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Wave propagation in anisotropic layered media

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THE PROPAGATION of time-harmonic waves in a continuously stratified anisotropic, viscoelastic layer bounded by two homogeneous anisotropic solid half-spaces, is studied analytically. A plane wave is assumed to impinge on the boundary of the layer, and the resulting field, inside and outside of the layer, is described according to the causality principle and formal wave-splitting. Reflection and transmission coefficients are derived for arbitrary angle of incidence, together with a formal expression of the wave field within the layer. A local reflectivity is defined as a function of the depth and used to obtain up and down-going modes in the layer. Reduction of the model to particular material symmetries allows for scalar fields whose properties generalize known results concerning the isotropic media. Numerical results are given to illustrate the method in the scalar case.

1. Introduction

WAVE PROPAGATION in stratified layers has been extensively investigated in connection with a wide range of constitutive and geometric models which are mainly motivated by geophysical applications. Beside the frequent approaches based on homogeneous waves in elastic isotropic materials (see for ex. [1, 2]), inhomogeneous waves have also been exploited in order to account for dissipative effects [3, 4], and anisotropic materials have been considered in the multilayered case [5, 6]. However, in these last works each layer is assumed to be homogeneous, thus allowing for an effective use of the propagator matrix.

The aim of the present paper is to investigate wave propagation across a continuously stratified (and hence inhomogeneous) viscoelastic solid layer with arbitrary material symmetry. A time-harmonic inhomogeneous plane wave, coming from a homogeneous anisotropic half-space, is assumed to impinge on the outset of the layer, giving rise to a reflected field. A transmitted wave field propagates along the edge of the layer within a second homogeneous anisotropic half-space. For arbitrary angle of incidence, three reflected modes and three transmitted modes are, in general, possible within the homogeneous half-spaces. Although forward and backward plane waves are allowed in the first solid half-space, the causality reasons imply that only forward waves propagate in the second solid half-space. Transmitted modes are then exploited to infer a formal wave-splitting within the layer, where the wave field is described by three independent components whose amplitudes and polarizations are functions of the depth. Continuity requirements imposed on the displacement and on the traction are used to obtain the reflection and the transmission matrices, and to get boundary conditions in order to integrate the differential equation for the displacement. A wave-splitting is then introduced for each wave component in the layer. To this end, a reflectivity matrix is defined which satisfies suitable conditions at the boundaries. As a result, the wave field within the layer is given as the superposition of three pairs of up and down-going modes. A notable simplification of the present model is achieved by considering special material symmetries. In Sec. 6 of this paper it is shown that some crystal systems such as orthorhombic, tetragonal, cubic and hexagonal systems allow for a decoupling of the governing differential equation, which splits into a scalar equation for horizontally polarized waves and a vector equation for vertically polarized waves. The first one is analyzed in detail to stress the comparison with the known results on isotropic layers [7]. In particular, the reflectivity is shown to satisfy a Riccati equation as occurs in scalar theories of wave propagation in isotropic layered media [8, 9]. The scalar problem for horizontally polarized waves is also solved numerically to explicitly obtain the split wave-field. Two examples are considered of the dependence of the constitutive properties on the depth.

2. Stratified anisotropic layers

We are here concerned with an inhomogeneous anisotropic solid layer $\mathcal{L}$ bounded by two plane parallel surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. A Cartesian coordinate system is chosen in such a way that $\mathcal{S}_1$ and $\mathcal{S}_2$ correspond to the planes $z = 0$ and $z = d$, where d is the thickness of the layer. Inhomogeneity in $\mathcal{L}$ is assumed to consist of a continuously stratified structure along the z direction. Two homogeneous anisotropic solid media $\mathcal{B}_1$ and $\mathcal{B}_2$ occupy, respectively, the half-spaces $z < 0$ and $z > d$. All the media $\mathcal{B}_1$, $\mathcal{L}$, $\mathcal{B}_2$ are supposed to behave as viscoelastic materials where the Cauchy stress $\mathbf{T}$ has a linear dependence on the strain history and an arbitrary dependence on the space coordinates. More precisely, denoting by $\widehat{\mathbf{e}} = \mathrm{Sym}(\nabla\widehat{\mathbf{u}})$ the infinitesimal strain tensor, $\widehat{\mathbf{u}}$ being the displacement, we assume (cf. [10])

$$\mathbf{T}(\mathbf{x},t) = \widetilde{\mathbf{G}}(z,0)\widehat{\mathbf{e}}(\mathbf{x},t) + \int_0^\infty \widetilde{\mathbf{G}}_s(z,s)\,\widehat{\mathbf{e}}(\mathbf{x},t-s)\,ds, \tag{2.1}$$

where $\widetilde{\mathbf{G}} : \mathbf{R}\times\mathbf{R}^+ \to \mathrm{Lin(Sym)}$ is the relaxation function and $\widetilde{\mathbf{G}}_s = \partial\widetilde{\mathbf{G}}/\partial s$. It is convenient, in elastic theories of anisotropic solids, to adopt a double-indices notation for the relaxation function (see [11]), introducing the indicial correspondence $(ij) \to \alpha$ $(i,j = 1,2,3;\ \alpha = 1,...,6)$ given by $(11) \to 1$, $(22) \to 2$, $(33) \to 3$, $(23) \to 4$, $(13) \to 5$, $(12) \to 6$. The corresponding six-dimensional relaxation matrix $\Gamma(z,s)$ is assumed to be non-singular for any $z \in \mathbf{R}$ and $s \in \mathbf{R}^+$. In the following we shall assume that the displacement $\widehat{\mathbf{u}}$ has a time-harmonic dependence

$$\widehat{\mathbf{u}}(\mathbf{x},t) = \overline{\mathbf{u}}(\mathbf{x})\exp(-i\omega t),$$

with $\omega \in \mathbf{R}^{++}$. Hence, assuming $\widehat{\mathbf{e}}(\mathbf{x},-\infty) = 0$, integration by parts reduces Eq. (2.1) to

$$\overline{\mathbf{T}}(\mathbf{x},\omega) = \mathbf{G}(z,\omega)\,\overline{\mathbf{e}}(\mathbf{x},\omega), \tag{2.2}$$

where $\overline{\mathbf{T}} = \widehat{\mathbf{T}}\exp(i\omega t)$ and $\mathbf{G}(z,\omega) = -i\omega\int_0^\infty \widetilde{\mathbf{G}}(z,s)\exp(i\omega s)\,ds$. Accounting for the model we are dealing with, the constitutive parameters take the form

$$\Gamma_{\alpha\beta} = \begin{cases} \Gamma^{(1)}_{\alpha\beta}(\omega) & \text{for } z < 0, \\ \Gamma_{\alpha\beta}(z,\omega) & \text{for } z \in [0,d], \\ \Gamma^{(2)}_{\alpha\beta}(\omega) & \text{for } z > d, \end{cases} \tag{2.3}$$

where $\Gamma_{\alpha\beta} = -i\omega\int_0^\infty \widetilde{\Gamma}_{\alpha\beta}\exp(i\omega s)\,ds$. We also assume that $\Gamma_{\alpha\beta}$ are continuous throughout z and sufficiently smooth in $[0,d]$. Additional restrictions hold if particular material symmetries are allowed for the solid media. A classification of such symmetries can be achieved by the determination of the planes of reflective symmetry (see [12]) and of the consequent non-vanishing elastic constants. For the future purposes we observe that most of the crystal systems (such as orthorhombic, tetragonal, cubic and hexagonal) can be characterized by the nine non-vanishing parameters

$$\begin{aligned} &\Gamma_{\alpha\beta} && \text{with} \quad \alpha,\beta = 1,2,3, \\ &\Gamma_{\gamma\gamma} && \text{with} \quad \gamma = 4,5,6. \end{aligned} \tag{2.4}$$

These, in turn, may reduce to a lower number of independent entries for particular crystal classes (see for ex. [13]).

3. The governing differential equation

Accounting for layer's inhomogeneities along the z-axis, we assume that the displacement $\overline{\mathbf{u}}(\mathbf{x})$ has a plane-wave-like dependence on x and y, that is

$$\overline{\mathbf{u}}(x,y,z) = \mathbf{u}(z)\exp[i(k_x x + k_y y)], \tag{3.1}$$

where k_x and k_y are complex-valued wave-numbers and where $\mathbf{u} \in \mathbf{C}^3$. Avoiding inessential formal difficulties, we can choose the x-axis in such a way that the real part of k_x vanishes. This can be accomplished by a suitable orthogonal transformation applied to the constitutive tensor $\mathbf{G}$ (see e.g. [5]). We also neglect the imaginary part of k_x. This amounts to assume that the incident wave-number bivector lies on the (y,z) plane. Hence, putting $k_y = k$, Eq. (3.1) takes the form

$$\overline{\mathbf{u}}(y,z) = \mathbf{u}(z)\exp(iky), \tag{3.2}$$

which, according to the Snell's law, holds at any point in $\mathcal{B}_1$, $\mathcal{L}$, and $\mathcal{B}_2$. In view of the time-harmonic dependence, the equation of motion for $\overline{\mathbf{u}}$ reads

$$\nabla\cdot\overline{\mathbf{T}} + \varrho\omega^2\overline{\mathbf{u}} = 0,$$

where ϱ is the mass density. By exploiting Eqs. (2.2) and (3.2) and accounting for the description in terms of Γ, we arrive at

$$[\mathbf{C} + \varrho\omega^2\mathbf{I}]\mathbf{u} = 0, \tag{3.3}$$

where $\mathbf{C}$ is a linear symmetric differential operator whose entries are expressed by

$$\begin{aligned}
C_{11} &= \partial_z(\Gamma_{55}\partial_z) + ik(\Gamma_{56,z} + 2\Gamma_{56}\partial_z) - k^2\Gamma_{66},\\
C_{12} &= \partial_z(\Gamma_{45}\partial_z) + ik(\Gamma_{25,z} + 2\Gamma_{46}\partial_z) - k^2\Gamma_{26},\\
C_{13} &= \partial_z(\Gamma_{35}\partial_z) + ik(\Gamma_{45,z} + 2\Gamma_{36}\partial_z) - k^2\Gamma_{46},\\
C_{22} &= \partial_z(\Gamma_{44}\partial_z) + ik(\Gamma_{24,z} + 2\Gamma_{24}\partial_z) - k^2\Gamma_{22},\\
C_{23} &= \partial_z(\Gamma_{34}\partial_z) + ik(\Gamma_{44,z} + 2\Gamma_{23}\partial_z) - k^2\Gamma_{24},\\
C_{33} &= \partial_z(\Gamma_{33}\partial_z) + ik(\Gamma_{34,z} + 2\Gamma_{34}\partial_z) - k^2\Gamma_{44}.
\end{aligned} \tag{3.4}$$

Equation (3.3) is a second order homogeneous linear differential equation for $\mathbf{u}$. More explicitly, it can be written as

$$\mathbf{L}\mathbf{u}'' + (\mathbf{L}' + 2ik\mathbf{M}_1)\mathbf{u}' + (ik\mathbf{M}_2' - k^2\mathbf{Q} + \varrho\omega^2\mathbf{I})\mathbf{u} = 0, \tag{3.5}$$

where prime denotes differentiation with respect to z, and where

$$\mathbf{L} = \begin{pmatrix} \Gamma_{55} & \Gamma_{45} & \Gamma_{35}\\ \Gamma_{45} & \Gamma_{44} & \Gamma_{34}\\ \Gamma_{35} & \Gamma_{34} & \Gamma_{33}\end{pmatrix}, \qquad \mathbf{M}_1 = \begin{pmatrix} \Gamma_{56} & \Gamma_{46} & \Gamma_{36}\\ \Gamma_{46} & \Gamma_{24} & \Gamma_{23}\\ \Gamma_{36} & \Gamma_{23} & \Gamma_{34}\end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} \Gamma_{56} & \Gamma_{25} & \Gamma_{45}\\ \Gamma_{25} & \Gamma_{24} & \Gamma_{44}\\ \Gamma_{45} & \Gamma_{44} & \Gamma_{34}\end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} \Gamma_{66} & \Gamma_{26} & \Gamma_{46}\\ \Gamma_{26} & \Gamma_{22} & \Gamma_{24}\\ \Gamma_{46} & \Gamma_{24} & \Gamma_{44}\end{pmatrix}.$$

Since Γ is non-singular, the operator $\mathbf{L}(z)$ is invertible for any $z \in \mathbf{R}$, hence Eq. (3.5) may be rewritten in the following form

$$\mathbf{u}'' + \mathbf{A}\mathbf{u}' + \mathbf{B}\mathbf{u} = 0, \tag{3.6}$$

where

$$\mathbf{A} = \mathbf{L}^{-1}(\mathbf{L}' + 2ik\mathbf{M}_1), \qquad \mathbf{B} = \mathbf{L}^{-1}(ik\mathbf{M}_2' - k^2\mathbf{Q} + \varrho\omega^2\mathbf{I}). \tag{3.7}$$

Before developing a procedure to obtain a representation of the displacement within the layer $\mathcal{L}$, we look for solutions of Eq. (3.6) in the homogeneous regions. In $\mathcal{B}_1$ and $\mathcal{B}_2$ the tensor Γ is taken to be independent of z, whence

$$\begin{aligned}
\mathbf{A}^{(1,2)} &= 2ik(\mathbf{L}^{(1,2)})^{-1}\mathbf{M}_1^{(1,2)},\\
\mathbf{B}^{(1,2)} &= (\mathbf{L}^{(1,2)})^{-1}[\varrho\omega^2\mathbf{I} - k^2\mathbf{Q}^{(1,2)}],
\end{aligned} \tag{3.8}$$

with obvious meaning of the superscripts 1, 2. In view of a formal splitting of the wave field into elementary modes, we first look for non-dissipative, normal incident waves, such that $k = 0$. Owing to (3.8) we have $\mathbf{A}^{(1,2)} = 0$, and the solutions of Eq. (3.6) take the form

$$\mathbf{u} = \sum_{h=1}^{3} [\mathbf{u}_{h+} \exp(i\zeta_h z) + \mathbf{u}_{h-} \exp(-i\zeta_h z)], \tag{3.9}$$

where ζ_h $(h = 1, 2, 3)$ are those solutions of the bi-cubic secular equation

$$\det [\varrho\omega^2 \mathbf{L}^{-1} - \zeta^2 \mathbf{I}] = 0, \tag{3.10}$$

which have positive real parts. From Eq. (3.9) the displacement of normal incident waves in homogeneous regions consists of three pairs of up and down-going modes. For an arbitrary incidence and possible dissipation ($k \neq 0$), Eqs. (3.9) and (3.10) must be replaced by

$$\mathbf{u} = \sum_{h=1}^{6} \mathbf{u}_h \exp(i\zeta_h z), \tag{3.11}$$

$$\det[\varrho\omega^2 \mathbf{I} - k^2 \mathbf{Q} - 2k\zeta \mathbf{M}_1 - \zeta^2 \mathbf{L}] = 0. \tag{3.12}$$

The left-hand side of Eq. (3.12) is a sixth-degree polynomial with constant, complex-valued coefficients, parametrized by k. Its zeroes ζ_h $(h = 1, \ldots, 6)$ appear in the representation (3.11). If the solutions $\pm\zeta_h, (h = 1, 2, 3)$ of Eq. (3.10) are distinct, there will be a neighbourhood C_k of $k = 0$ in the complex k-plane where each solution of Eq. (3.12) has a one-to-one correspondence with each value $\pm\zeta_h$ and keeps its own sign. Then, assuming $k \in C_k$, solutions of Eq. (3.12) may be represented by the set

$$\{\zeta_{1+}, \zeta_{1-}, \zeta_{2+}, \zeta_{2-}, \zeta_{3+}, \zeta_{3-}\},$$

and Eq. (3.11) becomes

$$\mathbf{u} = \sum_{h=1}^{3} [\mathbf{u}_{h+} \exp(i\zeta_{h+} z) + \mathbf{u}_{h-} \exp(i\zeta_{h-} z)], \tag{3.13}$$

thus yelding three couples of "up" and "down-going" modes. In the following we shall assume that the eigenvalue problem (3.12) and the corresponding eigenvector problem have been solved in $\mathcal{B}_1$ and in $\mathcal{B}_2$ so that the constant amplitudes $\mathbf{u}_{h\pm}$ are known. In view of further developments, we represent these vectors in the form

$$\mathbf{u}_{h\pm} = \alpha_h^{\pm} \begin{pmatrix} 1 \\ p_{h\pm} \\ q_{h\pm} \end{pmatrix}. \tag{3.14}$$

4. Formal wave-splitting and the field within the layer

A plane harmonic wave, coming from the homogeneous region $\mathcal{B}_1$, is supposed to impinge on the boundary $\mathcal{S}_1$ of the inhomogeneous layer ($z = 0$). Owing to the superposition principle, we can restrict our attention to one of the possible up-going modes, labelled by $l+$, ($l = 1, 2, 3$) and study the reflected and transmitted modes at the respective surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Each impinging mode allows for a superposition of all the possible down-going reflected modes in $\mathcal{B}_1$, and all the possible up-going transmitted modes in $\mathcal{B}_2$, according to (3.13). The causality principle implies that no down-going modes arise in $\mathcal{B}_2$, that is $\mathbf{u}_{h-}^{(2)} = 0$ for $h = 1, 2, 3$. Hence the wave fields in $\mathcal{B}_1$ and $\mathcal{B}_2$ can be expressed, respectively, as

$$(4.1) \qquad \mathbf{u}^{(1)} = \begin{pmatrix} 1 \\ p_{l+}^{(1)} \\ q_{l+}^{(1)} \end{pmatrix} \exp(i\zeta_{l+}^{(1)} z) + \sum_{h=1}^{3} V_{lh} \begin{pmatrix} 1 \\ p_{h-}^{(1)} \\ q_{h-}^{(1)} \end{pmatrix} \exp(i\zeta_{h-}^{(1)} z) \qquad \text{for} \quad z < 0,$$

$$(4.2) \qquad \mathbf{u}^{(2)} = \sum_{h=1}^{3} W_{lh} \begin{pmatrix} 1 \\ p_{h+}^{(2)} \\ q_{h+}^{(2)} \end{pmatrix} \exp(i\zeta_{h+}^{(2)} z) \qquad \text{for} \quad z > d,$$

for any impinging wave ($l = 1, 2, 3$). Equations (4.1) and (4.2) can also be viewed as a definition of the complex-valued reflection and transmission coefficients V_{lh} and W_{lh}. Compatibly with the causality principle, each component mode of the transmitted field (4.2) may be thought of as being originated by a corresponding field within the layer. Specifically, we decompose the field in $\mathcal{L}$ as

$$(4.3) \qquad \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

and impose continuity requirements on $\mathcal{S}_2$, pertinent to each component separately. To this end we observe that, in view of (2.2), the traction $\mathbf{t} = \mathbf{T}\mathbf{e}_3$ ($\mathbf{e}_3$ being the unit vector along the z-direction) is given by

$$(4.4) \qquad \mathbf{t} = ik\mathbf{P}\mathbf{u} + \mathbf{L}\mathbf{u}',$$

where

$$\mathbf{P} = \begin{pmatrix} \Gamma_{56} & \Gamma_{25} & \Gamma_{45} \\ \Gamma_{46} & \Gamma_{24} & \Gamma_{44} \\ \Gamma_{36} & \Gamma_{23} & \Gamma_{34} \end{pmatrix},$$

and hence

$$(4.5) \qquad \mathbf{u}' = \mathbf{L}^{-1}(\mathbf{t} - ik\mathbf{P}\mathbf{u}).$$

According to the regularity conditions on Γ at the boundaries, the continuity of the displacement and of the traction across $\mathcal{S}_1$ and $\mathcal{S}_2$ implies continuity of the derivative of $\mathbf{u}$ at layer's boundaries. As a consequence, from (4.2) and (4.3) we have, at $z = d$,

$$\mathbf{u}_j(d) = W_{lj} \begin{pmatrix} 1 \\ p_{j+}^{(2)} \\ q_{j+}^{(2)} \end{pmatrix} \exp(i\zeta_{j+}^{(2)} d) \qquad (j = 1,2,3), \tag{4.6}$$

$$\mathbf{u}'_j(d) = i\zeta_{j+}^{(2)} W_{lj} \begin{pmatrix} 1 \\ p_{j+}^{(2)} \\ q_{j+}^{(2)} \end{pmatrix} \exp(i\zeta_{j+}^{(2)} d) \qquad (j = 1,2,3), \tag{4.7}$$

for any $l = 1,2,3$. Now we introduce a triad of second-rank matrices $\mathbf{N}^{[j]}$ $(j = 1,2,3)$ such that

$$\mathbf{u}'_j = i\mathbf{N}^{[j]}\mathbf{u}_j \qquad (j = 1,2,3). \tag{4.8}$$

Substituting this into the governing differential equation (3.6) we obtain the following first-order Riccati-type differential equations for the matrices $\mathbf{N}^{[j]}$

$$(\mathbf{N}^{[j]})' = i\mathbf{B} - \mathbf{A}\mathbf{N}^{[j]} - i\mathbf{N}^{[j]}\mathbf{N}^{[j]}. \tag{4.9}$$

Boundary conditions, in order to integrate (4.9), may be obtained from (4.6)–(4.8) as

$$\mathbf{N}^{[j]}(d) = \zeta_{j+}^{(2)}\mathbf{I} \qquad (j = 1,2,3). \tag{4.10}$$

According to (3.14) we assume

$$\mathbf{u}_j(z) = \alpha_j(z) \begin{pmatrix} 1 \\ p_j(z) \\ q_j(z) \end{pmatrix} \qquad (j = 1,2,3), \tag{4.11}$$

where $\alpha_j(z)$ are scalar, complex-valued amplitudes and $p_j(z)$, $q_j(z)$ characterize the polarization of the field. Substitution of (4.11) into (4.8) yields a first-order differential equation for a two-dimensional polarization vector, and the expression of the scalar amplitudes in terms of the entries of the matrices $\mathbf{N}^{[j]}$. Explicitly

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix}' = i \begin{pmatrix} N_{21}^{[j]} \\ N_{31}^{[j]} \end{pmatrix} + i \begin{pmatrix} N_{22}^{[j]} - N_{11}^{[j]} & N_{23}^{[j]} \\ N_{32}^{[j]} & N_{33}^{[j]} - N_{11}^{[j]} \end{pmatrix} \begin{pmatrix} p_j \\ q_j \end{pmatrix} - i \begin{pmatrix} p_j \\ q_j \end{pmatrix} \left(N_{12}^{[j]},\, N_{13}^{[j]} \right) \begin{pmatrix} p_j \\ q_j \end{pmatrix}, \tag{4.12}$$

$$(4.13)\qquad \alpha_j(z) = \alpha_j(0)\exp\left[i\int_0^z \left(N_{11}^{[j]} + N_{12}^{[j]}p_j + N_{13}^{[j]}q_j\right)d\tau\right],$$

with $j = 1,2,3$. Equation (4.12) has the form of a Riccati equation. Boundary conditions are obtained from (4.6) and (4.11) as

$$(4.14)\qquad \begin{pmatrix} p_j(d) \\ q_j(d) \end{pmatrix} = \begin{pmatrix} p_{j+}^{(2)} \\ q_{j+}^{(2)} \end{pmatrix}.$$

Consequently, integration of equations (4.9) and (4.12), together with Eq. (4.13) allows us to obtain the field in the layer. In order to complete the picture, we have to determine the constants of integration $\alpha_j(0)$ in Eq. (4.13). This can be performed by imposing the continuity of **u** and **t** at the surface $\mathcal{S}_1$ ($z = 0$). As a result, we also obtain the reflection and transmission matrices V_{lh} and W_{lh}, ($h, l = 1,2,3$). Just like the previous conditions at $\mathcal{S}_2$, we require the continuity of **u** and **u**$'$ at $\mathcal{S}_1$. According to (4.11), we obtain, for any impinging mode l,

$$(4.15)\qquad \begin{aligned} 1 + \sum_{j=1}^{3} V_{lj} &= \sum_{j=1}^{3}\alpha_j(0), \\ p_{l+}^{(1)} + \sum_{j=1}^{3} p_{j-}^{(1)}V_{lj} &= \sum_{j=1}^{3} p_j(0)\alpha_j(0), \\ q_{l+}^{(1)} + \sum_{j=1}^{3} q_{j-}^{(1)}V_{lj} &= \sum_{j=1}^{3} q_j(0)\alpha_j(0); \end{aligned}$$

$$(4.16)\qquad \begin{aligned} \zeta_{l+}^{(1)} + \sum_{j=1}^{3} \zeta_{j-}^{(1)}V_{lj} &= \sum_{j=1}^{3} \Omega_1^{[j]}(0)\alpha_j(0), \\ \zeta_{l+}^{(1)}p_{l+}^{(1)} + \sum_{j=1}^{3} \zeta_{j-}^{(1)}p_{j-}^{(1)}V_{lj} &= \sum_{j=1}^{3} \Omega_2^{[j]}(0)\alpha_j(0), \\ \zeta_{l+}^{(1)}q_{l+}^{(1)} + \sum_{j=1}^{3} \zeta_{j-}^{(1)}q_{j-}^{(1)}V_{lj} &= \sum_{j=1}^{3} \Omega_3^{[j]}(0)\alpha_j(0), \end{aligned}$$

where Eq. (4.12) has been used in working out the last two of Eqs. (4.16), and where

$$(4.17)\qquad \Omega_k^{[j]}(z) = N_{k1}^{[j]}(z) + N_{k2}^{[j]}(z)p_j(z) + N_{k3}^{[j]}(z)q_j(z) \qquad (k = 1,2,3),$$

for any $j = 1,2,3$. From Eq. (4.15) we have, for any l,

$$(4.18)\qquad \alpha_j(0) = \nu_{jl}^{+}(0) + \sum_{h=1}^{3} V_{lh}\nu_{jh}^{-}(0) \qquad (j = 1,2,3),$$

with

$$\nu_{jm}^{\pm} = \frac{1}{r}\left[q_{m\pm}^{(1)}(p_{j+1}-p_{j+2}) + q_{j+1}(p_{j+2}-p_{m\pm}^{(1)}) + q_{j+2}(p_{m\pm}^{(1)}-p_{j+1})\right] \qquad (j, m = 1, 2, 3),$$

$$r = \sum_{j=1}^{3} q_j(p_{j+1}-p_{j+2}),$$

and where a cyclic permutation of the indices j is understood. Substitution of Eq. (4.18) into (4.16) yields, after some manipulations, the reflection matrix as

$$\mathbf{V} = -\mathbf{H}_{+}^{-1}\mathbf{H}_{-}\,, \tag{4.19}$$

where $\mathbf{H}_+$ and $\mathbf{H}_-$ are matrices whose entries are given by

$$\begin{aligned}(H_\pm)_{1h} &= \zeta_{h\mp}^{(1)} - \sum_{j=1}^{3}\nu_{jh}^{\mp}(0)\Omega_1^{[j]}(0),\\ (H_\pm)_{2h} &= \zeta_{h\mp}^{(1)}p_{h\mp}^{(1)} - \sum_{j=1}^{3}\nu_{jh}^{\mp}(0)\Omega_2^{[j]}(0),\\ (H_\pm)_{3h} &= \zeta_{h\mp}^{(1)}q_{h\mp}^{(1)} - \sum_{j=1}^{3}\nu_{jh}^{\mp}(0)\Omega_3^{[j]}(0),\end{aligned} \tag{4.20}$$

with $h = 1, 2, 3$. The transmission coefficients may be obtained from the reflection matrix $\mathbf{V}$ by simply observing that Eq. (4.13) can be also written as

$$\alpha_j(z) = W_{lj}\exp\left[-i\int_z^d \Omega_1^{[j]}(\tau)\,d\tau\right]\exp(i\zeta_{j+}^{(2)}d).$$

Hence we obtain

$$\mathbf{W} = \mathbf{K}_+\mathbf{V} + \mathbf{K}_-, \tag{4.21}$$

where

$$(K_\pm)_{jh} = \nu_{jh}^{\mp}(0)\exp\left[i\int_0^d\left[\Omega_1^{[j]}(\tau) - \zeta_{j+}^{(2)}\right]d\tau\right] \qquad (j, k = 1, 2, 3).$$

5. Local reflectivity and couples of opposite modes

The aim of the present section is to give a representation of the displacement within the layer $\mathcal{L}$, as a set of pairs of up-going and down-going modes. Accounting for the formal splitting (4.3), we write

$$\mathbf{u}_j = \alpha_j \begin{pmatrix} 1 \\ p_j \\ q_j \end{pmatrix} = \alpha_j^+ \begin{pmatrix} 1 \\ p_j^+ \\ q_j^+ \end{pmatrix} + \alpha_j^- \begin{pmatrix} 1 \\ p_j^- \\ q_j^- \end{pmatrix} \qquad (j = 1, 2, 3), \tag{5.1}$$

where the dependence on z of the amplitudes and polarizations is understood. Let us introduce the local reflectivity matrix $\mathbf{R}(z)$ as

$$(\alpha_1^-, \alpha_2^-, \alpha_3^-) = \mathbf{R}\,(\alpha_1^+, \alpha_2^+, \alpha_3^+). \tag{5.2}$$

From Eqs. (5.1) and (5.2) we can express the amplitudes $\alpha_j^{\pm}$ in terms of the amplitudes α_j, which have been derived in the previous section. We get

$$(\alpha_1^+, \alpha_2^+, \alpha_3^+) = (\mathbf{I} + \mathbf{R})^{-1}(\alpha_1, \alpha_2, \alpha_3), \tag{5.3}$$

$$(\alpha_1^-, \alpha_2^-, \alpha_3^-) = \mathbf{R}(\mathbf{I} + \mathbf{R})^{-1}(\alpha_1, \alpha_2, \alpha_3). \tag{5.4}$$

In order to match the wave-splitting given by (5.1) and (5.2) with the solutions of Eq. (3.6) in $\mathcal{B}_1$ and $\mathcal{B}_2$, according to the analysis of the previous section, we must impose the following conditions at the boundaries

$$(\alpha_1^-, \alpha_2^-, \alpha_3^-)|_{z=0} = \mathbf{V}\,(\alpha_1^+, \alpha_2^+, \alpha_3^+)|_{z=0}\,, \tag{5.5}$$

$$(\alpha_1^-, \alpha_2^-, \alpha_3^-)|_{z=d} = 0. \tag{5.6}$$

In view of Eqs. (5.2) and (5.4), this implies that the matrix function $\mathbf{R}(z)$ must satisfy the conditions

$$\mathbf{R}(0) = \mathbf{V}, \tag{5.7}$$

$$\mathbf{R}(d) = \mathbf{0}. \tag{5.8}$$

Let us consider the matrix function

$$\mathbf{R}(z) = -\mathbf{H}_+^{-1}(z)\,\mathbf{H}_-(z), \tag{5.9}$$

where $\mathbf{H}_{\pm}(z)$ are given by (4.20) being $\nu_{jh}^{\mp}$ and $\Omega_k^{[j]}$ evaluated at the depth z in the layer. It is a simple matter to show that (5.9) satisfies conditions (5.7) and (5.8). In fact, Eq. (5.7) is the obvious consequence of (4.19) and (5.8) follows from the fact that $\mathbf{H}_-(d) = 0$, in view of (4.20), (4.17) and (4.10). Hence Eqs. (5.3), (5.4) and (5.9) yield the appropriate representation of the split field within the layer. However, this description is not the only one, since other representations are possible for different matrix functions which satisfy Eqs. (5.7) and (5.8). As to the polarizations $p_j^{\pm}(z)$, $q_j^{\pm}(z)$, we can apply the previous analysis in view of the formulae

$$(\alpha_1^- p_1^-, \alpha_2^- p_2^-, \alpha_3^- p_3^-) = \mathbf{R}(\alpha_1^+ p_1^+, \alpha_2^+ p_2^+, \alpha_3^+ p_3^+),$$

$$(\alpha_1^- q_1^-, \alpha_2^- q_2^-, \alpha_3^- q_3^-) = \mathbf{R}(\alpha_1^+ q_1^+, \alpha_2^+ q_2^+, \alpha_3^+ q_3^+),$$

$$\alpha_j^+ p_j^+ + \alpha_j^- p_j^- = \alpha_j p_j, \qquad \alpha_j^+ q_j^+ + \alpha_j^- q_j^- = \alpha_j q_j,$$

with $j = 1, 2, 3$.

6. Horizontally polarized waves for particular symmetries

According to Eq. (2.6), orthorhombic, tetragonal, cubic and hexagonal systems are characterizerd by the following restrictions

$$\begin{aligned} &\Gamma_{14} = \Gamma_{15} = \Gamma_{16} = \Gamma_{24} = \Gamma_{25} = \Gamma_{26} = 0, \\ &\Gamma_{34} = \Gamma_{35} = \Gamma_{36} = \Gamma_{45} = \Gamma_{46} = \Gamma_{56} = 0. \end{aligned} \tag{6.1}$$

For waves incident on the plane (y, z), Eq. (3.5) splits into

$$\Gamma_{55} u_1'' + \Gamma_{55}' u_1' - (k^2 \Gamma_{66} - \varrho\omega^2) u_1 = 0, \tag{6.2}$$

$$\begin{aligned} &\begin{pmatrix} \Gamma_{44} & 0 \\ 0 & \Gamma_{33} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}'' + \left[\begin{pmatrix} \Gamma_{44}' & 0 \\ 0 & \Gamma_{33}' \end{pmatrix} + 2ik \begin{pmatrix} 0 & \Gamma_{23} \\ \Gamma_{23} & 0 \end{pmatrix} \right] \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}' \\ &\quad + \left[ik \begin{pmatrix} 0 & \Gamma_{44}' \\ \Gamma_{44}' & 0 \end{pmatrix} - k^2 \begin{pmatrix} \Gamma_{22} & 0 \\ 0 & \Gamma_{44} \end{pmatrix} + \varrho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = 0. \end{aligned} \tag{6.3}$$

Equation (6.2) is the governing equation for waves polarized along the x-direction, i.e. horizontally polarized waves, and Eq. (6.3) accounts for waves whose amplitude lies on the vertical propagation plane, i.e. vertically polarized waves. The analysis of Sec. 4 may be applied separately to Eq. (6.2) and Eq. (6.3). Here we remark some peculiar features of horizontally polarized waves. Let us note that, according to (6.1)

$$t_1 = \Gamma_{55} u_1',$$

hence continuity of t_1 at the boundaries of the layer is equivalent to continuity of u_1'. The continuity requirements reduce to

$$\begin{aligned} u_1(0) &= 1 + V, \\ u_1'(0) &= i(\zeta_+^{(1)} + \zeta_-^{(1)} V), \\ u_1(d) &= W \exp(i\zeta_+^{(2)} d), \\ u_1'(d) &= i\zeta_+^{(2)} W \exp(i\zeta_+^{(2)} d). \end{aligned} \tag{6.4}$$

Introducing the complex-valued function $\sigma(z)$ such that

$$u_1' = i\sigma u_1, \tag{6.5}$$

Eq. (6.2) yields

$$\sigma' = \frac{i}{\Gamma_{55}}(\varrho\omega^2 - k^2\Gamma_{66}) - \frac{\Gamma_{55}'}{\Gamma_{55}}\sigma - i\sigma^2. \tag{6.6}$$

In addition, from (6.4) we get

$$\sigma(d) = \zeta_+^{(2)}. \tag{6.7}$$

If Eq. (6.6) is solved together with the boundary condition (6.7), the horizontal displacement $u_1(z)$ may be given in the form

$$u_1(z) = \frac{\zeta_-^{(1)} - \zeta_+^{(1)}}{\zeta_-^{(1)} - \sigma(0)} \exp\left[i\int_0^z \sigma(\tau)\,d\tau\right]. \tag{6.8}$$

As to the reflection coefficient V, Eqs. (6.4) yield

$$V = -\frac{\zeta_+^{(1)} - \sigma(0)}{\zeta_-^{(1)} - \sigma(0)}. \tag{6.9}$$

The scattering problem has been reduced to the solution of the first-order Riccati equation (6.6) for the function $\sigma(z)$.

Consider now the splitting of horizontally polarized waves and introduce the up and down-going modes $u_1^+(z)$, $u_1^-(z)$ and a local reflectivity $R(z)$ such that

$$u_1 = u_1^+ + u_1^-, \qquad u_1^- = Ru_1^+. \tag{6.10}$$

$$R(0) = V, \qquad R(d) = 0. \tag{6.11}$$

It is easy to show that the function

$$R(z) = -\frac{\zeta_+(z) - \sigma(z)}{\zeta_-(z) - \sigma(z)}, \tag{6.12}$$

where the functions $\zeta_\pm(z)$ are defined according to

$$\begin{aligned} \zeta_+(z) + \zeta_-(z) &= i\frac{\Gamma_{55}'(z)}{\Gamma_{55}(z)}, \\ \zeta_+(z)\zeta_-(z) &= -\frac{1}{\Gamma_{55}(z)}\left[\varrho\omega^2 - k^2\Gamma_{66}(z)\right], \end{aligned} \tag{6.13}$$

satisfies restrictions (6.11). This fact is a direct consequence of Eqs. (6.9) and (6.7). We finally show that, in this case, the reflectivity $R(z)$ satisfies a first-order Riccati differential equation. To this end we observe that, owing to (6.13), Eq. (6.6) can be rewritten as

$$\sigma' = -i(\sigma - \zeta_+)(\sigma - \zeta_-). \tag{6.14}$$

Then, differentiating Eq. (6.12) and accounting for Eq. (6.14), we obtain

$$R' = \frac{\zeta'_+}{\zeta_+ - \zeta_-} - \left[i(\zeta_+ - \zeta_-) - \frac{\zeta'_+ + \zeta'_-}{\zeta_+ - \zeta_-} \right] R + \frac{\zeta'_-}{\zeta_+ - \zeta_-} R^2. \tag{6.15}$$

Integration of Eq. (6.15) with the boundary condition $(6.11)_2$ turns out to be an alternative approach in deriving the reflection coefficient. The result (6.15) is a generalization of recent results on isotropic layers [7]. More generally, a Riccati-type equation for the reflectivity is a common feature of scalar theories in layered media (see for ex. [9]).

7. Numerical examples

In order to varify the method previously outlined, we give a numerical solution for the wave-field inside a solid layer with known constitutive properties. We restrict our computations to the scalar problem developed in Sec. 6; extension to the more general case may be performed without qualitative changes. Two different examples are considered for the dependence of the constitutive parameters on the depth within the layer. In each instance, the quantities Γ_{55}, Γ_{66}, ϱ have the same dependence on z and, according to the present model, are C^1 throughout z. The first example accounts for a monotone increasing dependence on z as

$$(\varrho, \Gamma_{55}, \Gamma_{66}) = (\varrho_0, \Gamma^0_{55}, \Gamma^0_{66})[1 + Q(1 - \cos(\pi Z))], \qquad Z \in [0, 1], \tag{7.1}$$

where ϱ_0, Γ^0_{55}, Γ^0_{66} are constant quantities pertaining to $\mathcal{B}_1$, $2Q$ is the ratio between the maximum and the minimum value of the constitutive parameters and where the dimensionless variable $Z = z/d$ has been introduced. In the second example a symmetric layer is considered, with

$$(\varrho, \Gamma_{55}, \Gamma_{66}) = (\varrho_0, \Gamma^0_{55}, \Gamma^0_{66})[1 + Q(1 - \cos(2\pi Z))], \qquad Z \in [0, 1], \tag{7.2}$$

so that $\mathcal{B}_1$ and $\mathcal{B}_2$ are mechanically equivalent.

Effective wave propagation within the layer requires a non-zero real part of $\zeta_\pm$. According to (6.13), this implies

$$\varrho\omega^2 - \Gamma_{66}k^2 > \frac{{\Gamma'_{55}}^2}{4\Gamma_{55}}, \qquad \forall\ Z. \tag{7.3}$$

In view of (7.1) and (7.2), the inequality (7.3) amounts to the following restriction on ω and k,

$$\varrho\omega^2 - \Gamma_{66}^0 k^2 > c\Gamma_{55}^0 \frac{\pi^2 Q^2}{1+2Q}, \tag{7.4}$$

with $c = 1/4$ or $c = 1$ depending on the alternative use of (7.1) or (7.2), respectively.

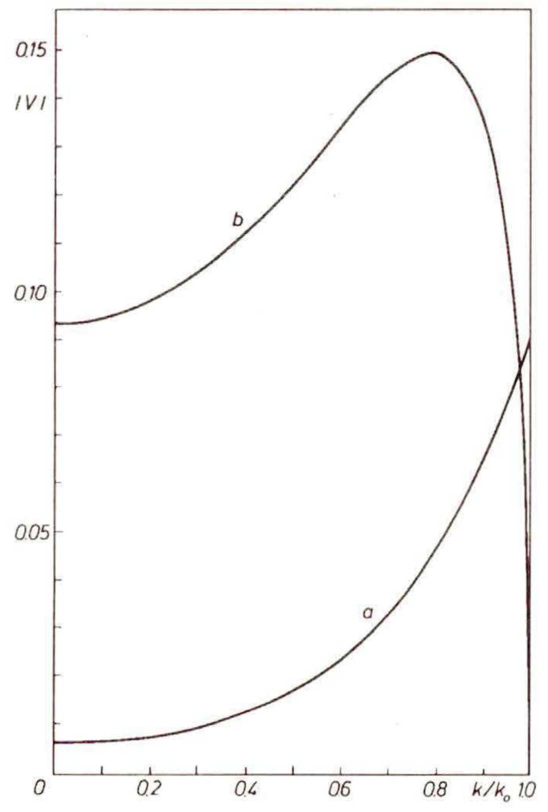

FIG. 1. Reflection coefficient $|V| = |R(0)|$ as a function of k/k_0 for a layer described by Eq. (7.1) (curve a) or by Eq. (7.2) (curve b).

Equation (6.15) has been numerically integrated along with the boundary condition $R(Z = 1) = 0$, adopting Eqs. (7.1), (7.2) and accounting for (7.4). The reflection coefficient $|V| = |R(0)|$ has been derived for all possible values of k ($0 \leq k < k_0$, with $k_0 = \left[\frac{\varrho\omega^2}{\Gamma_{66}^0} - c\frac{\Gamma_{55}^0}{\Gamma_{66}^0}\frac{\pi^2 Q^2}{1+2Q}\right]^{1/2}$). The values of V have been substituted into the boundary conditions $(6.4)_{1,2}$ in order to integrate Eq. (6.2). Then, both solutions for u_1 and R have been exploited to obtain the wave splitting within the layer, according to (6.10). The results are shown in Figs. 1–5 for

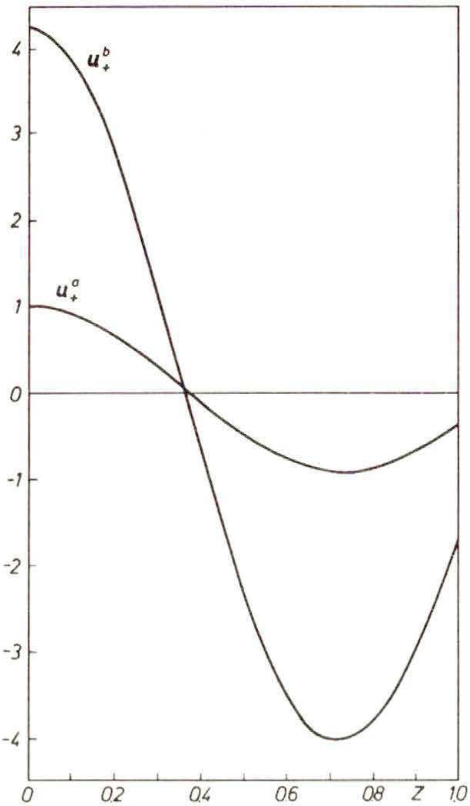

FIG. 2. Real and imaginary parts of the forward wave component within the "monotone" layer (see Eq. (7.1)).

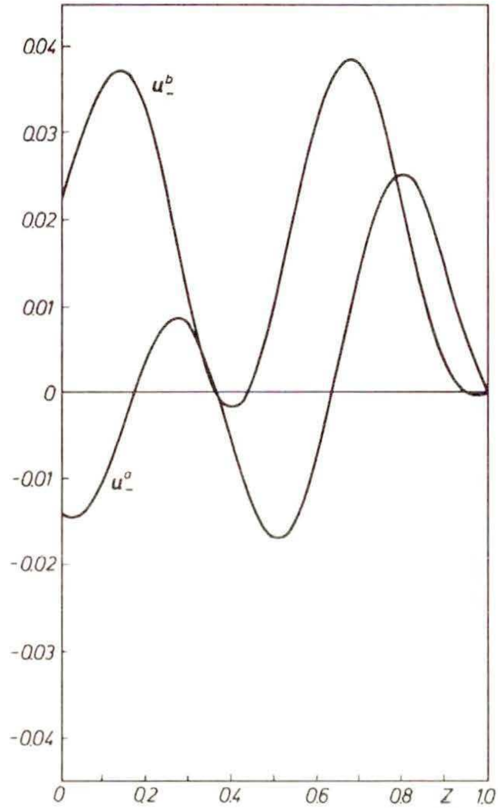

FIG. 3. Real and imaginary parts of the backward wave component within the "monotone" layer (see Eq. (7.1)).

a layer of zinc ($\varrho_0 = 7135\,\mathrm{kg/m^3}$, $\Gamma_{55}^0 = 39\cdot 10^9\,\mathrm{Pa}$, $\Gamma_{66}^0 = 63\cdot 10^9\,\mathrm{Pa}$) with $Q = 0.1$ and $\omega = 10^4\,\mathrm{Hz}$. In particular, Fig. 1 shows $|V|$ versus k for the "monotone" layer described by Eq. (7.1) (curve a), and for the symmetric layer described by Eq. (7.2) (curve b). Figures 2 and 3 show the real and the imaginary parts $u_\pm^a$ and $u_\pm^b$ of the opposite modes in the split wave-field for normal incidence ($k = 0$) in the "monotone" layer (see Eq. (7.1)). Analogous results are shown in Figs. 4, 5 for the symmetric layer (see Eq. (7.2)). From Figs. 3 and 5 is evident the phase shift between u_-^a and u_-^b which shows the mixing effect of the reflectivity R on the real and imaginary parts of the field inside the layer. We also observe that the reflection coefficient $|V|$ for normal incidence in the symmetric layer is by one order of magnitude greater than that of the "monotone" layer (Fig. 1). This fact, which is also evident from the results of the reflected amplitudes $u_-^{a,b}$ (Figs. 3, 5), is due to the steeper profile of the constitutive properties in the symmetric layer. We note, however, that this behaviour is reversed when incidences are considered which are close to the limiting value k_0.

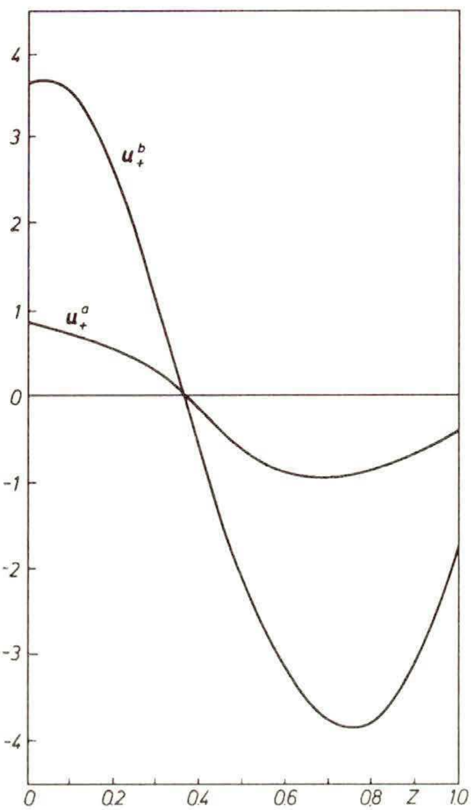

FIG. 4. Real and imaginary parts of the forward wave component within the symmetric layer (see Eq. (7.2)).

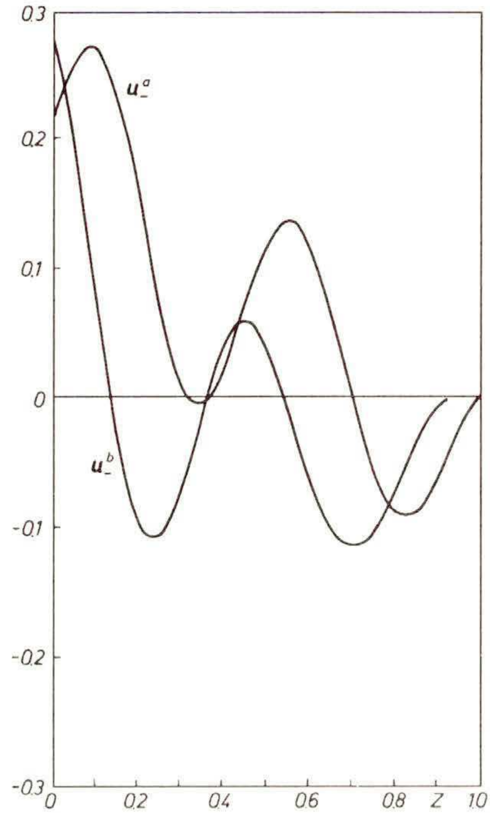

FIG. 5. Real and imaginary parts of the backward wave component within the symmetric layer (see Eq. (7.2)).

References

1. W.W. HAGER and R. ROSTAMIAN, *Reflection and refraction of elastic waves for stratified materials*, Wave Motion, **10**, pp. 333–348, 1988.
2. R.B. EVANS, *The decoupling of seismic waves*, Wave Motion, **8**, pp. 321–328, 1986.
3. G. CAVIGLIA and A. MORRO, *Wave propagation in a dissipative stratified layer*, Wave Motion, **19**, pp. 51–66, 1994.
4. M. DESCHAMPS and P. CHEVÉE, *Reflection and refraction of a heterogeneous plane wave by a solid layer*, Wave Motion, **15**, pp. 61–75, 1992.
5. A.H. NAYFEH, *The general problem of elastic wave propagation in multilayered anisotropic media*, J. Acoust. Soc. Am., **89**, pp. 1521–1531, 1991.
6. A.M.B. BRAGA and G. HERMANN, *Floquet waves in anisotropic periodically layered composites*, J. Acoust. Soc. Am., **91**, pp. 1211–1227, 1992.
7. G. CAVIGLIA, P. CERMELLI and A. MORRO, *Time-harmonic waves in continuously layered media*, Atti Sem. Mat. Fis. Univ. Modena, **41**, pp. 393–408, 1993.
8. L.M. BREKHOVSKIKH, *Waves in layered media*, Academic Press, New York 1980.
9. R.G. NEWTON, *Inversion of reflection data for layered media: a review of exact methods*, Geophys. J. R. Astr. Soc., **65**, pp. 191–215, 1981.
10. G. CAVIGLIA and A. MORRO, *Inhomogeneous waves in solids and fluids*, World Scientific, Singapore 1992.

11. M.M. MEHRABADI and S.C. COWIN, *Eigentensors of linear anisotropic elastic materials*, Q. J. Mech. Appl. Math., **43**, pp. 15–41, 1990.

12. S.C. COWIN and M.M. MEHRABADI, *On the identification of material symmetry for anisotropic elastic materials*, Q. J. Mech. Appl. Math., **40**, pp. 452–476, 1987.

13. M.J.P. MUSGRAVE, *Crystal acoustics*, Holden-Day, London 1970.

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