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On the existence of solutions for two-dimensional Stokes flows past rigid obstacles

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IN THIS PAPER we obtain some existence and uniqueness properties for the solution corresponding to the problem of the plane unbounded Stokes flow past rigid obstacles. The stream function of the flow is represented in the form of simple layer potentials.

1. Introduction

IN SOME PREVIOUSLY published papers [5, 6, 7], the authors treated the problem of an unbounded two-dimensional viscous flow past an arbitrary obstacle, using the method of matched inner and outer expansions of the corresponding solution. These results were then generalized to the three-dimensional case.

The purpose of this paper is to present a method for studying the problem of the Stokes flow past some rigid two-dimensional obstacles, using the properties of simple layer potentials.

Let $N \geq 2$ be the number of obstacles denoted by Ω_i, $i = \overline{1, N}$, Ω denoting the region exterior to these obstacles. The flow is described by the velocity $\mathbf{u}$ and the pressure p. We suppose that $\mathbf{u} \to U\mathbf{i}$, $p \to p_\infty$ as $|x| \to \infty$, where $x = x_1\mathbf{i} + x_2\mathbf{j}$, and U, p are prescribed constants. Using the dimensionless variables: $x' = x/l$, $\mathbf{u}' = \mathbf{u}/U$, $p' = l(p - p_\infty)/\mu U$ and the Reynolds number $\mathrm{Re} = \varrho l U/\mu$, where l is a characteristic length, μ the dynamic viscosity, and ϱ the fluid density, then $\mathbf{u}'$ and p' are solutions of the Navier–Stokes problem (disregarding the primes over u and p)

$$
\begin{aligned}
\Delta \mathbf{u} - \nabla p &= \mathrm{Re}(\mathbf{u}\cdot\nabla)\mathbf{u} \qquad \text{in } \Omega, \\
\nabla\cdot\mathbf{u} &= 0, \\
\mathbf{u} &= \mathbf{f}^i \qquad \text{on } C^i = \partial\Omega_i, \quad i = \overline{1, N}, \\
\mathbf{u} &\to \mathbf{i}, \qquad p \to 0, \qquad \text{as } |x| \to \infty.
\end{aligned}
\tag{1.1}
$$

Here Δ and ∇ denote the two-dimensional Laplacean and the gradient operator, respectively. We require the given velocities $\mathbf{f}^i$, $i = \overline{1, N}$ to satisfy the zero outflow conditions:

$$
\int\limits_{C^i} \mathbf{f}^i\cdot\mathbf{n}^i \, ds = 0,
\tag{1.2}
$$

where $\mathbf{n}^i$ is the exterior vector normal to Ω_i, $i = \overline{1, N}$.

We suppose that the Reynolds number defined above is sufficiently small.

The Navier–Stokes problem (1.1), for the case $N = 1$, is singular in the sense that the linearized Stokes form:

$$\begin{aligned} \Delta \mathbf{u} - \nabla p &= 0, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \tag{1.3}$$

together with the same conditions as in $(1.1)_{3,4}$, has no solution in view of the Stokes paradox. But, in this case, it is possible to obtain a solution, if the condition at infinity is replaced by:

$$\mathbf{u} = \mathbf{A} \ln |x| + \mathcal{O}(1), \qquad \text{as} \quad |x| \to \infty, \tag{1.4}$$

for any given constant vector $\mathbf{A}$ [6, 7]. Also, in the case of $N \geq 2$, we prove that there exists a constant vector $\mathbf{A}$ such that the problem (1.3) has a solution, if the condition at infinity is replaced with (1.4).

2. Integral equation of the first kind

The equation of continuity $\nabla \cdot \mathbf{u} = 0$ implies the existence of a stream function ψ such that

$$\mathbf{u} = (\nabla \psi)^{\perp}, \tag{2.1}$$

where $\mathbf{v}^{\perp}$ denotes the vector obtained by rotating the vector $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j}$ by $\pi/2$ counterclockwise, so that $\mathbf{v}^{\perp} = -v_2 \mathbf{i} + v_1 \mathbf{j}$. Because the domain Ω is not simply connected, the condition (1.4) is only local, i.e. ψ might not be a single-valued function. But the following arguments prove that ψ is necessarily a single-valued function.

Let C be any closed curve bounding the domain $\Omega^0 \subset \Omega$ and $\Omega^* = (\Omega \backslash \Omega^0) \cap B_R$, where B_R is a large disk of radius R. Applying the Green's formula, we obtain:

$$0 = \int_{\Omega^*} \operatorname{div} \mathbf{u} \, dx = \sum_{i=1}^{N} \int_{C^i} \mathbf{u} \cdot \mathbf{n} \, ds + \int_{C} \mathbf{u} \cdot \mathbf{n} \, ds - \int_{\partial B_R} \mathbf{u} \cdot \mathbf{n} \, ds . \tag{2.2}$$

From $(1.1)_3$ and (1.2), it results that $\int_{C^i} \mathbf{u} \cdot \mathbf{n} \, ds = 0$, $i = \overline{1, N}$.

From Green's formula in $\Omega_R = \Omega \cap B_R$, we have:

$$0 = \int_{\Omega_R} \operatorname{div} \mathbf{u} \, dx = \sum_{i=1}^{N} \int_{C^i} \mathbf{u} \cdot \mathbf{n} \, ds + \int_{\partial B_R} \mathbf{u} \cdot \mathbf{n} \, ds . \tag{2.3}$$

Hence (2.3) implies that $\int\limits_{\partial B_R} \mathbf{u}\cdot\mathbf{n}\, ds = 0$. The above arguments show that

$$\int\limits_{C} \mathbf{u}\cdot\mathbf{n}\, ds = 0, \qquad \text{so} \qquad \int\limits_{C} \mathbf{u}^{\perp}\cdot d\mathbf{s} = 0.$$

Then we express ψ in the form

$$\psi(x) = -\int\limits_{x_0}^{x} \mathbf{u}^{\perp}\cdot d\mathbf{s}, \qquad x \in \Omega, \tag{2.4}$$

where x_0 is a fixed point in Ω, x is an arbitrary point in Ω, and the integral is evaluated along an arbitrary polygonal line between x_0 and x. Also, it is easy to establish the condition (2.1).

Using (1.3) and (2.1), we obtain the Stokes problem for stream function ψ:

$$\begin{aligned} \Delta^2\psi &= 0 \quad \text{in } \Omega, \\ \nabla\psi(x) &= \mathbf{j} - \mathbf{f}^{i\perp}(x), \qquad x \in C^i, \quad i = \overline{1,N}. \end{aligned} \tag{2.5}$$

We shall prove that there exists a real constant vector $\mathbf{A}$ such that

$$\nabla\psi(x) = \mathbf{A}\ln|x| + \mathcal{O}(1), \qquad \text{as} \quad |x| \to \infty, \tag{2.6}$$

and that the problem (2.5)–(2.6) has a solution.

For these purposes, we represent the stream function ψ in the form:

$$\psi(x) = \sum_{i=1}^{N} \int\limits_{C^i} \nabla_y F(x,y)\cdot\boldsymbol{\phi}^i(y)\, ds_y^i, \qquad x \in \Omega \cup \left(\bigcup_{i=1}^{N} C^i\right), \tag{2.7}$$

where s_y^i denotes the arc length measured along C^i, $i = \overline{1,N}$ and F is the fundamental solution of biharmonic equation:

$$F(x,y) = \frac{1}{8\pi}|x-y|^2[\ln|x-y| - 1]. \tag{2.8}$$

It is easy to show that ψ given by (2.7), satisfies the equation $(2.5)_1$ and will be a solution of the boundary conditions $(2.5)_2$, if the density function $\widetilde{\phi}$, with $\widetilde{\phi}(x) = \boldsymbol{\phi}^i(x)$, $x \in C^i$, $i = \overline{1,N}$, satisfies the following system of integral equations of the first kind:

$$\sum_{i=1}^{N} \int\limits_{C^i} \nabla_x\nabla_y F(x^k,y)\boldsymbol{\phi}^i(y)\, ds_y^i = \mathbf{g}^k(x^k), \qquad x^k \in C^k, \qquad k = \overline{1,N}, \tag{2.9}$$

where

$$\mathbf{g}^k := \mathbf{j} - \mathbf{f}^{k\perp}. \tag{2.10}$$

The integral operator V^i defined by

$$V^i \boldsymbol{\phi}^i(x) := \int_{C^i} \nabla_x \nabla_y F(x,y) \boldsymbol{\phi}^i(y)\, ds_y^i, \qquad x \in C^i$$

has a kernel with logarithmic singularity.

Differentiating (2.9) with respect to the arc length s_x^k, $k = \overline{1,N}$, we obtain the set of integral equations with a Cauchy singularity:

$$\sum_{i=1}^{N} \int_{C^i} \frac{\partial}{\partial s_x^k} \nabla_x \nabla_y F(x^k, y) \cdot \boldsymbol{\phi}^i(y)\, ds_y^i = \frac{d}{ds_x^k} \mathbf{g}^k(x^k), \qquad k = \overline{1,N}. \tag{2.11}$$

Because F is a function of $|x-y|$ only, it is seen that the adjoint homogeneous system of (2.11) has the form:

$$\sum_{i=1}^{N} \int_{C^i} \frac{\partial}{\partial s_x} \nabla_x \nabla_y F(x, y^k) \cdot \mathbf{S}^i(x)\, ds_x = 0, \qquad y^k \in C^k, \qquad k = \overline{1,N}. \tag{2.12}$$

We remark that the functions $\widetilde{S}^i \colon \bigcup_{j=1}^{N} C^j \to \mathbb{R}^2$, given by

$$\widetilde{S}^i(x) = a_j^i x + \mathbf{b}_j^i, \qquad x \in C^j, \qquad j = \overline{1,N}, \tag{2.13}$$

with a_j^i, $\mathbf{b}_j^i$ denoting constants, are the solutions of the system (2.12). These functions determine a linear space with $3N$ dimensions, which implies that the dimension of to solution space corresponding to the homogeneous system (2.11) is at least $3N$. We use here the fact that the homogeneous system (2.11) and the adjoint system (2.12) have the same number of linearly independent solutions (see [10]).

THEOREM 1. *There exist at most $3N$ linearly independent solutions of the homogeneous system* (2.11).

P r o o f. The functions

$$\widetilde{\tau}^i \;:\; \bigcup_{j=1}^{N} C^j \to \mathbb{R}^2, \qquad \widetilde{\tau}^i(x) = \begin{cases} 0, & x \in C^j, \quad j \neq i, \\ \boldsymbol{\tau}^i(x), & x \in C^i, \end{cases}$$

$i = \overline{1,N}$, where $\boldsymbol{\tau}^i(x)$ denotes the unit tangent vector in the point $x \in C^i$, are N linearly independent solutions of the homogeneous system (2.11).

Let $\widetilde{\varphi}^i\colon \bigcup_{j=1}^{N} C^j \to \mathbb{R}^2$ be any $2N+1$ solutions of the homogeneous system (2.11), and $\psi^i = \psi^i(\widetilde{\varphi}^i)$, $i = \overline{1,\ 2N+1}$, denote the corresponding stream functions, as in (2.7). Then functions ψ^i satisfy the equations

$$
\begin{aligned}
&\Delta^2\psi^i = 0 \qquad \text{in } \Omega,\\
&\nabla\psi^i\Big|_{C^j} = \mathbf{C}_i^j, \qquad j = \overline{1,N},\\
&\nabla\psi^i(x) = \mathbf{A}^i \ln|x| + \mathcal{O}(1), \qquad \text{as } |x| \to \infty,
\end{aligned}
\tag{2.14}
$$

where $\mathbf{C}_i^j$ is a constant vector and

$$
\mathbf{A}^i = \frac{1}{4\pi}\sum_{j=1}^{N}\int_{C^j} \boldsymbol{\varphi}_j^i(y)\, ds_y^j\,.
$$

We define the function $\widetilde{\varphi}^i$ as $\widetilde{\varphi}^i(x) = \boldsymbol{\varphi}_j^i(x)$ for $x \in C^j$, $j = \overline{1,N}$.

We can choose real constants $\alpha_1, \ldots, \alpha_{2N+1}$, not all equal to zero, and the vector $\mathbf{c}(c_1, c_2)$, such that:

$$
\begin{aligned}
&\sum_{i=1}^{2N+1} \alpha_i \mathbf{c}_i^j - \mathbf{c} = 0, \qquad j = \overline{1,N},\\
&\sum_{i=1}^{N} \alpha_i \mathbf{A}^i = 0
\end{aligned}
\tag{2.15}
$$

because we have here $2N+2$ homogeneous equations with $2N+3$ unknowns.

Let the functions ψ_0 and $\widetilde{\phi}_0$ be defined by:

$$
\psi_0 = \sum_{i=1}^{2N+1} \alpha_i \psi^i\,, \qquad \widetilde{\phi}_0 = \sum_{i=1}^{N} \alpha_i \widetilde{\varphi}^i. \tag{2.16}
$$

Then ψ_0 satisfies the equation

$$
\begin{aligned}
&\Delta^2\psi_0 = 0 \qquad \text{in } \Omega,\\
&\nabla\psi_0(x) = \mathbf{C}, \qquad x \in C^j, \quad j = \overline{1,N},\\
&\nabla\psi_0(x) = \mathcal{O}(1), \qquad \text{as } |x| \to \infty.
\end{aligned}
\tag{2.17}
$$

The problem (2.17) has a solution of linear form $\psi_0(x) = \mathbf{c}\cdot\mathbf{x}$. From the uniqueness theorem of the solution corresponding to the exterior Stokes problem (see Theorem 3), we deduce that ψ_0 is the unique solution of (2.17). The function ψ_0 given by (2.16) is also biharmonic in each domain Ω_i and is continuous together with its first derivatives on C^i, $i = \overline{1, N}$. Using the uniqueness result of the inner Stokes problem, we conclude that ψ_0 has also a linear form in Ω_i, $i = \overline{1, N}$.

Using [5], it is easy to prove that on each contour C^j, $j = \overline{1, N}$, the stream function ψ given by (2.7) has the properties:

$$
\begin{aligned}
(\Delta\psi)^+ - (\Delta\psi)^- &= \lambda^j \mathbf{n}^j \cdot \boldsymbol{\phi}^j, \\
\left(\frac{\partial}{\partial n^j}\Delta\psi\right)^+ - \left(\frac{\partial}{\partial n^j}\Delta\psi\right)^- &= \lambda^j \frac{d}{ds^j}\left(\boldsymbol{\tau}^j \cdot \boldsymbol{\phi}^j\right),
\end{aligned}
\tag{2.18}
$$

where the symbols $+$, $-$ denote the limits in Ω and Ω_j, respectively, and $\partial/\partial n^j$ is the normal derivative on C^j, $j = \overline{1, N}$.

Since ψ_0 has a linear form in Ω and Ω_j, respectively, from (2.18) we obtain that there exists a constant β^j such that:

$$
\boldsymbol{\phi}_0^j(x) = \beta^j \boldsymbol{\tau}^j(x), \qquad x \in C^j, \quad j = \overline{1, N}, \tag{2.19}
$$

where the function $\boldsymbol{\phi}_0^j$ is defined by $\widetilde{\phi}_0(x) = \boldsymbol{\phi}_0^j(x)$, $x \in C^j$, $j = \overline{1, N}$.

Hence we deduce that

$$
\widetilde{\phi}_0(x) - \sum_{j=1}^{N} \beta^j \widetilde{\tau}^j(x) = 0, \qquad x \in \bigcup_{j=1}^{N} C^j \tag{2.20}
$$

or

$$
\sum_{i=1}^{2N+1} \alpha_i \widetilde{\varphi}^i(x) - \sum_{j=1}^{N} \beta^j \widetilde{\tau}^j(x) = 0, \qquad x \in \bigcup_{j=1}^{N} C^j, \tag{2.21}
$$

with the functions $\widetilde{\tau}^j$ defined above. It results that the functions $\widetilde{\varphi}^i$, $\widetilde{\tau}^j$, $i = \overline{1, 2N+1}$, $j = \overline{1, N}$, are linearly dependent.

So, we have proved that the dimension of the solutions space of the homogeneous system (2.12) equals exactly $3N$, and each solution $\widetilde{S}$ has the form:

$$
\widetilde{S}(x) = a^i x + \mathbf{b}^i, \qquad x \in C^i, \qquad i = \overline{1, N}, \tag{2.22}
$$

where a^i, $\mathbf{b}^i$ are constants.

Using the theory of singular integral equations (the Fredholm alternative, [10]), the system (2.11) has solutions if and only if

$$
\sum_{i=1}^{N} \int_{C^i} \frac{d}{ds_x^i} \mathbf{g}^i(x) \cdot \mathbf{S}^i(x)\, ds_x^i = 0, \tag{2.23}
$$

where $\widetilde{S}$, with $\widetilde{S}(x) = \mathbf{S}^i(x)$, $x \in C^i$, $i = \overline{1, N}$, is a solution of the adjoint system (2.12).

From (2.22), (2.10) and (1.2) the conditions (2.23) follow immediately.

Let $\widetilde{\phi}^0$ be a solution of the system (2.11), with $\widetilde{\phi}^0\big|_{C^j} = \boldsymbol{\phi}^0_j$, $j = \overline{1,N}$. The corresponding stream function $\psi^0 = \psi^0(\widetilde{\phi}^0)$ satisfies:

$$
\begin{aligned}
&\Delta^2\psi^0 = 0 \qquad \text{in } \Omega,\\
(2.24)\qquad &\nabla\psi^0(x) = \mathbf{g}^i(x) + \mathbf{k}^i, \qquad x\in C^i, \quad i = \overline{1,N},\\
&\nabla\psi^0(x) = \mathbf{A}^0 \ln|x| + \mathcal{O}(1), \qquad \text{as } |x|\to\infty,
\end{aligned}
$$

where

$$
\mathbf{A}^0 = \frac{1}{4\pi}\sum_{j=1}^{N}\int_{C^j}\boldsymbol{\phi}^0_j(x)\,ds^j_x\,,
$$

and $\mathbf{k}^i$, $i = \overline{1,N}$ are constant vectors. Let $\widetilde{k}^0\colon \bigcup_{j=1}^{N} C^j \to \mathbb{R}^2$ be defined by $\widetilde{k}^0\big|_{C^j} = \mathbf{k}^j$, $j = \overline{1,N}$.

Also let $\widetilde{\varphi}^i$, $i = \overline{1,2N}$ and $\widetilde{\tau}^j$, $j = \overline{1,N}$, be the $3N$ linearly independent solutions of the homogeneous system (2.11). Then the stream functions $\psi^i = \psi^i(\widetilde{\varphi}^i)$, $i = \overline{1,2N}$ satisfy the equations

$$
\begin{aligned}
&\Delta^2\psi^i = 0 \qquad \text{in } \Omega,\\
(2.25)\qquad &\nabla\psi^i(x) = \mathbf{k}^i_j, \qquad x\in C^j, \quad j = \overline{1,N},\\
&\nabla\psi^i(x) = \mathbf{A}^i \ln|x| + \mathcal{O}(1), \qquad \text{as } |x|\to\infty,
\end{aligned}
$$

with

$$
\mathbf{A}^i = \frac{1}{4\pi}\sum_{j=1}^{N}\int_{C^j}\boldsymbol{\varphi}^i_j(x)\,ds^j_x\,, \qquad \widetilde{\varphi}^i\big|_{C^j} = \boldsymbol{\varphi}^i_j\,, \quad j = \overline{1,N} \quad \text{and} \quad \mathbf{k}^i_j, \quad j = \overline{1,N},
$$

are the constant vectors, $i = \overline{1,2N}$. Let $\widetilde{k}^i\colon \bigcup_{j=1}^{N} C^j \to \mathbb{R}^2$, be given by $\widetilde{k}^i\big|_{C^j} = \mathbf{k}^i_j$, $j = \overline{1,N}$, $i = \overline{1,2N}$.

Let V be the set defined by:

$$
V = \left\{ \widetilde{k} : \bigcup_{j=1}^{N} C^j \to \mathbb{R}^2 \mid \widetilde{k}(x) = \mathbf{k}^j, \ \ x\in C^j, \ \ \mathbf{k}^j \text{ a constant vector}, \ \ j = \overline{1,N} \right\}.
$$

V is a linear space with $\dim V = 2N$, and the functions $\widetilde{k}^0$, $\widetilde{k}^i$, $i = \overline{1,2N}$ belong to V. Hence, there exist the real constants $\alpha_1,\ldots,\alpha_{2N}$ with the property:

$$
(2.26)\qquad \sum_{i=1}^{2N}\alpha_i\widetilde{k}^i(x) + \widetilde{k}^0(x) = 0, \qquad x\in\bigcup_{j=1}^{2N} C^j,
$$

if we suppose that the functions $\widetilde{\varphi}^i$, $i = \overline{1, 2N}$, satisfy:

$$\mathbf{A}^i = 0, \qquad i = \overline{1, 2N}, \tag{2.27}$$

since $\widetilde{k}^i$ are linearly independent functions.

Using (2.24), (2.25) and (2.26), we deduce that the function

$$\psi = \psi^0 + \sum_{i=1}^{2N} \alpha_i \psi^i$$

is a solution of the Stokes problem (2.5). At infinity ψ satisfies the condition:

$$\psi(x) = \mathbf{A}^0 \ln|x| + \mathcal{O}(1), \qquad \text{as } |x| \to \infty, \tag{2.28}$$

where $\mathbf{A}^0$ is defined in (2.24).

So, we obtain the following result:

THEOREM 2. *If the functions* $\mathbf{f}^i$*:* $C^i \to \mathbb{R}^2$, $i = \overline{1, N}$ *satisfy the conditions* (1.2), *then in the hypothesis* (2.27), *there exists a constant vector* $\mathbf{A}$ *such that the problem* (2.5) *with the condition* (2.28) *at infinity, has a solution* ψ.

In the proof of the Theorem 1, we used the uniqueness property of solution for the exterior Stokes problem. This result is given by:

THEOREM 3. *The Stokes problem* (2.5) *has at most one solution (up to an additive constant), under the condition that*

$$\psi(x) = \mathcal{O}\left(|x|^{-1}\right), \qquad D^m \psi(x) = \mathcal{O}\left(|x|^{-2}\right), \quad m \geq 1, \quad \text{as } |x| \to \infty, \tag{2.29}$$

and

$$\int_{C^i} \frac{\partial \omega}{\partial n}(x)\, ds_x^i = 0, \qquad i = \overline{1, N}, \tag{2.30}$$

where $\omega = \Delta\psi$.

P r o o f. We suppose that there exist two solutions ψ^1 and ψ^2 of the problem (2.5). If we consider the difference $\psi = \psi^1 - \psi^2$, then ψ satisfies the equation

$$\begin{aligned} \nabla\psi &= 0 \qquad \text{in } \Omega, \\ \nabla\psi\Big|_{C^i} &= 0, \qquad i = \overline{1, N}, \end{aligned} \tag{2.31}$$

with the additional conditions (2.29) and (2.30).

Let $\Omega_R = \Omega \cap B_R$, where B_R is a large disk of radius R. From Green's formula we obtain:

$$(2.32)\qquad \int_{\Omega_R} \left[\psi(x)\Delta^2\psi(x) - (\Delta\psi(x))^2\right] dx$$
$$= \sum_{i=1}^{N} \int_{C^i} \left[\psi(x)\frac{\partial\omega}{\partial n}(x) - \omega(x)\frac{\partial\psi}{\partial n}(x)\right] ds_x^i + \int_{\partial B_R} \left[\psi(x)\frac{\partial\omega}{\partial n}(x) - \omega(x)\frac{\partial\psi}{\partial n}(x)\right] ds_x ,$$

where ∂B_R denotes the boundary of the disk B_R.

From (2.32), it results that the integrals taken along ∂B_R are zero, for $R \to \infty$. From the homogeneous conditions $(3.31)_2$ we have

$$\int_{C^i} \omega(x)\frac{\partial\psi}{\partial n}(x)\, ds_x^i = 0, \qquad i = \overline{1, N}.$$

Also $\psi(x) = c_i$, for $x \in C^i$, where c_i is a real constant, $i = \overline{1, N}$.

Now, if we use the conditions (2.30), we deduce:

$$\sum_{i=1}^{N} \int_{C^i} \psi(x)\frac{\partial\omega}{\partial n}(x)\, ds_x^i = \sum_{i=1}^{N} c_i \int_{C^i} \frac{\partial\omega}{\partial n}(x)\, ds_x^i = 0.$$

Hence the above identity (2.32) implies $\Delta\psi = 0$ in Ω.

Applying again the Green's formula, we obtain:

$$(2.33)\qquad 0 = \int_{\Omega_R} \psi(x)\Delta\psi(x)\, dx$$
$$= \int_{\partial B_R} \psi(x)\frac{\partial\psi}{\partial n}(x)\, ds_x + \sum_{i=1}^{N} \int_{C^j} \psi(x)\frac{\partial\psi}{\partial n}(x)\, ds_x^i - \int_{\Omega_R} (\nabla\psi(x))^2\, dx.$$

Using the conditions (2.29), (2.30), $(2.31)_2$ we obtain $\nabla\psi = 0$ in Ω, hence ψ is a constant in Ω and $\psi_1 = \psi_2$ (up to an additive constant).

REMARK. Since we determine the stream function ψ in the form (2.7), the conditions (2.30) are easily obtained as a consequence of Green's identity.

Using the stream function ψ determined above, we obtain the velocity $\mathbf{u} = (\nabla\psi)^{\perp}$, and the pressure p as the harmonic conjugate of $\omega = \Delta\psi$, but only locally, because the domain Ω is not simply connected.

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