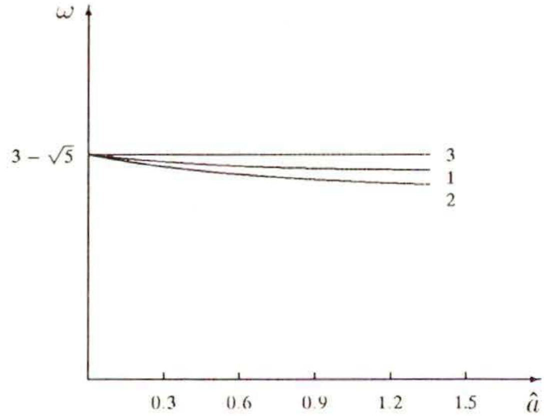

FIG. 2.

Appendix I

I_a: To obtain (1.23) from (1.22) we calculate the integral

$$(A.1)\qquad I_a = \int_0^\infty dx \left[\int_x^\infty a(\xi - x) b(\xi)\, d\xi\right]^2,$$

where

$$(A.2)\qquad \begin{aligned} a(p) &= pe^{-s\sqrt{1-\Omega_0\kappa_1 p}}, \\ b(p) &= |A(p,t)| \qquad (p > 0). \end{aligned}$$

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Changing the variables in (A.1) and using the Fubini theorem, we obtain

$$\text{(A.3)}\qquad I_a = \int_0^\infty dx \left[\int_0^\infty a(p)b(p+x)dp\right]^2$$

$$= \int_0^\infty dx \left[\int_0^\infty a(p)b(p+x)dp\right] \cdot \left[\int_0^\infty a(\widehat{p})b(\widehat{p}+x)\,d\widehat{p}\right]$$

$$= \int_0^\infty a(p)\,dp \cdot \int_0^\infty a(\widehat{p})\,d\widehat{p} \cdot \int_0^\infty dx[b(p+x)b(\widehat{p}+x)].$$

From the Schwartz inequality it follows

$$\text{(A.4)}\qquad \int_0^\infty b(p+x)b(\widehat{p}+x)\,dx \le \left[\int_0^\infty b^2(p+x)dx\right]^{1/2} \cdot \left[\int_0^\infty b^2(\widehat{p}+x)\,dx\right]^{1/2}$$

$$= \left[\int_p^\infty b^2(\xi)\,d\xi\right]^{1/2} \cdot \left[\int_{\widehat{p}}^\infty b^2(\xi)\,d\xi\right]^{1/2} \le \int_0^\infty b^2(\xi)\,d\xi.$$

Finally we obtain

$$\text{(A.5)}\qquad I_a \le \left[\int_0^\infty a(p)\,dp\right]^2 \cdot \int_0^\infty b^2(\xi)\,d\xi.$$

Similiarly we estimate the integrals I_b, I_c, I_d:

$$\text{(A.6)}\qquad I_b = \int_0^\infty dx \left\{\int_0^x (x-\xi)e^{-s\sqrt{1-\Omega_0\kappa_1}(x-\xi)}\cdot|A(\xi,t)|\,d\xi\right.$$

$$\left.\cdot\int_0^x (x-\xi')e^{-s\sqrt{1-\Omega_0\kappa_1}(x-\xi')}\cdot|A(\xi',t)|\,d\xi'\right\}$$

$$\le s^{-4}(1-\Omega_0\kappa_1)^{-2}\int_0^\infty |A(x,t)|^2\,dx,$$

$$\text{(A.7)}\qquad I_c = \int_0^\infty |a(x,t;\Omega_0,s)|\,dx \le \frac{1}{2}\Omega_0 m(1-\Omega_0 x_1)^{-3/2}s^{-1}\int_0^\infty |A(x,t)|\,dx,$$

$$\text{(A.8)}\qquad I_d = \int_0^\infty |b(x,t;\Omega_0,s)|\,dx \le \frac{1}{2}\Omega_0 m(1-\Omega_0 x_1)^{-3/2}s^{-1}\int_0^\infty |A(x,t)|\,dx.$$

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Integrating the inequality (cf. (1.19))

$$|NA(x,t)| \le |a(x,t;\Omega_0,s)| + |b(x,t;\Omega_0,s)| \tag{A.9}$$

with respect to x over the interval $[0,\infty)$ and using the estimate of the integral I_c and I_d, we obtain

$$\begin{aligned}\int_0^\infty |NA(x,t)|\,dx &\le \int_0^\infty |a(x,t;\Omega_0,s)|\,dx + \int_0^\infty |b(x,t;\Omega_0,s)|\,dx \\ &\le \Omega_0 m(1-\Omega_0 x_1)^{-3/2}s^{-1}\int_0^\infty |A(x,t)|\,dx,\end{aligned} \tag{A.10}$$

and finally

$$\begin{aligned}\|NA(x,t)\|^2_{X_2^{(1)}} &= \sup_{t\in[0,\infty)}\int_0^\infty |NA(x,t)|\,dx \\ &\le \Omega_0 m(1-\Omega_0\kappa_1)^{3/2}s^{-1}\sup_{t\in[0,\infty)}\int_0^\infty |A(x,t)| \\ &= \Omega_0 m(1-\Omega_0\kappa_1)^{3/2}s^{-1}\|A(x,t)\|^2_{X_2^{(1)}}.\end{aligned} \tag{A.11}$$

From the last inequality it results that NA is a contraction operator in $X_2^{(1)}$, if

$$q_1 = \sqrt{q} < 1. \tag{A.12}$$

Let us consider the integral

$$I_e = \int_0^\infty \frac{b^2(x,t;\Omega_0,s)}{s^2(1-\Omega_0\kappa)}\,dx.$$

Due to (A.12) we get

$$\begin{aligned}I_e \le \frac{s^2\Omega_0^2 m^2}{4(1-\Omega_0\kappa)}\int_0^\infty \Bigg\{&\frac{1}{s\sqrt{1-\Omega_0\kappa}} \\ &\cdot \int_0^x (x-\xi)\exp\left[-s\sqrt{1-\Omega_0\kappa_1}(x-\xi)\right]|A(\xi,t)|\,d\xi\Bigg\}^2 dx.\end{aligned} \tag{A.13}$$

Hence, by making estimates similar to those for the integral I_b we obtain

$$I_e \le \frac{1}{4}\Omega_0^2 m^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\int_0^\infty \frac{|A(x,t)^2|}{s^2(1-\Omega_0\kappa)}\,dx, \tag{A.14}$$

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and

$$\text{(A.15)} \qquad \|b(x,t;\Omega_0,s)\|^2_{X_1^{(-1/2)}} = \sup_{t\in[0,\infty)} I_e \le \frac{1}{4}\Omega_0^2 m^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\|A(x,t)\|^2_{X_1^{-1/2}}.$$

Applying a similar procedure to that used for integral I_a, we get

$$\text{(A.16)} \qquad I_f = \int_0^\infty \frac{a^2(x,t;\Omega_0,s)}{s^2(1-\Omega_0\kappa(x))}dx \le \frac{1}{4}\Omega_0^2 m^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\int_0^\infty \frac{|A(x,t)^2|}{s^2(1-\Omega_0\kappa(x))}dx.$$

Since

$$\text{(A.17)} \qquad NA(x,t) = a(x,t;\Omega_0,s) + b(x,t;\Omega_0,s)$$

and

$$\text{(A.18)} \qquad \begin{aligned} \|a(x,t;\Omega_0,s)\|^2_{X_1^{(-1/2)}} &= \sup_{t\in[0,\infty)} I_e \le \frac{1}{4}\Omega_0^2 m^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\|A(x,t)\|^2_{X_1^{(-1/2)}},\\ \|b(x,t;\Omega_0,s)\|^2_{X_1^{(-1/2)}} &= \sup_{t\in[0,\infty)} I_e \le \frac{1}{4}\Omega_0^2 m^2(1-\Omega_0\kappa_1)^{-3}s^{-2}\|A(x,t)\|^2_{X_1^{(-1/2)}}, \end{aligned}$$

the operator N is a contraction in $X_1^{(-1/2)}$ if the following condition is fulfilled:

$$\text{(A.19)} \qquad q = \Omega_0 m(1-\Omega_0\kappa_1)^{-3/2}s^{-1} < 1.$$

Appendix II

$$\begin{aligned} P_0(\tilde\omega) &= 2\tilde\omega^2(1-\tilde\omega) + \frac{1}{8}(2-\tilde\omega)^4\tilde\omega,\\ P_1(\tilde\omega) &= \frac{1}{2}\tilde\omega^2(1-\tilde\omega)(2-\tilde\omega)^2 + \frac{1}{2}(2-\tilde\omega)^4,\\ P_2(\tilde\omega) &= -[4\tilde\omega^2(1-\tilde\omega)(2-\tilde\omega) + (2-\tilde\omega)^4\tilde\omega], \end{aligned}$$

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$$P_3(\tilde{\omega}, \kappa_1) = 8 - 4\tilde{\omega} + \tilde{\omega}^2 - 4\tilde{\omega}\kappa_1,$$
$$P_4(\tilde{\omega}, \kappa_1) = -32 + 8\tilde{\omega} + 24\tilde{\omega}\kappa_1 - 4\tilde{\omega}^2\kappa_1,$$
$$P_5(\tilde{\omega}, \kappa_1) = 8 + \kappa_1\tilde{\omega}^2 - 8\tilde{\omega}\kappa_1,$$

$$k_1^2 = \tilde{\omega}^{-1}[4s^2(1-\tilde{\omega})(1-\kappa_1)(\tilde{\omega} - \tilde{\omega}\kappa_1 - 1)],$$
$$k_2^2 = s^2(1-\tilde{\omega})(1-2\kappa_1)\tilde{\omega}[2 - 2\tilde{\omega} - 2\kappa_1 + \tilde{\omega}\kappa_1]^{-1},$$
$$l_1 = [2s^3\tilde{\omega}(1-\kappa_1)(1-\tilde{\omega}\kappa_1)^{1/2}]^{-1},$$
$$l_2 = -[2s^3\tilde{\omega}(1-\kappa_1)(1-\tilde{\omega})^{1/2}]^{-1},$$
$$l_3 = [2(1-\tilde{\omega}\kappa_1) + \tilde{\omega}][2s^3\tilde{\omega}(1-\kappa_1)(\tilde{\omega}-2)(1-\tilde{\omega}\kappa_1)^{1/2}]^{-1},$$
$$l_4 = [2s^3\tilde{\omega}(1-\kappa_1)(1-\tilde{\omega})^{1/2}]^{-1},$$

$$l_5 = [-2(1-\tilde{\omega}\kappa_1)(1-\kappa_1)\tilde{\omega} + (1-2\kappa_1)(\tilde{\omega}-2)\tilde{\omega} - 4(\tilde{\omega}-2) - \tilde{\omega}^2(1-\kappa_1)] \times[8s^3(1-\kappa_1)^2(\tilde{\omega}-1)(\tilde{\omega}-2)\tilde{\omega}]^{-1},$$
$$l_6 = [8s^3(1-\kappa_1)^2(\tilde{\omega}-1)(\tilde{\omega}-2)\tilde{\omega}]^{-1}[-2(\tilde{\omega}-1)\tilde{\omega} - (1-2\kappa_1)(\tilde{\omega}-2)\tilde{\omega} + 4(\tilde{\omega}-2) + \tilde{\omega}^2(1-\kappa_1)],$$
$$l_7 = (1-\tilde{\omega}\kappa_1)^{1/2}[4(1-\kappa_1)(\tilde{\omega}-1)(\tilde{\omega}-2)]^{-1}[2\tilde{\omega}(1-\tilde{\omega}\kappa_1) - \tilde{\omega}(1-2\kappa_1)(\tilde{\omega}-2) + 4(1-\kappa_1)(\tilde{\omega}-2) + \tilde{\omega}^2],$$
$$l_8 = (1-\tilde{\omega}\kappa_1)^{1/2}[16(1-\kappa_1)(\tilde{\omega}-1)(\tilde{\omega}-2)(1-\tilde{\omega}\kappa_1)]^{-1} \times[8\tilde{\omega}(1-\tilde{\omega}\kappa_1)^2 - \tilde{\omega}(1-2\kappa_1)(\tilde{\omega}-2)^3 + 4(1-\kappa_1)(\tilde{\omega}-2)^3 + \tilde{\omega}^2(\tilde{\omega}-2)^2],$$
$$a_1 = [8s^3(1-\kappa_1)^2(1-\tilde{\omega}\kappa_1)^{1/2}(\tilde{\omega}-1)\tilde{\omega}]^{-1}[2(1-\tilde{\omega}\kappa_1)\tilde{\omega} - (\tilde{\omega}-2)(1-2\kappa_1)\tilde{\omega} + 2(\tilde{\omega}-2)(\tilde{\omega}+6)(1-\kappa_1)],$$
$$a_2 = [16s^3(1-\kappa_1)^2(1-\tilde{\omega}\kappa_1)^{1/2}(\tilde{\omega}-1)\tilde{\omega}(\tilde{\omega}-2)^2]^{-1}[-4(1-\tilde{\omega}\kappa_1)\tilde{\omega}(\tilde{\omega}-2)^2 + 2(\tilde{\omega}-2)^3(1-2\kappa_1)\tilde{\omega} + 16(1-\tilde{\omega}\kappa_1)(\tilde{\omega}-1)(1-\kappa_1) - 8(\tilde{\omega}-2)^3(1-\kappa_1) - \tilde{\omega}^2(\tilde{\omega}-2)(1-\kappa_1)],$$
$$b_1 = -[16s^3(1-\kappa_1)^2(1-\tilde{\omega}\kappa_1)^{1/2}(\tilde{\omega}-1)\tilde{\omega}][4(1-\tilde{\omega}\kappa_1)\tilde{\omega}(\tilde{\omega}-1) - 2(\tilde{\omega}-2)(1-2\kappa_1)\tilde{\omega} + 8(\tilde{\omega}-2)(1-\kappa_1) + \tilde{\omega}^2)(1-\kappa_1) + 4(\tilde{\omega}-1)(1-\tilde{\omega}\kappa_1)],$$
$$b_2 = [16s^3(1-\kappa_1)^2(1-\tilde{\omega}\kappa_1)^{1/2}(\tilde{\omega}-1)\tilde{\omega}(\tilde{\omega}-2)^2]^{-1} \times [4(1-\tilde{\omega}\kappa_1)\tilde{\omega}(\tilde{\omega}-2)^2 - 2(\tilde{\omega}-2)^3(1-2\kappa_1)\tilde{\omega} + 8(\tilde{\omega}-2)^3(1-\kappa_1)\tilde{\omega} + \tilde{\omega}^2(\tilde{\omega}-2)^2(1-\kappa_1) + 16(1-\tilde{\omega}\kappa_1)(\tilde{\omega}-1)(1-\kappa_1)].$$

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