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Stress tensors associated with deformation tensors via duality

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THE CONCEPT of dual variables, initially introduced by HAUPT and TSAKMAKIS [3], enables us to relate to each other strain and stress tensors, as well as associated rates, independently of particular material properties. Generally, it is different than the method of conjugate variables, as defined e.g. by MAC VEAN [2] or HILL [4–6]. The duality concept postulated by HAUPT and TSAKMAKIS [3] deals only with two classes of dual stress and strain tensors. The second Piola–Kirchhoff stress tensor and the Green strain tensor, as well as the negative convected stress tensor and the Piola strain tensor, are respectively the Lagrangean stress and strain tensors included in the two classes of dual stress and strain tensors. However, there are further (infinitely many) Lagrangean stress and strain tensors, which may be taken into consideration. The aim of the present paper is to develop further the concept of dual variables to take into account the whole set of Lagrangean stress and strain tensors. Doing this, we obtain a specific mathematical structure in the sets of all strain and stress tensors, which makes it possible to relate strain and stress tensors, as well as associated rates, independently of the particular material properties.

1. Introduction

IT IS WELL-KNOWN that in the theory of finite deformations, several stress and strain tensors can be introduced in various ways. These stress and strain tensors are not *a priori* related to each other, raising the question of whether or not there exists a method to associate with each stress tensor a strain tensor independently of specific material properties. The stress power is usually the convenient framework for answering this question.

According to ZIEGLER and MAC VEAN [1, 2], a stress tensor is assigned to a given strain tensor, if the stress power can be represented by this stress tensor and an appropriate rate of the given strain tensor. We call stress and strain tensors related in this way conjugate in the sense of Ziegler and MacVean. Note in passing that this definition of conjugancy was also adopted by HAUPT and TSAKMAKIS [3]. However, in HAUPT and TSAKMAKIS [3], it was also shown that the above definition brings out the difficulty that arises because the stress and strain tensors associated in such a manner are not unique. For example, consider the strain tensor $\underline{\mathbf{K}} = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-1}\mathbf{F}^{T-1}\right)$. $\underline{\mathbf{K}}$ is conjugate in the sense of Ziegler and MacVean, on the one hand, to the stress tensor $\overline{\mathbf{T}} = (\det \mathbf{F})\mathbf{F}^T\mathbf{T}\mathbf{F}$, with respect to the material time derivative $\dot{\underline{\mathbf{K}}}$, and on the other hand, to the stress tensor $\overline{\mathbf{S}} = (\det \mathbf{F})\mathbf{R}^T\mathbf{T}\mathbf{R}$, with respect to the rate $\overset{\triangle}{\underline{\mathbf{K}}} = \dot{\underline{\mathbf{K}}} + (\dot{\mathbf{U}}\mathbf{U}^{-1})\underline{\mathbf{K}} + \underline{\mathbf{K}}(\dot{\mathbf{U}}\mathbf{U}^{-1})$,

$$W = \overline{\mathbf{T}} \cdot \dot{\underline{\mathbf{K}}} = \overline{\mathbf{S}} \cdot \overset{\triangle}{\underline{\mathbf{K}}} .$$

In these relations (1), $\mathbf{F}$ denotes the deformation gradient tensor, with polar decomposition $\mathbf{F} = \mathbf{RU}$, $\mathbf{T}$ is the Cauchy stress tensor, and W the stress power per unit volume of the reference configuration.

Another concept used to relate stress and strain tensors within the framework of the stress power is due to Hill (see e.g. HILL [4–6] as well as HAVNER [7], OGDEN [8, Sec. 3.5.2] and WANG and TRUESDELL [9, Secs. 3.8 and 3.9]). According to this concept, a stress tensor $\mathbf{t}$ is postulated to be conjugate (in the sequel called conjugate in Hill's sense) to a given strain tensor $\mathbf{e}$ if the inner product of $\mathbf{t}$ with the material time derivative of $\mathbf{e}$ yields the stress power W, i.e., if

$$W = \mathbf{t} \cdot \dot{\mathbf{e}} .$$

Clearly, all pairs of stress and strain variables conjugate in Hill's sense are also conjugate in the sense of Ziegler and MacVean, but the converse is generally not true.

Hill's concept of conjugancy has the characteristic feature that there exist stress tensors which do not necessarily have any conjugate strain tensor associated with them having an integrable strain rate. Strain rate tensors are called integrable (2) (not-integrable) if they are expressible (not-expressible) as material time derivatives of some strain tensors, which are defined as functions of the deformation. It is well-known that the strain rate $\mathbf{D}$, representing the symmetrical part of the velocity gradient tensor $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, is a non-integrable rate in general. Thus the weighted Cauchy stress tensor $\mathbf{S} = (\det \mathbf{F})\mathbf{T}$, having the property $W = \mathbf{S} \cdot \mathbf{D}$, is e.g. not conjugate in Hill's sense to a strain tensor which possesses an integrable rate. The same is also true for the stress tensor $\overline{\mathbf{S}}$. On the other hand, if a strain tensor is given, it must not necessarily have a conjugate stress tensor associated with it. As an example of strain tensors to which no stress tensor conjugate in Hill's sense exists, we mention the Almansi strain tensor $\mathbf{A} = \frac{1}{2}(\mathbf{1} - \mathbf{F}^{T-1}\mathbf{F}^{-1})$. These issues have also been discussed e.g. by OGDEN [8, p. 159].

A further possibility for associating stress and strain tensors within the framework of the stress power has been proposed by HAUPT and TSAKMAKIS [3], and referred to as the concept of dual variables (3). Several mathematical aspects from a local differential geometric point of view were discussed by SVENDSEN and TSAKMAKIS [11]. The relation between stress and strain tensors within the duality concept of HAUPT and TSAKMAKIS [3] is unique; in fact, this constitutes the

(1) The nomenclature is introduced in the Secs. 2 and 3.

(2) The term integrable (not-integrable) strain rate is adopted from PALGEN and DRUCKER [10].

(3) We take this opportunity to correct some misleading and erroneous statements in HAUPT and TSAKMAKIS [3]. The notion of conjugancy used in this reference should be understood in the sense of Ziegler and MacVean, even though in some places this notion was attributed to Hill. Further, on page 184 in HAUPT and TSAKMAKIS [3], the interpretation of the term "direct flux" in Hill's expression "$\mathbf{R}^T\mathbf{DR}$ is not a direct flux", as the specification of a strain tensor with the associated rate $\mathbf{R}^T\mathbf{DR}$, is not correct. Indeed, the term "direct flux" as used by Hill must be interpreted to mean the material time derivative. Furthermore, the statement on p. 174 that $\mathbf{\Psi}$, which is not necessarily assumed to be the gradient of a vector field, induces a system of spatial coordinates, is not true in general.

differences to the conjugancy concept according to Ziegler and MacVean. In addition, concerning dual pairs of variables, use is made not only of the material time derivative, but also e.g. of the so-called objective derivatives. This clarifies the differences compared with the conjugancy concept due to Hill. In the present work, the concept of duality will appropriately be generalized, to include the generalized Lagrangean strain tensors, which are introduced in Sec. 5.1. To be definite, the duality concept postulated in HAUPT and TSAKMAKIS [3] deals only with two classes of dual stress and strain tensors, called family 1 and 2. Representative (Lagrangean) strain tensors are the Green strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1})$ (family 1) and the Piola strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{F}^{-1}\mathbf{F}^{T-1} - \mathbf{1})$ (family 2). The purpose of the present paper is to complete the duality concept of HAUPT and TSAKMAKIS [3] by introducing further classes of dual strain and stress tensors, which include the whole set of generalized Lagrangean strain tensors.

After introducing the notation and some background relations in Secs. 2 and 3 we show in Sec. 4 how various so-called objective time derivatives can be assigned to the Cauchy stress tensor. To each of these objective time derivatives of the Cauchy stress tensor corresponds a Lagrangean stress tensor. It turns out that, among all these derivatives, only two possess the structure of generalized Oldroyd time derivatives (the term "generalized" Oldroyd time derivative is specified in Chapter 3). In other words, among all Lagrangean stress tensors, only two are associated to the Cauchy stress tensor with respect to the definition of the generalized Oldroyd time derivatives. This result motivates in Sec. 5 the introduction of a set of generalized strain and stress tensors respectively. Considering various scalar quantities, which are required to be form-invariant with respect to the chosen configuration, the above sets can be partitioned into equivalence classes of generalized strains and associated generalized dual stress tensors, respectively. The concept of duality used here is a generalization of that in HAUPT and TSAKMAKIS [3]. Furthermore, to each strain and stress tensor, a time derivative can be associated, having the form of "generalized" Oldroyd time derivative. This way, we obtain a specific mathematical structure in the sets of all strain and stress tensors, which enables us to relate strain and stress tensors, as well as the associated rates, independently of particular material properties. Some examples formulated using strain and associated dual stress tensors, are briefly discussed in Sec. 6. Finally, in Sec. 7, the duality concept is appropriately extended to take into account two-point tensor fields, as well.

2. Preliminaries

We denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of real and natural numbers, respectively. The absolute value of $c \in \mathbb{R}$ is $|c|$. We use the letter t for the time variable. If φ is a function of t we write $\dot{\varphi}$ or $d\varphi/dt$ for its material time derivative. For the

n-th material time derivative of φ we write also $d^n\varphi/dt^n$, where $n \in \mathbb{N}$, $n \geq 0$. If x is a scalar variable other than t and $f(x)$ is a function of x, then we denote the derivative of $f(x)$ with respect to x by $f'(x)$. In particular, we write $f'(a)$, $a \in \mathbb{R}$, instead of $f'(x)|_{x=a}$. Commonly the same symbol is used to designate a function and the value of that function at a point. However, if we deal with different representations of the same function, then use will be made of different symbols.

Given two sets A and B, the Cartesian product of A and B is denoted by $A \times B$. In particular, we write

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n-\text{times}},$$

$n \in \mathbb{N}$, $n \geq 1$. Let $\mathbf{a}$ and $\mathbf{b}$ be elements of a three-dimensional Euclidean vector space $\mathbb{V}$. By $\mathbf{a} \otimes \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b}$ we denote the tensor product, the vector product and the inner product, respectively. The magnitude of $\mathbf{a}$ is denoted by $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. In this work, we identify the vector space $\mathbb{V}$ with its dual space $\mathbb{V}^*$, the identification being specified with the help of the metric tensor induced by the inner product in $\mathbb{V}$. Thus, any n-order tensor $\mathbf{T}$ on $\mathbb{V}$ is regarded as an n-linear function from $\mathbb{V}^n$ to $\mathbb{R}$, denoted by $\mathbf{T} \in \mathsf{L}(\mathbb{V}^n, \mathsf{R})$. In the following, second-order tensors (like vectors) are denoted by boldface letters, whereas for fourth-order tensors we use script letters. For example, $\mathbf{A}, \mathbf{B}, \ldots$ denote second-order tensors, whereas $\mathcal{A}, \mathcal{B}, \ldots$ denote fourth-order tensors, respectively.

Let $\mathbf{A}$, $\mathbf{B}$ be second-order tensors. We write $\operatorname{tr}\mathbf{A}$, $\det\mathbf{A}$ and $\mathbf{A}^T$ for the trace, the determinant and the transpose of $\mathbf{A}$, respectively, while $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr}(\mathbf{A}\mathbf{B}^T)$ is the inner product between $\mathbf{A}$ and $\mathbf{B}$. We write $\mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ for the identity tensor of second order, where δ_{ij} is the Kronecker delta symbol and $\{\mathbf{e}_i\}$, $i = 1, 2, 3$, is an orthonormal basis in $\mathbb{V}$. Further, we use the notations $\mathbf{AB} = A_{ij}B_{jk}\mathbf{e}_i \otimes \mathbf{e}_k$ and $\mathbf{A}^{T-1} = (\mathbf{A}^{-1})^T$, provided $\mathbf{A}^{-1}$ exists. In these relations the convention of summation over repeated indices is employed.

If $\mathbf{A}$ is a symmetric and positive definite second-order tensor having eigenvalues λ_i and corresponding eigenvectors $\mathbf{a}_i$, then the spectral decomposition (see e.g. GURTIN [12, Ch. I.2])

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \mathbf{a}_i \otimes \mathbf{a}_i$$

applies. In this case, we denote by $\mathbf{A}^m$, $m \in \mathsf{R}$, the second-order tensor

$$\mathbf{A}^m = \sum_{i=1}^{3} \lambda_i^m \mathbf{a}_i \otimes \mathbf{a}_i \,.$$

Let $\mathcal{K}$, $\mathcal{P}$ be two fourth-order tensors, i.e., linear transformations from the space of second-order tensors into itself. With respect to the orthonormal basis $\{\mathbf{e}_i\}$, the following rules apply: if $\mathcal{K}$, $\mathcal{P}$ and $\mathbf{A}$ are represented by $\mathcal{K} = K_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$, $\mathcal{P} = P_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$, and $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, respectively, the relations

$$\begin{aligned} \mathcal{K}\mathcal{P} &= K_{ijmn}P_{mnkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \,, \\ \mathcal{K}^T &= K_{ijkl}\mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_i \otimes \mathbf{e}_j \,, \\ \mathcal{K}[\mathbf{A}] &= K_{ijmn}A_{mn}\mathbf{e}_i \otimes \mathbf{e}_j \end{aligned}$$

hold. In addition, if $\mathbf{B}$ is a second-order tensor, we have $\mathbf{A} \cdot \mathcal{K}[\mathbf{B}] = \mathbf{B} \cdot \mathcal{K}^T[\mathbf{A}]$. We write $\mathcal{I}$ for the fourth-order identity tensor,

$$\mathcal{I} = \delta_{ij}\delta_{mn}\mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_j \otimes \mathbf{e}_n \,.$$

The tensor $\mathcal{I}$ can be decomposed in the form

$$\mathcal{I} = \mathcal{E} + \mathcal{J} \,,$$

where

$$\mathcal{E} = \frac{1}{2}(\delta_{ij}\delta_{mn} + \delta_{in}\delta_{mj})\,\mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_j \otimes \mathbf{e}_n$$

and

$$\mathcal{J} = \frac{1}{2}(\delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj})\,\mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_j \otimes \mathbf{e}_n \,.$$

This implies $\mathcal{E}[\mathbf{A}] = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$, $\mathcal{J}[\mathbf{A}] = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$, and $\mathcal{I}[\mathbf{A}] = \mathbf{A}$.

3. Background relations

Consider a material body B which occupies the region R_0 in the three-dimensional Euclidean point space E in some reference configuration. Choosing a fixed point (origin) in E, we identify each particle of B by the position vector $\mathbf{X}$ to the place X in R_0 occupied by the considered particle. We write $\mathbf{x}$ for the position vector to the place x occupied by the same material particle in the (current) configuration at time t. In this configuration, the body B occupies the region R_t in E.

A motion of B in E, i.e., an one-parameter family of configurations parameterized by the time t, is a mapping

$$\bar{\mathbf{x}} : (\mathbf{X}, t) \longmapsto \mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t), \tag{3.1}$$

which has an inverse $\mathbf{X} = \overline{\mathbf{X}}(\mathbf{x}, t)$ for fixed time t. In what follows, it is assumed that all functions possess continuous derivatives up to any desired order with respect to the space variables and the time t.

The deformation gradient tensor corresponding to (3.1) is denoted by

$$\mathbf{F} = \frac{\partial \bar{\mathbf{x}}}{\partial \mathbf{X}} = \mathrm{GRAD}\,\bar{\mathbf{x}} \,. \tag{3.2}$$

We distinguish between GRAD and grad, representing the gradient operator with respect to **X** and **x**, respectively. Furthermore, $\det \mathbf{F} > 0$ is assumed.

The right Cauchy-Green tensor **C** and the left Cauchy-Green tensor **B** are given by

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 , \tag{3.3}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 , \tag{3.4}$$

in which **U** and **V** are the right and the left stretch tensor, respectively, appearing in the polar decomposition of **F**:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} . \tag{3.5}$$

Here, **R** represents a proper orthogonal second-order tensor. Since **U** and **V** are symmetric and positive definite, they possess the spectral decompositions

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{M}_i \otimes \mathbf{M}_i , \tag{3.6}$$

and

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \ \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_i \tag{3.7}$$

respectively, with

$$\boldsymbol{\mu}_i = \mathbf{R}\,\mathbf{M}_i . \tag{3.8}$$

λ_i $(i = 1, 2, 3)$ are positive eigenvalues and $\mathbf{M}_i$, as well as $\boldsymbol{\mu}_i$ are the corresponding unit eigenvectors. It is common (see e.g. OGDEN [8, Sec. 2.2.5]) to call $\mathbf{M}_i$ and $\boldsymbol{\mu}_i$ the Lagrangean and Eulerian principal axes, respectively. Note that the spectral decomposition (3.6) implies

$$\mathbf{U}^{-1} = \sum_{i=1}^{3} \frac{1}{\lambda_i} \mathbf{M}_i \otimes \mathbf{M}_i . \tag{3.9}$$

Let X be the place of a material particle in R_0 and denote by y the place of the same material particle in an arbitrary configuration, in which B occupies the region M. Further, we denote by $T_y\mathsf{M}$ the tangent space of M at y. Note that M does not need to be an Euclidean manifold. This is for example the case for the non-Euclidean intermediate configuration in plasticity. An n-order tensor **A** is called a tensor at $y \in \mathsf{M}$ if $\mathbf{A} \in \mathsf{L}((T_y\mathsf{M})^n, \mathbb{R})$. If $\mathsf{M} = \mathsf{R}_0$, **A** is called a Lagrangean tensor at $y \in \mathsf{R}_0$. In the case when M is different than R_0, the tensor **A** is called a spatial tensor at $y \in \mathsf{M}$. In particular, if $\mathsf{M} = \mathsf{R}_t$, then **A** is called an Eulerian tensor (4) at $y \in \mathsf{R}_t$. In the following, we denote by

$$\boldsymbol{\Psi} = \widehat{\boldsymbol{\Psi}}(\mathbf{X}, t) \in \mathrm{Lin}^+ \tag{3.10}$$

(4) The definition on spatial tensors given here is not standard. The definitions on Lagrangean and Eulerian tensors are taken from OGDEN [8, Sec. 2.4.1].

a space- and time-dependent linear transformation (second-order two-point tensor field (5)) mapping vectors from $T_X\mathsf{R}_0$ onto $T_y\mathsf{M}$ ($\mathbf{\Psi} \in \mathsf{L}(T_X\mathsf{R}_0 \times T_y\mathsf{M}, \mathbb{R})$) and having a positive determinant.

Let X^k ($k = 1, 2, 3$) be a system of material coordinates, and let

$$\mathbf{X} = \tilde{\mathbf{X}}(X^k) \tag{3.11}$$

be the position vector of a material particle in the reference configuration. The coordinate system induces the local basis of tangent vectors $\{\mathbf{G}_k\}$,

$$\mathbf{G}_k = \frac{\partial \tilde{\mathbf{X}}}{\partial X^k}\,, \tag{3.12}$$

and the gradient vectors $\{\mathbf{G}^k\}$,

$$\mathbf{G}^k = \mathrm{GRAD}\,\tilde{X}^k(\mathbf{X}), \tag{3.13}$$

being the reciprocal basis of the tangent vectors $\{\mathbf{G}_k\}$, where

$$X^k = \tilde{X}^k(\mathbf{X}) \tag{3.14}$$

are the relations inverse to (3.11). With respect to (3.10), (3.12) and (3.13), various bases $\{\mathbf{g}_k^{(\Psi)}\}$ in $T_y\mathsf{M}$, with reciprocal basis $\{\mathbf{g}^{(\Psi)k}\}$, can be defined by

$$\mathbf{g}_k^{(\Psi)} := \mathbf{\Psi}\mathbf{G}_k\,, \tag{3.15}$$

$$\mathbf{g}^{(\Psi)k} := \mathbf{\Psi}^{T-1}\mathbf{G}^k\,. \tag{3.16}$$

Note that the special case $\mathbf{\Psi} = \mathbf{F}$ defines the so-called convected coordinate systems. From (3.15), (3.16),

$$\overset{\bullet}{\mathbf{g}}{}_k^{(\Psi)} := \dot{\mathbf{\Psi}}\,\mathbf{\Psi}^{-1}\mathbf{g}_k^{(\Psi)}\,, \tag{3.17}$$

$$\overset{\bullet}{\mathbf{g}}{}^{(\Psi)k} = -(\dot{\mathbf{\Psi}}\,\mathbf{\Psi}^{-1})^T\mathbf{g}^{(\Psi)k}\,. \tag{3.18}$$

Next consider the spatial, time-dependent tensor field $\mathbf{u}$, having the representation

$$\mathbf{u} = u^k\mathbf{g}_k^{(\Psi)} = u_k\,\mathbf{g}^{(\Psi)k}\,. \tag{3.19}$$

The relations

$$\frac{\delta^{(\cdot)}}{\delta t}\mathbf{u} := \overset{\bullet}{u}{}^k\,\mathbf{g}_k^{(\Psi)}\,, \tag{3.20}$$

$$\frac{\delta_{(\cdot)}}{\delta t}\mathbf{u} := \overset{\bullet}{u}_k\,\mathbf{g}^{(\Psi)k}\,, \tag{3.21}$$

(5) $\mathbf{\Psi}$ can be interpreted to be related with a local deformation process.

define time derivatives which are called generalized Oldroyd time derivatives of $\mathbf{u}$. Clearly, from (3.17)–(3.21),

$$\frac{\delta^{(\cdot)}}{\delta t}\mathbf{u} = \dot{\mathbf{u}} - \dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1}\mathbf{u}, \tag{3.22}$$

$$\frac{\delta_{(\cdot)}}{\delta t}\mathbf{u} = \dot{\mathbf{u}} + (\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1})^{T}\mathbf{u}. \tag{3.23}$$

Note that the time derivatives $\delta_{(\cdot)}\mathbf{u}/\delta t$ and $\delta^{(\cdot)}\mathbf{u}/\delta t$ are related to the material time derivative of the Lagrangean vectors $\mathbf{u}^{(\mathrm{L})}$, $\mathbf{u}_{(\mathrm{L})}$,

$$\mathbf{u}^{(\mathrm{L})} := \boldsymbol{\Psi}^{-1}\mathbf{u}, \tag{3.24}$$

$$\mathbf{u}_{(\mathrm{L})} := \boldsymbol{\Psi}^{T}\mathbf{u}, \tag{3.25}$$

through

$$\dot{\mathbf{u}}^{(\mathrm{L})} = \boldsymbol{\Psi}^{-1}\frac{\delta^{(\cdot)}}{\delta t}\mathbf{u}, \tag{3.26}$$

$$\dot{\mathbf{u}}_{(\mathrm{L})} := \boldsymbol{\Psi}^{T}\frac{\delta_{(\cdot)}}{\delta t}\mathbf{u}, \tag{3.27}$$

respectively. These definitions of generalized Oldroyd time derivatives for vector fields can easily be extended to introduce generalized Oldroyd time derivatives for tensor fields. For example, for a spatial symmetric second-order tensor

$$\mathbf{A} = A^{kl}\mathbf{g}_k^{(\Psi)} \otimes \mathbf{g}_l^{(\Psi)} = A_{kl}\mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}, \tag{3.28}$$

the corresponding symmetric generalized Oldroyd rates are defined by

$$\frac{\delta^{(\cdot\cdot)}}{\delta t}\mathbf{A} = \dot{A}^{kl}\mathbf{g}_k^{(\Psi)} \otimes \mathbf{g}_l^{(\Psi)}, \tag{3.29}$$

$$\frac{\delta_{(\cdot\cdot)}}{\delta t}\mathbf{A} = \dot{A}_{kl}\mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}. \tag{3.30}$$

It follows that

$$\frac{\delta^{(\cdot\cdot)}}{\delta t}\mathbf{A} = \dot{\mathbf{A}} - \dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1}\mathbf{A} - \mathbf{A}(\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1})^{T}, \tag{3.31}$$

$$\frac{\delta_{(\cdot\cdot)}}{\delta t}\mathbf{A} = \dot{\mathbf{A}} + (\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1})^{T}\mathbf{A} + \mathbf{A}\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1}, \tag{3.32}$$

and that

$$\dot{\mathbf{A}}^{(\mathrm{L})} = \boldsymbol{\Psi}^{-1}\left(\frac{\delta_{(\cdot\cdot)}}{\delta t}\mathbf{A}\right)\boldsymbol{\Psi}^{T-1}, \tag{3.33}$$

$$\dot{\mathbf{A}}_{(\mathrm{L})} = \boldsymbol{\Psi}^{T}\left(\frac{\delta^{(\cdot\cdot)}}{\delta t}\mathbf{A}\right)\boldsymbol{\Psi}, \tag{3.34}$$

where

$$\text{(3.35)} \qquad \mathbf{A}^{(\mathrm{L})} := \boldsymbol{\Psi}^{-1}\mathbf{A}\boldsymbol{\Psi}^{T-1},$$

$$\text{(3.36)} \qquad \mathbf{A}_{(\mathrm{L})} := \boldsymbol{\Psi}\mathbf{A}\boldsymbol{\Psi}^{T}.$$

Next we note that with respect to the basis $\{\mathbf{M}_i\}$, various strain tensors can be defined. In order to obtain the Lagrangean strain tensors introduced by Hill ([6]), we consider monotonic scalar functions $g: (0,\infty) \to \mathbb{R}$, such that

$$\text{(3.37)} \qquad g(1) = 0, \qquad g'(1) = 1.$$

Then, the symmetric tensors $\mathbf{G}_{(g)}$, defined by means of the isotropic tensor-valued function $\mathbf{g}(\cdot)$,

$$\text{(3.38)} \qquad \mathbf{g}: \mathbf{U} \longmapsto \mathbf{G}_{(g)} = \mathbf{g}(\mathbf{U}) := \sum_{i=1}^{3} g(\lambda_i)\,\mathbf{M}_i \otimes \mathbf{M}_i,$$

represent Lagrangean measures of strain, referred to as Hill's Lagrangean strain tensors. Examples of such functions are given by ($m \in \mathbb{R}$)

$$\text{(3.39)} \qquad g_{(m)}(\lambda_i) := \begin{cases} \dfrac{1}{m}(\lambda_i^m - 1) & \text{if} \quad m \neq 0, \\ \ln \lambda_i & \text{if} \quad m = 0, \end{cases}$$

inducing the strain tensors ([7])

$$\text{(3.40)} \qquad \mathbf{G}_{(m)} := \mathbf{G}_{(g_{(m)})} := \begin{cases} \displaystyle\sum_{i=1}^{3} \frac{1}{m}(\lambda_i^m - 1)\mathbf{M}_i \otimes \mathbf{M}_i = \frac{1}{m}(\mathbf{U}^m - \mathbf{1}) & \text{if } m \in (\mathbb{R}\backslash 0), \\ \displaystyle\sum_{i=1}^{3} (\ln \lambda_i)\mathbf{M}_i \otimes \mathbf{M}_i = \ln \mathbf{U} & \text{if } m = 0. \end{cases}$$

In relating stress tensors to the given strain tensors, we will employ the stress power per unit volume of the reference configuration W, which can also be written in the form

$$\text{(3.41)} \qquad W = \mathbf{T}_R \cdot \dot{\mathbf{F}}.$$

In this formula, $\mathbf{T}_R$ stands for the first Piola–Kirchhoff stress tensor, i.e.,

$$\text{(3.42)} \qquad \mathbf{T}_R = (\det \mathbf{F})\mathbf{T}\mathbf{F}^{T-1} = \mathbf{S}\mathbf{F}^{T-1},$$

([6]) The treatment of Hill's Lagrangean strain tensors given here is taken from OGDEN [8, Sec. 2.2.7] as well as WANG and TRUESDELL [9, Sec. 3.8].

([7]) These Lagrangean strain tensors were introduced for the first time by DOYLE and ERICKSEN [13, Ch. 4].

where $\mathbf{T}$ and $\mathbf{S} = (\det \mathbf{F})\mathbf{T}$ are the Cauchy and the weighted Cauchy (or Kirchhoff) stress tensor ([8]), respectively. Furthermore, we have

$$W = \mathbf{S} \cdot \mathbf{D}, \tag{3.43}$$

where $\mathbf{D}$ represents the symmetric part of the velocity gradient tensor $\mathbf{L}$ (the antisymmetric part of $\mathbf{L}$ being $\mathbf{W}$):

$$\mathbf{L} = \operatorname{grad} \dot{\mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{D} + \mathbf{W}, \tag{3.44}$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \tag{3.45}$$

$$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T). \tag{3.46}$$

4. Objective rates for S

In this section, we shall consider the Lagrangean stress tensors conjugate (in Hill's sense) to the strain tensors (3.37)–(3.40). As a first step towards the development of a general duality concept for associating strain and stress tensors, we will derive the relations between these Lagrangean stress tensors and the weighted Cauchy stress tensor $\mathbf{S}$. These relations are expressed in terms of linear transformations and using the same transformations, we shall establish various so-called objective rates for $\mathbf{S}$. It turns out that, among all the transformations corresponding to arbitrary m, only those for $m = \pm 2$ lead to objective rates for $\mathbf{S}$ having the structure of a generalized Oldroyd time derivative.

In order to derive this result, we turn to the strain tensors $\mathbf{G}_{(g)}$ defined by (3.38), where $g(\lambda_i)$ may be specified by (3.29). First of all, the stress power W is rewritten as

$$W = \mathbf{T}_{(\mathrm{BS})} \cdot \dot{\mathbf{U}}, \tag{4.1}$$

where

$$\mathbf{T}_{(\mathrm{BS})} := \frac{1}{2}\left(\mathbf{T}_R^T\mathbf{R} + \mathbf{R}^T\mathbf{T}_R\right) \tag{4.2}$$

is the symmetric part of the Biot stress tensor

$$\mathbf{T}_{(\mathrm{B})} := \mathbf{R}^T\mathbf{T}_R \tag{4.3}$$

(see OGDEN [8, Sec. 3.5.2]). The definition of the stress tensors $\mathbf{T}_{(g)}$, conjugate in Hill's sense to $\mathbf{G}_{(g)}$, should be based on the identity

$$\mathbf{T}_{(\mathrm{BS})} \cdot \dot{\mathbf{U}} = \mathbf{T}_{(g)} \cdot \dot{\mathbf{G}}_{(g)}. \tag{4.4}$$

([8]) We are concerned here only with nonpolar materials, so that T, and therefore S, is symmetric.

In view of $\mathbf{G}_{(g)} = \mathbf{g}(\mathbf{U})$, by (3.38), we obtain

$$\dot{\mathbf{G}}_{(g)} = \mathcal{U}_{(g)}[\dot{\mathbf{U}}] \tag{4.5}$$

where

$$\mathcal{U}_{(g)} = \frac{\partial \mathbf{g}}{\partial \mathbf{U}} \tag{4.6}$$

and

$$\mathcal{U}_{(g)} = \mathcal{U}_{(g)}^T, \tag{4.7}$$

i.e., $\mathcal{U}_{(g)}$ is symmetric. Furthermore, there exists a symmetric fourth-order tensor $\mathcal{T}_{(g)}$ satisfying the relation

$$\mathcal{U}_{(g)}\mathcal{T}_{(g)} = \mathcal{T}_{(g)}\mathcal{U}_{(g)} = \mathcal{E}\,. \tag{4.8}$$

Thus,

$$\mathbf{T}_{(\mathrm{BS})} \cdot \dot{\mathbf{U}} = \mathbf{T}_{(g)} \cdot \mathcal{U}_{(g)}[\dot{\mathbf{U}}] = \mathcal{U}_{(g)}[\mathbf{T}_{(g)}] \cdot \dot{\mathbf{U}} \tag{4.9}$$

and

$$\mathbf{T}_{(g)} = \mathcal{T}_{(g)}[\mathbf{T}_{(\mathrm{BS})}]. \tag{4.10}$$

With respect to the basis $\{\mathbf{M}_i\}$, the following representations hold:

$$\begin{aligned}\mathcal{U}_{(g)} = {} & \sum_{i=1}^{3} g'(\lambda_i)\mathbf{M}_i \otimes \mathbf{M}_i \otimes \mathbf{M}_i \otimes \mathbf{M}_i \\ & + \sum_{i \neq j} \ell_{(g)ij} \left(\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_i \otimes \mathbf{M}_j + \mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_j \otimes \mathbf{M}_i\right),\end{aligned} \tag{4.11}$$

$$\begin{aligned}\mathcal{T}_{(g)} = {} & \sum_{i=1}^{3} \frac{1}{g'(\lambda_i)}\mathbf{M}_i \otimes \mathbf{M}_i \otimes \mathbf{M}_i \otimes \mathbf{M}_i \\ & + \frac{1}{4}\sum_{i \neq j} \frac{1}{\ell_{(g)ij}} \left(\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_i \otimes \mathbf{M}_j + \mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_j \otimes \mathbf{M}_i\right),\end{aligned} \tag{4.12}$$

where

$$\ell_{(g)ij} := \begin{cases} \dfrac{1}{2}\dfrac{g(\lambda_j) - g(\lambda_i)}{\lambda_j - \lambda_i} & \text{if} \quad \lambda_i \neq \lambda_j\,, \quad i \neq j, \\ \dfrac{1}{2}g'(\lambda_i) & \text{if} \quad \lambda_i = \lambda_j, \quad i \neq j. \end{cases} \tag{4.13}$$

(For a more detailed derivation of the relations (4.1)–(4.13) see OGDEN [8, Sec. 3.5.2]).

In order to express the dependence on the weighted Cauchy stress tensor $\mathbf{S}$, we note that in view of (4.2) and $(3.42)_2$, (3.5), the equation

$$\mathbf{T}_{(\mathrm{BS})} = \frac{1}{2}\left(\mathbf{U}^{-1}\mathbf{R}^T\mathbf{S}\mathbf{R} + \mathbf{R}^T\mathbf{S}\mathbf{R}\mathbf{U}^{-1}\right) =: \mathcal{K}[\mathbf{S}] \tag{4.14}$$

applies. Taking into account the relations (3.6)–(3.9), it is not difficult to derive for $\mathcal{K}$ the representation

$$(4.15)\qquad \mathcal{K} = \frac{1}{4}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\lambda_i+\lambda_j}{\lambda_i\lambda_j}\left(\mathbf{M}_i\otimes\mathbf{M}_j\otimes\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_j + \mathbf{M}_i\otimes\mathbf{M}_j\otimes\boldsymbol{\mu}_j\otimes\boldsymbol{\mu}_i\right).$$

Inserting $(4.14)_2$ in (4.10) yields

$$(4.16)\qquad \mathbf{T}_{(g)} = \mathcal{T}_{(g)}\mathcal{K}[\mathbf{S}] = \mathcal{A}_{(g)}[\mathbf{S}]$$

with

$$(4.17)\qquad \mathcal{A}_{(g)} := \mathcal{T}_{(g)}\mathcal{K}\,.$$

From (4.12) and (4.15) we obtain

$$(4.18)\qquad \mathcal{A}_{(g)} = \sum_{i=1}^{3}\frac{1}{\lambda_i g'(\lambda_i)}\mathbf{M}_i\otimes\mathbf{M}_i\otimes\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_i + \sum_{i\neq j}\alpha_{(g)ij}\left(\mathbf{M}_i\otimes\mathbf{M}_j\otimes\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_j + \mathbf{M}_i\otimes\mathbf{M}_j\otimes\boldsymbol{\mu}_j\otimes\boldsymbol{\mu}_i\right),$$

where (no summation over $i, j, i \neq j$)

$$(4.19)\qquad \alpha_{(g)ij} := \begin{cases} \dfrac{1}{4}\dfrac{\lambda_i^2-\lambda_j^2}{\lambda_j\lambda_i(g(\lambda_i)-g(\lambda_j))} & \text{if}\quad \lambda_i\neq\lambda_j,\quad i\neq j,\\[2ex] \dfrac{1}{2}\dfrac{1}{\lambda_i g'(\lambda_i)} & \text{if}\quad \lambda_i=\lambda_j,\quad i\neq j.\end{cases}$$

Introducing the fourth-order tensor $\mathcal{P}_{(g)}$ by

$$(4.20)\qquad \mathcal{A}_{(g)}\mathcal{P}_{(g)} = \mathcal{P}_{(g)}\mathcal{A}_{(g)} = \mathcal{E}\,,$$

where

$$(4.21)\qquad \mathcal{P}_{(g)} = \sum_{i=1}^{3}\lambda_i g'(\lambda_i)\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_i\otimes\mathbf{M}_i\otimes\mathbf{M}_i + \frac{1}{4}\sum_{i\neq j}\frac{1}{\alpha_{(g)ij}}\left(\boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_j\otimes\mathbf{M}_i\otimes\mathbf{M}_j + \boldsymbol{\mu}_i\otimes\boldsymbol{\mu}_j\otimes\mathbf{M}_j\otimes\mathbf{M}_i\right),$$

in view of (4.18), we deduce from $(4.16)_2$ that

$$(4.22)\qquad \mathbf{S} = \mathcal{P}_{(g)}[\mathbf{T}_{(g)}].$$

The tensors $\mathcal{P}_{(g)}$ and $\mathcal{A}_{(g)}$ induce transformations relating the stress tensors $\mathbf{T}_{(g)}$ and $\mathbf{S}$, respectively. This enables us to associate with each function $g(\cdot)$ an objective rate of $\mathbf{S}$, defined by application of the same transformation $\mathcal{P}_{(g)}$ to the material time derivative of the Lagrangean stress tensor $\mathbf{T}_{(g)}$:

$$(4.23) \qquad \frac{D_{(g)}}{Dt}\mathbf{S} := \mathcal{P}_{(g)}[\dot{\mathbf{T}}_{(g)}] \qquad \Longleftrightarrow \qquad \dot{\mathbf{T}}_{(g)} = \mathcal{A}_{(g)}\left[\frac{D_{(g)}}{Dt}\mathbf{S}\right].$$

From this, as well as from $(4.16)_2$, we obtain

$$(4.24) \qquad \frac{D_{(g)}}{Dt}\mathbf{S} = \mathcal{P}_{(g)}\left[\left(\mathcal{A}_{(g)}[\mathbf{S}]\right)^{\bullet}\right] = \mathcal{P}_{(g)}\left[\mathcal{A}_{(g)}[\dot{\mathbf{S}}] + \dot{\mathcal{A}}_{(g)}[\mathbf{S}]\right].$$

Inserting herein the relation

$$(4.25) \qquad \mathcal{P}_{(g)}\dot{\mathcal{A}}_{(g)} = -\dot{\mathcal{P}}_{(g)}\mathcal{A}_{(g)},$$

which follows from (4.20), and taking into account (4.20), we see that

$$(4.26) \qquad \frac{D_{(g)}}{Dt}\mathbf{S} = \dot{\mathbf{S}} - \dot{\mathcal{P}}_{(g)}\mathcal{A}_{(g)}[\mathbf{S}].$$

It is verified in Appendix A that the rate $D_{(g)}\mathbf{S}/Dt$ constitutes an objective Eulerian tensor.

Next, we discuss the requirement that the objective stress rate $D_{(g)}\mathbf{S}/Dt$ should fit into the structure of a generalized Oldroyd time derivative. We see, that this requirement implies a special structure of the fourth-order tensor $\mathcal{A}_{(g)}$, namely the property

$$(4.27) \qquad \mathcal{A}_{(g)}[\mathbf{S}] = \mathbf{\Psi}^T_{(g)}\mathbf{S}\mathbf{\Psi}_{(g)},$$

valid for all Eulerian second-order tensors $\mathbf{S}$, where $\mathbf{\Psi}_{(g)} \in \mathrm{Lin}^+$. Indeed, if this relation is true, $(4.16)_2$ reads

$$(4.28) \qquad \mathbf{T}_{(g)} = \mathcal{A}_{(g)}[\mathbf{S}] = \mathbf{\Psi}^T_{(g)}\mathbf{S}\mathbf{\Psi}_{(g)},$$

and $(4.23)_2$ implies

$$(4.29) \qquad \dot{\mathbf{T}}_{(g)} = \mathcal{A}_{(g)}\left[\frac{D_{(g)}}{Dt}\mathbf{S}\right] = \mathbf{\Psi}^T_{(g)}\left(\frac{D_{(g)}}{Dt}\mathbf{S}\right)\mathbf{\Psi}_{(g)},$$

with

$$(4.30) \qquad \frac{D_{(g)}}{Dt}\mathbf{S} = \dot{\mathbf{S}} + \left(\dot{\mathbf{\Psi}}_{(g)}\mathbf{\Psi}^{-1}_{(g)}\right)^T\mathbf{S} + \mathbf{S}\dot{\mathbf{\Psi}}_{(g)}\mathbf{\Psi}^{-1}_{(g)}.$$

Using the representation

$$(4.31) \qquad \mathbf{\Psi}_{(g)} = \Psi_{(g)ij}\boldsymbol{\mu}_i \otimes \mathbf{M}_j,$$

we conclude from (4.28) that

$$\mathcal{A}_{(g)} = \frac{1}{2}\Psi_{(g)pq}\Psi_{(g)mn}\left(\mathbf{M}_q \otimes \mathbf{M}_n \otimes \boldsymbol{\mu}_p \otimes \boldsymbol{\mu}_m + \mathbf{M}_q \otimes \mathbf{M}_n \otimes \boldsymbol{\mu}_m \otimes \boldsymbol{\mu}_p\right). \tag{4.32}$$

Comparing this with (4.18), (4.19) yields

$$\boldsymbol{\Psi}_{(g)} = \sum_{i=1}^{3} \Psi_{(g)ii}\,\boldsymbol{\mu}_i \otimes \mathbf{M}_i\,, \tag{4.33}$$

and therefore

$$\begin{aligned}\mathcal{A}_{(g)} = {} & \sum_{i=1}^{3}(\Psi_{(g)ii})^2\mathbf{M}_i \otimes \mathbf{M}_i \otimes \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_i \\ & + \frac{1}{2}\sum_{i\neq j}\Psi_{(g)ii}\Psi_{(g)jj}\left(\mathbf{M}_i \otimes \mathbf{M}_j \otimes \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_j + \mathbf{M}_i \otimes \mathbf{M}_j \otimes \boldsymbol{\mu}_j \otimes \boldsymbol{\mu}_i\right).\end{aligned} \tag{4.34}$$

Hence, through (4.34) and (4.18), (4.19), it follows (no summation over i, j)

$$(\Psi_{(g)ii})^2 = \frac{1}{\lambda_i g'(\lambda_i)}\,, \tag{4.35}$$

$$\Psi_{(g)ii}\Psi_{(g)jj} = \frac{1}{2}\,\frac{\lambda_i^2 - \lambda_j^2}{\lambda_i\lambda_j(g(\lambda_i) - g(\lambda_j))} \qquad (\lambda_i \neq \lambda_j, \quad i \neq j). \tag{4.36}$$

If $i \neq j$ and $\lambda_i = \lambda_j$, only (4.35) applies, so that it suffices to concentrate on $\lambda_i \neq \lambda_j$. Since $\{(\lambda_i^2 - \lambda_j^2)/(g(\lambda_i) - g(\lambda_j))\} > 0$, from (4.35), (4.36), we have

$$\frac{1}{\sqrt{\lambda_i\lambda_j g'(\lambda_i) g'(\lambda_j)}} = \frac{1}{2}\,\frac{\lambda_i^2 - \lambda_j^2}{\lambda_i\lambda_j(g(\lambda_i) - g(\lambda_j))}. \tag{4.37}$$

We recall that λ_i, being eigenvalues of the positive definite second-order tensor **U**, are positive. Thus, if (4.29) holds, the function $g(\cdot)$ has to satisfy the relation (4.37) for all positive λ_i, λ_j.

Now, suppose $g(\cdot)$ belonging to the one-parameter set of functions $g_{(m)}(\cdot)$, defined by (3.39). It is readily seen that for $m = 0$, equation (4.37) cannot be satisfied. For $m \neq 0$, on use of $(3.39)_1$, we obtain from (4.37), after some algebraic manipulations,

$$g_{(m)}(\lambda_i) - g_{(m)}(\lambda_j) = \frac{1}{2}\left(\lambda_i^{\frac{m}{2}+1}\lambda_j^{\frac{m}{2}-1} - \lambda_i^{\frac{m}{2}-1}\lambda_j^{\frac{m}{2}+1}\right). \tag{4.38}$$

On taking the derivative with respect to λ_i and then to λ_j, (4.38) reduces to

$$\left(\frac{m}{2} + 1\right)\left(\frac{m}{2} - 1\right)\left(\lambda_i^{\frac{m}{2}}\lambda_j^{\frac{m}{2}-2} - \lambda_i^{\frac{m}{2}-2}\lambda_j^{\frac{m}{2}}\right) = 0. \tag{4.39}$$

This formula must be satisfied for all $\lambda_i, \lambda_j > 0$ with $\lambda_i \neq \lambda_j$, which is possible if and only if

$$m = 2 \qquad \text{or} \qquad m = -2. \tag{4.40}$$

For $m = 2$, on the basis of $(3.40)_1$, we have

$$\mathbf{G}_{(2)} := \mathbf{E} = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{1}\right), \tag{4.41}$$

which is called the Green strain tensor, while for $m = -2$ we have

$$\mathbf{G}_{(-2)} := -\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{U}^{-2} - \mathbf{1}\right). \tag{4.42}$$

$\boldsymbol{\varepsilon}$ is called the Piola strain tensor. The corresponding conjugate stress tensors in the sense of Hill are given by

$$\mathbf{T}_{(2)} := \tilde{\mathbf{T}} = \mathbf{F}^{-1}\mathbf{S}\mathbf{F}^{T-1} \tag{4.43}$$

and

$$\mathbf{T}_{(-2)} := \overline{\mathbf{T}} = \mathbf{F}^{T}\mathbf{S}\mathbf{F}, \tag{4.44}$$

referred to as the second Piola - Kirchhoff stress tensor and the convected stress tensor, respectively. Clearly, along with $\mathbf{G}_{(-2)}$ and $\mathbf{T}_{(-2)}$, the variables $\boldsymbol{\varepsilon}$ and $-\overline{\mathbf{T}}$ form also a pair of conjugate (in Hill's sense) strain and stress tensors.

This general result suggests the following restriction on the choice of Lagrangean strain tensors: If we define the associated Lagrangean stress tensors which are conjugate in the sense of Hill, various objective time derivatives can be assigned to $\mathbf{S}$. If we require from these derivatives the structure of generalized Oldroyd time derivatives, then only two strain tensors are left, namely $\mathbf{E}$ and $\boldsymbol{\varepsilon}$. We remark that the Lagrangean variables $(\mathbf{E}, \tilde{\mathbf{T}})$ and $(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{\tau}})$, where

$$\tilde{\boldsymbol{\tau}} := -\overline{\mathbf{T}} = \mathbf{F}^{T}\boldsymbol{\varsigma}\mathbf{F} \tag{4.45}$$

and

$$\boldsymbol{\varsigma} := -\mathbf{S}, \tag{4.46}$$

were chosen in HAUPT and TSAKMAKIS [3] as basic pairs for introducing, by means of linear transformations, two different classes of pairs of spatial strain and stress tensors, referred to as family 1 and family 2, respectively. Strain and stress measures forming a pair belonging to one of the two classes were called dual variables. As it will be seen in what follows, the pairs of Lagrangean variables $(\mathbf{G}_{(m)}, \mathbf{T}_{(m)})$ if $m > 0$, or $(-\mathbf{G}_{(m)}, -\mathbf{T}_{(m)})$ if $m < 0$, are representatives of related classes, which can be interpreted as classes of generalized dual variables. Moreover, similar to the cases $m = \pm 2$, each of the Lagrangean stress tensors introduces a specific "generalized" Oldroyd time derivative for each of the stress tensors belonging to the same class. In fact, such a concept is established in the next section and essentially, it can be conceived as a generalization of the concept developed in HAUPT and TSAKMAKIS [3].

5. The concept of generalized dual variables

5.1. Generalized Lagrangean strain tensors

We remark that the set of strain tensors defined by (3.40) includes for each $m \neq 0$ the strain tensor $\mathbf{G}_{(m)}$ as well as its counterpart $\mathbf{G}_{(-m)}$. However, if $m = 0$, there is no such counterpart for $\ln \mathbf{U}$. This motivates the definition of a set of generalized Lagrangean strain tensors, slightly different from that introduced in (3.40), as follows.

The two-parameter set of functions

$$g_{(qm)}(\lambda_i) := \frac{1}{m}(\lambda_i^{qm} - 1), \tag{5.1}$$

where

$$q \in \{-1, 1\} \qquad \text{and} \qquad m \in (0, \infty), \tag{5.2}$$

introduces the strain tensors

$$\boldsymbol{\epsilon}_{(qm)} = \sum_{i=1}^{3} \frac{1}{m}(\lambda_i^{qm} - 1)\,\mathbf{M}_i \otimes \mathbf{M}_i\,. \tag{5.3}$$

Note that the functions $g_{(qm)}$, in contrast to (3.37), are monotonic but not necessarily increasing with

$$g_{(qm)}(1) = 0, \qquad |g'_{(qm)}(1)| = 1. \tag{5.4}$$

Since q is equal either to $+1$ or to -1, we have

$$\boldsymbol{\epsilon}_{(qm)}|_{q=-1} = \boldsymbol{\epsilon}_{(-m)}\,, \tag{5.5}$$

$$\boldsymbol{\epsilon}_{(qm)}|_{q=1} = \boldsymbol{\epsilon}_{(m)}\,. \tag{5.6}$$

Notice that, by taking the limit for $m \to 0$, we arrive at the strain tensors

$$\lim_{m\to 0} \boldsymbol{\epsilon}_{(qm)} = q \ln \mathbf{U}\,, \tag{5.7}$$

which is equivalent to

$$\lim_{m\to 0} \boldsymbol{\epsilon}_{(qm)} = \begin{cases} \ln \mathbf{U} & \text{if} \quad q = 1, \\ \ln \mathbf{U}^{-1} & \text{if} \quad q = -1. \end{cases} \tag{5.8}$$

We call the set of all strain tensors defined by (5.2), (5.3), together with the strain tensors $\ln \mathbf{U}$ and $\ln \mathbf{U}^{-1}$, the set of generalized Lagrangean strain (deformation) tensors and denote it by $\mathrm{D_L}$:

$$\mathrm{D_L} := \left\{\boldsymbol{\epsilon}_{(qm)} / q \in \{-1, 1\}, \;\; m > 0\right\} \cup \left\{\ln \mathbf{U}, \;\; \ln \mathbf{U}^{-1}\right\}. \tag{5.9}$$

In order to give a geometrical interpretation to the Lagrangean strain tensors included in D_L, it is convenient to introduce the generalized Green strain tensors $\mathbf{E}_{(m)}$, and the generalized Piola strain tensors $\boldsymbol{\epsilon}_{(m)}$, defined by

$$\mathbf{E}_{(m)} := \begin{cases} \boldsymbol{\epsilon}_{(m)} = \dfrac{1}{m}(\mathbf{U}^m - \mathbf{1}) & \text{if} \quad m > 0, \\ \ln \mathbf{U} & \text{if} \quad m = 0, \end{cases} \tag{5.10}$$

$$\boldsymbol{\varepsilon}_{(m)} := \begin{cases} \boldsymbol{\epsilon}_{(-m)} = \dfrac{1}{m}(\mathbf{U}^{-m} - \mathbf{1}) & \text{if} \quad m > 0, \\ \ln \mathbf{U}^{-1} & \text{if} \quad m = 0. \end{cases} \tag{5.11}$$

Further, we denote the set of all $\mathbf{E}_{(m)}$ by D_{LG} and the set of all $\boldsymbol{\varepsilon}_{(m)}$ by D_{LP},

$$D_{LG} := \left\{ \mathbf{E}_{(m)} / m \geq 0 \right\}, \tag{5.12}$$

$$D_{LP} := \left\{ \boldsymbol{\varepsilon}_{(m)} / m \geq 0 \right\}. \tag{5.13}$$

Clearly,

$$D_L = D_{LG} \cup D_{LP} \tag{5.14}$$

and

$$D_{LG} \cap D_{LP} = \emptyset\,. \tag{5.15}$$

Next, we give geometric interpretations for the Green strain tensor $\mathbf{E} \equiv \mathbf{E}_{(2)}$, defined by (4.41), and the Piola strain tensor $\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}_{(2)}$, defined by (4.42). As we shall see below, the geometric interpretation of the generalized strains $\mathbf{E}_{(m)}$ and $\boldsymbol{\varepsilon}_{(m)}$ is similar to that for $\mathbf{E}$ and $\boldsymbol{\varepsilon}$, respectively.

Let $d\mathbf{X}$ be a material line element in the reference configuration, which is transformed, under the deformation, into the material line element $d\mathbf{x}$ in the current configuration, i.e.,

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}\,. \tag{5.16}$$

Then we have the well-known formula

$$\Delta := \frac{1}{2}(d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X}) = d\mathbf{X} \cdot \mathbf{E} d\mathbf{X}\,. \tag{5.17}$$

To obtain a geometric interpretation for the Piola strain tensor $\boldsymbol{\varepsilon}$, we consider a material surface $\Phi(\mathbf{X}) = C = \text{const}$ in the reference configuration. In the current configuration this surface has the time-dependent form $\varphi(\mathbf{x}, t) = C$, where $\varphi(\overline{\mathbf{x}}(\mathbf{X}, t), t) = \Phi(\mathbf{X})$ holds for all $\mathbf{X}$ satisfying $\Phi(\mathbf{X}) = C$. It follows that

$$\boldsymbol{\xi} = \mathbf{F}^{T-1} \boldsymbol{\Xi}\,, \tag{5.18}$$

where

$$\boldsymbol{\xi} = \operatorname{grad} \varphi(\mathbf{x}, t), \tag{5.19}$$

$$\boldsymbol{\Xi} = \operatorname{GRAD} \Phi(\mathbf{X}) \tag{5.20}$$

and

$$\delta := \frac{1}{2}(\boldsymbol{\xi}\cdot\boldsymbol{\xi} - \boldsymbol{\Xi}\cdot\boldsymbol{\Xi}) = \boldsymbol{\Xi}\cdot\boldsymbol{\varepsilon}\boldsymbol{\Xi}. \tag{5.21}$$

Thus, the Green strain tensor $\mathbf{E}$ is used to refer to the reference configuration the difference Δ between material line elements in the current and the reference configuration. Analogously, one can make use of the Piola strain tensor $\boldsymbol{\varepsilon}$ in order to refer to the reference configuration the difference δ between normals to material surfaces in the current and the reference configuration.

Now, consider linear transformations described by $\mathbf{F}_{(m)}$, $\det \mathbf{F}_{(m)} > 0$, $m > 0$, where $\mathbf{F}_{(m)}$ is constructed as follows. From the polar decomposition theorem we have

$$\mathbf{F}_{(m)} = \mathbf{R}_{(m)}\mathbf{U}_{(m)} = \mathbf{V}_{(m)}\mathbf{R}_{(m)}, \tag{5.22}$$

$$\mathbf{U}^2_{(m)} = \mathbf{C}_{(m)} = \mathbf{F}^T_{(m)}\mathbf{F}_{(m)}, \tag{5.23}$$

$$\mathbf{V}^2_{(m)} = \mathbf{B}_{(m)} = \mathbf{F}_{(m)}\mathbf{F}^T_{(m)}, \tag{5.24}$$

where $\mathbf{R}_{(m)}$ denotes a proper orthogonal tensor. If we define

$$\mathbf{U}_{(m)} := \mathbf{U}^{m/2} = \sum_{i=1}^{3} \lambda_i^{m/2}\mathbf{M}_i \otimes \mathbf{M}_i, \tag{5.25}$$

($m > 0$), then it follows that $\mathbf{U}_{(m)}$ describes a class of right stretch tensors. Furthermore, defining $\mathbf{R}_{(2)} = \mathbf{R}$, we have $\mathbf{F}_{(2)} = \mathbf{F}$. Clearly $\mathbf{F}$, and so $\mathbf{U}$, must satisfy the compatibility conditions ([9]) in order to form a deformation gradient tensor derived from a deformation function. Although $\mathbf{U}$ and $\mathbf{F}$ satisfy the appropriate compatibility conditions, $\mathbf{U}_{(m)}$ and $\mathbf{F}_{(m)}$ in general do not.

Proceeding to complete the definition of $\mathbf{F}_{(m)}$, we note that all $\mathbf{U}_{(m)}$ possess the same principal vectors $\mathbf{M}_i$. This motivates to define all the corresponding left stretch tensors $\mathbf{V}_{(m)}$ to have the same principal vectors. Since the principal vectors of $\mathbf{V}_{(2)} = \mathbf{V}$ are $\boldsymbol{\mu}_{(i)}$ (see Eqs. (3.7), (3.8)), we have

$$\mathbf{V}_{(m)} := \mathbf{V}^{m/2} = \sum_{i=1}^{3} \lambda_i^{m/2}\boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_i \tag{5.26}$$

and

$$\mathbf{R}_{(m)} := \mathbf{R}. \tag{5.27}$$

Notice that $\mathbf{F}_{(m)}$ can be interpreted as a two-point tensor field which maps tangent spaces of material points in the reference configuration onto the corresponding tangent spaces of the same material points in configurations at time t. This fact

([9]) A detailed discussion on the compatibility conditions concerning $\mathbf{F}$, as well as $\mathbf{U}$ and $\mathbf{R}$, is given by NAGHDI and VONGSARNPIGOON [14].

follows by virtue of **U** and **V** (and therefore $\mathbf{U}_{(m)}$ and $\mathbf{V}_{(m)}$ too) operating within the tangent spaces of material points in the reference configuration and tangent spaces of material points in configurations at time t, respectively, and **R** mapping tangent spaces of material points in the reference configuration to the corresponding tangent spaces of the same material points in configurations at time t (see e.g. MARSDEN and HUGHES [15, pp. 51–52]).

Thus, analogous to **F** in (5.16), $\mathbf{F}_{(m)}$ transforms line elements $d\mathbf{X}$ in the reference configuration to vectors $d\mathbf{x}_{(m)}$ in configurations at time t:

$$d\mathbf{x}_{(m)} = \mathbf{F}_{(m)}\, d\mathbf{X}. \tag{5.28}$$

If we define

$$\Delta_{(m)} := \frac{1}{m}\left(d\mathbf{x}_{(m)} \cdot d\mathbf{x}_{(m)} - d\mathbf{X} \cdot d\mathbf{X}\right), \tag{5.29}$$

then we have, in view of the transformation rule (5.28), as well as the relations $(5.22)_1$, $(5.10)_1$ and (5.25),

$$\Delta_{(m)} := \frac{1}{m} d\mathbf{X} \cdot (\mathbf{U}^m - \mathbf{1})\, d\mathbf{X} = d\mathbf{X} \cdot \mathbf{E}_{(m)}\mathbf{X}, \tag{5.30}$$

with the property $d\mathbf{x}_{(2)} \equiv d\mathbf{x}$ and $\Delta_{(2)} \equiv \Delta$. On the other hand, $\mathbf{F}_{(m)}$ can be interpreted to transform normals $\boldsymbol{\Xi}$ on material surfaces in the reference configuration to vectors $\boldsymbol{\xi}_{(m)}$,

$$\boldsymbol{\xi}_{(m)} = \mathbf{F}_{(m)}^{T-1}\boldsymbol{\Xi}, \tag{5.31}$$

in configurations at time t, which generalizes the transformation formula (5.18). On defining

$$\delta_{(m)} := \frac{1}{m}\left(\boldsymbol{\xi}_{(m)} \cdot \boldsymbol{\xi}_{(m)} - \boldsymbol{\Xi} \cdot \boldsymbol{\Xi}\right), \tag{5.32}$$

we obtain, by virtue of $(5.22)_1$, (5.11) and (5.25),

$$\delta_{(m)} := \frac{1}{m}\boldsymbol{\Xi} \cdot (\mathbf{U}^{-m} - \mathbf{1})\boldsymbol{\Xi} = \boldsymbol{\Xi} \cdot \boldsymbol{\varepsilon}_{(m)}\boldsymbol{\Xi}. \tag{5.33}$$

Obviously, we have $\boldsymbol{\xi}_{(2)} \equiv \boldsymbol{\xi}$ and $\delta_{(2)} \equiv \delta$. This completes the geometrical interpretation of $\mathbf{E}_{(m)}$ and $\boldsymbol{\varepsilon}_{(m)}$. For arbitrary $m > 0$, these strain tensors represent the differences $\Delta_{(m)}$ and $\delta_{(m)}$ with respect to the reference configuration. We may extend the result to the limit case $m = 0$, by defining

$$\Delta_{(0)} := \lim_{m \to 0} \Delta_{(m)} = d\mathbf{X} \cdot \mathbf{E}_{(0)}\mathbf{X} \tag{5.34}$$

and

$$\delta_{(0)} := \lim_{m \to 0} \delta_{(m)} = \boldsymbol{\Xi} \cdot \boldsymbol{\varepsilon}_{(0)}\boldsymbol{\Xi}. \tag{5.35}$$

5.2. Generalized strain tensors and associated rates

Let $d\mathbf{X}$ and $\boldsymbol{\Xi}$ be material line elements and normals on material surfaces in the reference configuration, which are mapped by the linear transformation $\boldsymbol{\Psi}$, to vector fields $d\mathbf{x}^{(\Psi)}$ and $\boldsymbol{\Xi}^{(\Psi)}$, respectively (cf. (5.16) and (5.18) in Sec. 5.1):

$$d\mathbf{x}^{(\Psi)} := \boldsymbol{\Psi} d\mathbf{X}, \tag{5.36}$$

$$\boldsymbol{\xi}^{(\Psi)} := \boldsymbol{\Psi}^{T-1}\boldsymbol{\Xi}. \tag{5.37}$$

Next, consider for arbitrary but fixed $\boldsymbol{\Psi} \in \mathrm{Lin}^+$ and $m \geq 0$, the differences $\Delta_{(m)}$ and $\delta_{(m)}$. Requiring the derivatives $d^n\Delta_{(m)}/dt^n$ and $d^n\delta_{(m)}/dt^n$ ($n \in \mathbb{N}$, $n \geq 0$) to be form-invariant with respect to the chosen configuration, various symmetric strain tensors $\boldsymbol{\Pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\pi}^{(\Psi)}_{(m)}$, as well as the associated time derivatives (rates) $\overset{\triangle}{D}{}^n\boldsymbol{\Pi}^{(\Psi)}_{(m)}/Dt^n$ and $\overset{\triangle}{D}{}^n\boldsymbol{\pi}^{(\Psi)}_{(m)}/Dt^n$ can be defined ([10]):

$$\frac{d^n}{dt^n}\Delta_{(m)} = d\mathbf{x}^{(\Psi)} \cdot \left(\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\Pi}^{(\Psi)}_{(m)}\right) d\mathbf{x}^{(\Psi)}, \tag{5.38}$$

$$\frac{d^n}{dt^n}\delta_{(m)} = \boldsymbol{\xi}^{(\Psi)} \cdot \left(\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\pi}^{(\Psi)}_{(m)}\right) \boldsymbol{\xi}^{(\Psi)}. \tag{5.39}$$

These definitions imply

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\mathbf{E}_{(m)} \equiv \frac{d^n}{dt^n}\mathbf{E}_{(m)}, \tag{5.40}$$

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\varepsilon}_{(m)} \equiv \frac{d^n}{dt^n}\boldsymbol{\varepsilon}_{(m)}, \tag{5.41}$$

as well as

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\Pi}^{(\Psi)}_{(m)} = \mathcal{L}_{(\Psi)}\left[\frac{d^n}{dt^n}\mathbf{E}_{(m)}\right], \tag{5.42}$$

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\pi}^{(\Psi)}_{(m)} = \mathcal{M}_{(\Psi)}\left[\frac{d^n}{dt^n}\boldsymbol{\varepsilon}_{(m)}\right], \tag{5.43}$$

where $\mathcal{L}_{(\Psi)}$ and $\mathcal{M}_{(\Psi)}$ are fourth-order tensors operating on the set of all Lagrangean symmetric second-order tensors $\widetilde{\mathbf{S}}$:

$$\mathcal{L}_{(\Psi)}: \quad \widetilde{\mathbf{S}} \longmapsto \mathcal{L}_{(\Psi)}[\widetilde{\mathbf{S}}] = \boldsymbol{\Psi}^{T-1}\widetilde{\mathbf{S}}\boldsymbol{\Psi}^{-1}, \tag{5.44}$$

$$\mathcal{M}_{(\Psi)}: \quad \widetilde{\mathbf{S}} \longmapsto \mathcal{M}_{(\Psi)}[\widetilde{\mathbf{S}}] = \boldsymbol{\Psi}\widetilde{\mathbf{S}}\boldsymbol{\Psi}^{T}. \tag{5.45}$$

([10]) Here, and in what follows, symbol $\triangle$ denotes the associated time derivative for the strain tensor considered. In other words, $\triangle$ defines different time derivatives depending on the kind of the strain tensor considered.

In particular, for $n = 0$ we have

(5.46) $$\mathbf{\Pi}^{(\Psi)}_{(m)} = \mathcal{L}_{(\Psi)}[\mathbf{E}_{(m)}] = \mathbf{\Psi}^{T-1}\mathbf{E}_{(m)}\mathbf{\Psi}^{-1},$$

(5.47) $$\boldsymbol{\pi}^{(\Psi)}_{(m)} = \mathcal{M}_{(\Psi)}[\boldsymbol{\varepsilon}_{(m)}] = \mathbf{\Psi}\boldsymbol{\varepsilon}_{(m)}\mathbf{\Psi}^{T}.$$

It is readily seen that

(5.48) $$\begin{aligned} \overset{\triangle}{\mathbf{\Pi}}{}^{(\Psi)}_{(m)} &:= \frac{\overset{\triangle}{D}}{Dt}\mathbf{\Pi}^{(\Psi)}_{(m)} = \dot{\mathbf{\Pi}}{}^{(\Psi)}_{(m)} + (\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1})^T\mathbf{\Pi}^{(\Psi)}_{(m)} + \mathbf{\Pi}^{(\Psi)}_{(m)}\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1}, \\ \overset{\triangle\triangle}{\mathbf{\Pi}}{}^{(\Psi)}_{(m)} &:= \frac{\overset{\triangle}{D}{}^2}{Dt^2}\mathbf{\Pi}^{(\Psi)}_{(m)} = \left(\overset{\triangle}{\mathbf{\Pi}}{}^{(\Psi)}_{(m)}\right)^{\bullet} + (\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1})^T\,\overset{\triangle}{\mathbf{\Pi}}{}^{(\Psi)}_{(m)} + \overset{\triangle}{\mathbf{\Pi}}{}^{(\Psi)}_{(m)}\,\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1}, \\ &\;\;\vdots \end{aligned}$$

as well as

(5.49) $$\begin{aligned} \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)} &:= \frac{\overset{\triangle}{D}}{Dt}\boldsymbol{\pi}^{(\Psi)}_{(m)} = \dot{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)} - \dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1}\boldsymbol{\pi}^{(\Psi)}_{(m)} - \boldsymbol{\pi}^{(\Psi)}_{(m)}(\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1})^T, \\ \overset{\triangle\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)} &:= \frac{\overset{\triangle}{D}{}^2}{Dt^2}\boldsymbol{\pi}^{(\Psi)}_{(m)} = \left(\overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)}\right)^{\bullet} - \dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1}\,\overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)} - \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)}\,(\dot{\mathbf{\Psi}}\mathbf{\Psi}^{-1})^T, \\ &\;\;\vdots \end{aligned}$$

Further relations are obtained if we represent the various strain tensors with respect to the bases $\{\mathbf{g}^{(\Psi)}_k\}$ and $\{\mathbf{g}^{(\Psi)}_k\}$. From

(5.50) $$\mathbf{E}_{(m)} = E_{(m)kl}\mathbf{G}^k \otimes \mathbf{G}^l,$$

(5.51) $$\mathbf{\Pi}^{(\Psi)}_{(m)} = \Pi^{(\Psi)}_{(m)kl}\mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l},$$

as well as

(5.52) $$\boldsymbol{\varepsilon}_{(m)} = \varepsilon^{kl}_{(m)}\mathbf{G}_k \otimes \mathbf{G}_l,$$

(5.53) $$\boldsymbol{\pi}^{(\Psi)}_{(m)} = \pi^{(\Psi)kl}_{(m)}\mathbf{g}^{(\Psi)}_k \otimes \mathbf{g}^{(\Psi)}_l,$$

we infer that

(5.54) $$E_{(m)kl} = \Pi^{(\Psi)}_{(m)kl}$$

and

(5.55) $$\varepsilon^{kl}_{(m)} = \pi^{(\Psi)kl}_{(m)},$$

respectively. In addition, it holds

$$\text{(5.56)} \qquad \frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\Pi}^{(\Psi)}_{(m)} = \left(\frac{d^n}{dt^n}\Pi^{(\Psi)}_{(m)kl}\right)\mathbf{g}^{(\Psi)k}\otimes\mathbf{g}^{(\Psi)l}$$

and

$$\text{(5.57)} \qquad \frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\pi}^{(\Psi)}_{(m)} = \left(\frac{d^n}{dt^n}\pi^{(\Psi)kl}_{(m)}\right)\mathbf{g}^{(\Psi)}_k\otimes\mathbf{g}^{(\Psi)}_l\,,$$

which indicates, that the operators $\overset{\triangle}{D}{}^n(\cdot)/Dt^n$ induce generalized Oldroyd time derivatives. We call the strain tensors $\boldsymbol{\Pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\pi}^{(\Psi)}_{(m)}$ generalized strain tensors. The set of all generalized strain tensors is denoted by D,

$$\text{(5.58)} \qquad \mathrm{D} := \left\{\boldsymbol{\Pi}^{(\Psi)}_{(m)}, \boldsymbol{\pi}^{(\Psi)}_{(m)} \;/\; m \geq 0, \quad \boldsymbol{\Psi}\in\mathrm{Lin}^+\right\}.$$

Obviously, for arbitrary but fixed $m \geq 0$, the sets of all generalized strain tensors related to the differences $\Delta_{(m)}$ and $\delta_{(m)}$ constitute equivalence classes in D. We denote these equivalence classes by $\Theta^{(\Pi)}_{(m)}$ and $\Theta^{(\pi)}_{(m)}$, respectively,

$$\text{(5.59)} \qquad \Theta^{(\Pi)}_{(m)} := \left\{\boldsymbol{\Pi}^{(\Psi)}_{(m)} \;/\; \boldsymbol{\Psi}\in\mathrm{Lin}^+\right\},$$

$$\text{(5.60)} \qquad \Theta^{(\pi)}_{(m)} := \left\{\boldsymbol{\pi}^{(\Psi)}_{(m)} \;/\; \boldsymbol{\Psi}\in\mathrm{Lin}^+\right\}.$$

Then, for the system Ω_{D} of all equivalence classes in D,

$$\text{(5.61)} \qquad \Omega_{\mathrm{D}} := \left\{\Theta^{(\Pi)}_{(m)}, \Theta^{(\pi)}_{(m)} \;/\; m \geq 0\right\},$$

the equality holds

$$\text{(5.62)} \qquad \mathrm{D} = \bigcup_{\Theta\in\Omega_{\mathrm{D}}} \Theta\,.$$

5.3. Generalized stress tensors and associated rates

For defining the generalized strain tensors and their associated rates, use is made of the scalar quantities $d^n\Delta_{(m)}/dt^n$ and $d^n\delta_{(m)}/dt^n$. These scalars were required to be form-invariant with respect to the chosen configuration. In the following we consider the stress power as well as the material time derivatives d^nW/dt^n and require from these scalar quantities to be form-invariant with respect to the chosen configuration. This leads to the introduction of generalized stress tensors and the associated rates.

Proceeding to define generalized stress tensors, we draw attention to symmetric stress tensors only and assign to the generalized Lagrangean strain tensors

$\mathbf{E}_{(m)}$ and $\boldsymbol{\varepsilon}_{(m)}$, the symmetric generalized Lagrangean stress tensors $\widetilde{\mathbf{T}}_{(m)}$ and $\widetilde{\boldsymbol{\tau}}_{(m)}$, respectively, so that

$$W = \widetilde{\mathbf{T}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} = \widetilde{\boldsymbol{\tau}}_{(m)} \cdot \dot{\boldsymbol{\varepsilon}}_{(m)}, \tag{5.63}$$

for each $m \geq 0$. The set of all generalized Lagrangean stress tensors is denoted by $\mathrm{S_L}$

$$\mathrm{S_L} := \Big\{\widetilde{\mathbf{T}}_{(m)}, \widetilde{\boldsymbol{\tau}}_{(m)} \;/\; \widetilde{\mathbf{T}}_{(m)}, \widetilde{\boldsymbol{\tau}}_{(m)} : \quad \text{symmetric}, \quad W = \widetilde{\mathbf{T}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} = \widetilde{\boldsymbol{\tau}}_{(m)} \cdot \boldsymbol{\varepsilon}_{(m)}, \quad m \geq 0\Big\}. \tag{5.64}$$

The set $\mathrm{S_L}$ for the stress tensors is the counterpart of the set $\mathrm{D_L}$ for the strain tensors, while the sets

$$\Big\{\widetilde{\mathbf{T}}_{(m)} \;/\; m \geq 0\Big\} \tag{5.65}$$

and

$$\Big\{\widetilde{\boldsymbol{\tau}}_{(m)} \;/\; m \geq 0\Big\}, \tag{5.66}$$

are the counterparts of the sets $\mathrm{D_{LG}}$ and $\mathrm{D_{LP}}$, respectively. Moreover, to the generalized strain tensors $\boldsymbol{\Pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\pi}^{(\Psi)}_{(m)}$ the symmetric generalized stress tensors

$$\boldsymbol{\Sigma}^{(\Psi)}_{(m)} = \mathcal{M}_{(\Psi)}[\widetilde{\mathbf{T}}_{(m)}] = \boldsymbol{\Psi}\widetilde{\mathbf{T}}_{(m)}\boldsymbol{\Psi}^T \tag{5.67}$$

and

$$\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \mathcal{L}_{(\Psi)}[\widetilde{\boldsymbol{\tau}}_{(m)}] = \boldsymbol{\Psi}^{T-1}\widetilde{\boldsymbol{\tau}}_{(m)}\boldsymbol{\Psi}^{-1}, \tag{5.68}$$

can be assigned, respectively, so that

$$W = \boldsymbol{\Sigma}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\Pi}}{}^{(\Psi)}_{(m)} = \boldsymbol{\sigma}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)}, \tag{5.69}$$

for each $m \geq 0$ and $\boldsymbol{\Psi} \in \mathrm{Lin}^+$.

Notice that $(\mathbf{E}_{(m)}, \widetilde{\mathbf{T}}_{(m)})$, as well as $(\boldsymbol{\varepsilon}_{(m)}, \widetilde{\boldsymbol{\tau}}_{(m)})$, are pairs of variables which are conjugate in the sense of Hill. However, this is in general not true for the pairs of variables

$$\Big(\boldsymbol{\Pi}^{(\Psi)}_{(m)}, \boldsymbol{\Sigma}^{(\Psi)}_{(m)}\Big) \qquad \text{and} \qquad \Big(\boldsymbol{\pi}^{(\Psi)}_{(m)}, \boldsymbol{\sigma}^{(\Psi)}_{(m)}\Big). \tag{5.70}$$

For arbitrary $m \geq 0$ and $\boldsymbol{\Psi} \in \mathrm{Lin}^+$, the pairs of variables (5.70) are called *pairs of generalized dual variables*, or simply *dual variables*. Equivalently, the generalized stress tensors $\boldsymbol{\Sigma}^{(\Psi)}_{(m)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(m)}$ are said to be dual to the generalized strain tensors $\boldsymbol{\Pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\pi}^{(\Psi)}_{(m)}$, respectively, and *vice versa* ([11]).

([11]) This notation of generalized dual variables is just a generalization of the duality notation introduced in HAUPT and TSAKMAKIS [3].

If we write S for the set of all stress tensors $\boldsymbol{\Sigma}^{(\Psi)}_{(m)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(m)}$,

$$\text{(5.71)} \qquad \mathrm{S} := \left\{ \boldsymbol{\Sigma}^{(\Psi)}_{(m)}, \boldsymbol{\sigma}^{(\Psi)}_{(m)} \; / \; m \geq 0, \quad \boldsymbol{\Psi} \in \mathrm{Lin}^+ \right\}$$

then S can be partitioned into the equivalent classes

$$\text{(5.72)} \qquad \Theta^{(\Sigma)}_{(m)} := \left\{ \boldsymbol{\Sigma}^{(\Psi)}_{(m)} \; / \; \boldsymbol{\Psi} \in \mathrm{Lin}^+ \right\}$$

and

$$\text{(5.73)} \qquad \Theta^{(\sigma)}_{(m)} := \left\{ \boldsymbol{\sigma}^{(\Psi)}_{(m)} \; / \; \boldsymbol{\Psi} \in \mathrm{Lin}^+ \right\},$$

which for $m \geq 0$ cover S. Note that the counterpart of the sets D, $\Theta^{(\Pi)}_{(m)}$ and $\Theta^{(\pi)}_{(m)}$ for the strain tensors are the sets S, $\Theta^{(\Sigma)}_{(m)}$ and $\Theta^{(\sigma)}_{(m)}$ for the stress tensors, respectively.

To determine the time derivatives which are associated with the generalized stress tensors $\boldsymbol{\Sigma}^{(\Psi)}_{(m)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(m)}$, we next consider the quantity $\dot{W}$, which like W is required to be form-invariant with respect to the chosen configuration. On taking the material time derivative of (5.63), we obtain

$$\text{(5.74)} \qquad \begin{aligned} \dot{W} &= \dot{\tilde{\mathbf{T}}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} + \tilde{\mathbf{T}}_{(m)} \cdot \ddot{\mathbf{E}}_{(m)} \\ &= \dot{\tilde{\boldsymbol{\tau}}}_{(m)} \cdot \dot{\boldsymbol{\varepsilon}}_{(m)} + \tilde{\boldsymbol{\tau}}_{(m)} \cdot \ddot{\boldsymbol{\varepsilon}}_{(m)} \; . \end{aligned}$$

Using the stress and strain tensors included in the equivalence classes $\Theta^{(\Pi)}_{(m)}$, $\Theta^{(\Sigma)}_{(m)}$ and $\Theta^{(\pi)}_{(m)}$, $\Theta^{(\sigma)}_{(m)}$, respectively, as well as the associated strain rates defined by (5.48) and (5.49), the terms $\tilde{\mathbf{T}}_{(m)} \cdot \ddot{\mathbf{E}}_{(m)}$ and $\tilde{\boldsymbol{\tau}}_{(m)} \cdot \ddot{\boldsymbol{\varepsilon}}_{(m)}$ can be rewritten in the form

$$\text{(5.75)} \qquad \tilde{\mathbf{T}}_{(m)} \cdot \ddot{\mathbf{E}}_{(m)} = \boldsymbol{\Sigma}^{(\Psi)}_{(m)} \cdot \overset{\triangle\triangle}{\boldsymbol{\Pi}}{}^{(\Psi)}_{(m)},$$

$$\text{(5.76)} \qquad \tilde{\boldsymbol{\tau}}_{(m)} \cdot \ddot{\boldsymbol{\varepsilon}}_{(m)} = \boldsymbol{\sigma}^{(\Psi)}_{(m)} \cdot \overset{\triangle\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)}.$$

Thus, the quantities $\tilde{\mathbf{T}}_{(m)} \cdot \ddot{\mathbf{E}}_{(m)}$ and $\tilde{\boldsymbol{\tau}}_{(m)} \cdot \ddot{\boldsymbol{\varepsilon}}_{(m)}$ represent, for arbitrary but fixed $m \geq 0$, scalars which are form-invariant with respect to the chosen configuration. Consequently, the terms

$$\text{(5.77)} \qquad W^{\mathrm{incr}}_{(m)} := \dot{\tilde{\mathbf{T}}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)}$$

and

$$\text{(5.78)} \qquad w^{\mathrm{incr}}_{(m)} := \dot{\tilde{\boldsymbol{\tau}}}_{(m)} \cdot \dot{\boldsymbol{\varepsilon}}_{(m)},$$

which are called the incremental stress powers $W^{\text{incr}}_{(m)}$ and $w^{\text{incr}}_{(m)}$, respectively, must also be scalars which are form-invariant with respect to the chosen configuration. Indeed, we have

$$W^{\text{incr}}_{(m)} = \overset{\triangledown}{\boldsymbol{\Sigma}}{}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\Pi}}{}^{(\Psi)}_{(m)} \tag{5.79}$$

and

$$w^{\text{incr}}_{(m)} = \overset{\triangledown}{\boldsymbol{\sigma}}{}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)}, \tag{5.80}$$

where use is made of the definitions ([12])

$$\begin{aligned}\overset{\triangledown}{\boldsymbol{\Sigma}}{}^{(\Psi)}_{(m)} := \frac{\overset{\triangledown}{D}}{Dt}\boldsymbol{\Sigma}^{(\Psi)}_{(m)} &= \boldsymbol{\Psi}\left(\boldsymbol{\Psi}^{-1}\boldsymbol{\Sigma}^{(\Psi)}_{(m)}\boldsymbol{\Psi}^{T-1}\right)^{\bullet}\boldsymbol{\Psi}^{T} \\ &= \dot{\boldsymbol{\Sigma}}{}^{(\Psi)}_{(m)} - \dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1}\boldsymbol{\Sigma}^{(\Psi)}_{(m)} - \boldsymbol{\Sigma}^{(\Psi)}_{(m)}(\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1})^{T},\end{aligned} \tag{5.81}$$

$$\begin{aligned}\overset{\triangledown}{\boldsymbol{\sigma}}{}^{(\Psi)}_{(m)} := \frac{\overset{\triangledown}{D}}{Dt}\boldsymbol{\sigma}^{(\Psi)}_{(m)} &= \boldsymbol{\Psi}^{T-1}\left(\boldsymbol{\Psi}^{T}\boldsymbol{\sigma}^{(\Psi)}_{(m)}\boldsymbol{\Psi}\right)^{\bullet}\boldsymbol{\Psi}^{-1} \\ &= \dot{\boldsymbol{\sigma}}{}^{(\Psi)}_{(m)} + (\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1})^{T}\boldsymbol{\sigma}^{(\Psi)}_{(m)} + \boldsymbol{\Sigma}^{(\Psi)}_{(m)}\dot{\boldsymbol{\Psi}}\boldsymbol{\Psi}^{-1}.\end{aligned} \tag{5.82}$$

This way, by considering form-invariant scalar quantities, we can associate with each stress tensor $\boldsymbol{\Sigma}^{(\Psi)}_{(m)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(m)}$ a time derivative of the form (5.81) and (5.82), respectively. Similarly, by considering higher time derivatives d^nW/dt^n, associated time derivatives of higher order $\overset{\triangledown}{D}{}^n\, \boldsymbol{\Sigma}^{(\Psi)}_{(m)}/Dt^n$ and $\overset{\triangledown}{D}{}^n\, \boldsymbol{\sigma}^{(\Psi)}_{(m)}/Dt^n$ can be introduced in a natural way. In particular, we have

$$\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\tilde{\mathbf{T}}_{(m)} \equiv \frac{d^n}{dt^n}\tilde{\mathbf{T}}_{(m)} \tag{5.83}$$

and

$$\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\tilde{\boldsymbol{\tau}}_{(m)} \equiv \frac{d^n}{dt^n}\tilde{\boldsymbol{\tau}}_{(m)}, \tag{5.84}$$

as well as

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\Sigma}^{(\Psi)}_{(m)} = \mathcal{M}_{(\psi)}\left[\frac{d^n}{dt^n}\tilde{\mathbf{T}}_{(m)}\right] = \boldsymbol{\Psi}^{T-1}\left(\frac{d^n}{dt^n}\tilde{\mathbf{T}}_{(m)}\right)\boldsymbol{\Psi}^{-1} \tag{5.85}$$

and

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \mathcal{L}_{(\Psi)}\left[\frac{d^n}{dt^n}\tilde{\boldsymbol{\tau}}_{(m)}\right] = \boldsymbol{\Psi}\left(\frac{d^n}{dt^n}\tilde{\boldsymbol{\tau}}_{(m)}\right)\boldsymbol{\Psi}^{T}. \tag{5.86}$$

([12]) Similar to the notation of the symbol $\triangle$ for the strain tensors (see footnote 11), symbol $\triangledown$ denotes the associated time derivative for the stress tensor considered.

5.4. Properties of dual variables

Using the bases $\{\mathbf{G}_k\}$, $\{\mathbf{g}_k^{(\Psi)}\}$ and their reciprocal bases $\{\mathbf{G}^k\}$, $\{\mathbf{g}^{(\Psi)k}\}$, as well as the representations

$$\widetilde{\mathbf{T}}_{(m)} = \widetilde{T}^{kl}_{(m)}\mathbf{G}_k \otimes \mathbf{G}_l\,, \tag{5.87}$$

$$\mathbf{\Sigma}^{(\Psi)}_{(m)} = \Sigma^{(\Psi)kl}_{(m)}\mathbf{g}^{(\Psi)}_k \otimes \mathbf{g}^{(\Psi)}_l\,, \tag{5.88}$$

and

$$\widetilde{\boldsymbol{\tau}}_{(m)} = \widetilde{\tau}_{(m)kl}\mathbf{G}^k \otimes \mathbf{G}^l\,, \tag{5.89}$$

$$\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \sigma^{(\Psi)}_{(m)kl}\mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}\,, \tag{5.90}$$

we readily obtain

$$\widetilde{T}^{kl}_{(m)} = \Sigma^{(\Psi)kl}_{(m)}\,, \tag{5.91}$$

$$\widetilde{\tau}_{(m)kl} = \sigma^{(\Psi)}_{(m)kl} \tag{5.92}$$

and

$$\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\mathbf{\Sigma}^{(\Psi)}_{(m)} = \left(\frac{d^n}{dt^n}\Sigma^{(\Psi)kl}_{(m)}\right)\mathbf{g}^{(\Psi)}_k \otimes \mathbf{g}^{(\Psi)}_l\,, \tag{5.93}$$

$$\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \left(\frac{d^n}{dt^n}\sigma^{(\Psi)}_{(m)kl}\right)\mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}\,. \tag{5.94}$$

The relations (5.93), (5.94) together with (5.85) and (5.86) indicate that, similarly to the case of the generalized strain tensors, the operators $\overset{\triangledown}{D}{}^n(\cdot)/Dt^n$ induce generalized Oldroyd time derivatives.

We now compare the relations (5.56), (5.57), which concern the generalized strain tensors, and the relations (5.93), (5.94), which concern the generalized stress tensors. It turns out that $\mathbf{\Pi}^{(\Psi)}_{(m)}$ and $\mathbf{\Sigma}^{(\Psi)}_{(m)}$ or $\boldsymbol{\pi}^{(\Psi)}_{(m)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(m)}$, as well as the associated time derivatives, display their physical and geometrical properties in the context of a representation relative to a basis and the corresponding reciprocal (dual) basis, respectively. Moreover, the duality concept can also be verified by means of the following scalar products, which are form-invariant with respect to the chosen configuration:

$$I^{NM}_{(m)} := \left(\frac{\overset{\triangledown}{D}{}^N}{Dt^N}\mathbf{\Sigma}^{(\Psi)}_{(m)}\right)\cdot\left(\frac{\overset{\triangle}{D}{}^M}{Dt^M}\mathbf{\Pi}^{(\Psi)}_{(m)}\right), \tag{5.95}$$

$$i^{NM}_{(m)} := \left(\frac{\overset{\triangledown}{D}{}^N}{Dt^N}\boldsymbol{\sigma}^{(\Psi)}_{(m)}\right)\cdot\left(\frac{\overset{\triangle}{D}{}^M}{Dt^M}\boldsymbol{\pi}^{(\Psi)}_{(m)}\right), \tag{5.96}$$

where $m \geq 0$, and $N, M \in \mathbb{N}$ with $N, M \geq 0$. Some particular cases of (5.95) and (5.96) are:

1

$$I^{00}_{(m)} = \widetilde{\mathbf{T}}_{(m)} \cdot \mathbf{E}_{(m)} = \mathbf{\Sigma}^{(\Psi)}_{(m)} \cdot \mathbf{\Pi}^{(\Psi)}_{(m)}, \tag{5.97}$$

$$i^{00}_{(m)} = \widetilde{\boldsymbol{\tau}}_{(m)} \cdot \boldsymbol{\varepsilon}_{(m)} = \boldsymbol{\sigma}^{(\Psi)}_{(m)} \cdot \boldsymbol{\pi}^{(\Psi)}_{(m)}. \tag{5.98}$$

(Scalar product of dual stress and strain tensors).

2

$$I^{01}_{(m)} \equiv W = \widetilde{\mathbf{T}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} = \mathbf{\Sigma}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\mathbf{\Pi}}{}^{(\Psi)}_{(m)}, \tag{5.99}$$

$$i^{01}_{(m)} \equiv W = \widetilde{\boldsymbol{\tau}}_{(m)} \cdot \dot{\boldsymbol{\varepsilon}}_{(m)} = \boldsymbol{\sigma}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)}. \tag{5.100}$$

(Stress power per unit volume of the reference configuration).

3

$$I^{10}_{(m)} = \dot{\widetilde{\mathbf{T}}}_{(m)} \cdot \mathbf{E}_{(m)} = \overset{\triangledown}{\mathbf{\Sigma}}{}^{(\Psi)}_{(m)} \cdot \mathbf{\Pi}^{(\Psi)}_{(m)}, \tag{5.101}$$

$$i^{10}_{(m)} = \dot{\widetilde{\boldsymbol{\tau}}}_{(m)} \cdot \boldsymbol{\varepsilon}_{(m)} = \overset{\triangledown}{\boldsymbol{\sigma}}{}^{(\Psi)}_{(m)} \cdot \boldsymbol{\pi}^{(\Psi)}_{(m)}. \tag{5.102}$$

(Complementary stress powers).

4

$$I^{11}_{(m)} \equiv W^{\mathrm{incr}}_{(m)} = \dot{\widetilde{\mathbf{T}}}_{(m)} \cdot \dot{\mathbf{E}}_{(m)} = \overset{\triangledown}{\mathbf{\Sigma}}{}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\mathbf{\Pi}}{}^{(\Psi)}_{(m)}, \tag{5.103}$$

$$i^{11}_{(m)} \equiv w^{\mathrm{incr}}_{(m)} = \dot{\widetilde{\boldsymbol{\tau}}}_{(m)} \cdot \dot{\boldsymbol{\varepsilon}}_{(m)} = \overset{\triangledown}{\boldsymbol{\sigma}}{}^{(\Psi)}_{(m)} \cdot \overset{\triangle}{\boldsymbol{\pi}}{}^{(\Psi)}_{(m)}. \tag{5.104}$$

(Incremental stress powers).

6. Some examples

In most applications, m is chosen equal to 2. In such a case, the equivalence classes $\Theta^{(\Pi)}_{(2)}$ and $\Theta^{(\Sigma)}_{(2)}$ ($\Theta^{(\pi)}_{(2)}$ and $\Theta^{(\sigma)}_{(2)}$) are denoted as family 1 of strain tensors and family 1 of stress tensors (family 2 of strain tensors and family 2 of stress tensors), respectively. Some examples for particular choices of $\mathbf{\Psi}$ are given ([13]) in Tables 1 and 2 (the orthogonal second-order tensor $\mathbf{P}$ is given by $\dot{\mathbf{P}} = \mathbf{WP}$). Possible physical interpretations for the stress tensors $\mathbf{\Sigma}^{(\Psi)}_{(2)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(2)}$ are given in Appendix B.

([13]) For more details see HAUPT and TSAKMAKIS [3].

Table 1. Dual variables and associated derivatives: family 1.

Ψ	$\Pi^{(\Psi)}_{(2)} = \Psi^{T-1}E\Psi^{-1}$	$\overset{\triangle}{\Pi}{}^{(\Psi)}_{(2)} = \Psi^{T-1}\dot{E}\Psi^{-1}$	$\Sigma^{(\Psi)}_{(2)} = \Psi\tilde{T}\Psi^T$	$\overset{\nabla}{\Sigma}{}^{(\Psi)}_{(2)} = \Psi\dot{\tilde{T}}\Psi^T$
1	$E = \frac{1}{2}(C - 1)$	$\overset{\triangle}{E} \equiv \dot{E}$	$\tilde{T} = F^{-1}SF^{T-1}$	$\dot{\tilde{T}}$
F	$A = \frac{1}{2}(1 - B^{-1})$	$\overset{\triangle}{A} = \dot{A} + L^TA + AL = D$	$S = (\det F)T$	$\overset{\nabla}{S} = \dot{S} - LS - SL^T$
R	$\bar{K} = \frac{1}{2}(B - 1)$	$\overset{\triangle}{\bar{K}} = \dot{\bar{K}} - \dot{R}R^T\bar{K} + \bar{K}\dot{R}R^T$	$\underline{S} = R\tilde{T}R^T$	$\overset{\nabla}{\underline{S}} = \underline{S} - \dot{R}R^T\underline{S} - \underline{S}\dot{R}R^T$
U	$\underline{K} = \frac{1}{2}(1 - C^{-1})$	$\overset{\triangle}{\underline{K}} = \dot{\underline{K}} + (\dot{U}U^{-1})^T\underline{K}$ $+\underline{K}\dot{U}U^{-1}$	$\bar{S} = U\tilde{T}U$ $= R^TSR$	$\overset{\nabla}{\bar{S}} = \dot{\bar{S}} - \dot{U}U^{-1}\bar{S}$ $-\bar{S}(\dot{U}U^{-1})^T$
P	$\Pi_W = PEP^T$	$\overset{\triangle}{\Pi}_W = \dot{\Pi}_W - W\Pi_W$ $+\Pi_W W$	$\Sigma_W = P\tilde{T}P^T$	$\overset{\nabla}{\Sigma}_W = \dot{\Sigma}_W - W\Sigma_W$ $+\Sigma_W W$

Table 2. Dual variables and associated derivatives: family 2.

Ψ	$\pi^{(\Psi)}_{(2)} = \Psi\varepsilon\Psi^T$	$\overset{\triangle}{\pi}{}^{(\Psi)}_{(2)} = \Psi\dot{\varepsilon}\Psi^T$	$\sigma^{(\Psi)}_{(2)} = \Psi^{T-1}\tilde{\tau}\Psi^{-1}$	$\overset{\nabla}{\sigma}{}^{(\Psi)}_{(2)} = \Psi^{T-1}\dot{\tilde{\tau}}\Psi^{-1}$
1	$\varepsilon = \frac{1}{2}(C^{-1} - 1)$	$\overset{\triangle}{\varepsilon} \equiv \dot{\varepsilon}$	$\tilde{\tau} = F^T\varsigma F$	$\dot{\tilde{\tau}}$
F	$\alpha = \frac{1}{2}(1 - B)$	$\overset{\triangle}{\alpha} = \dot{\alpha} - L\alpha - \alpha L^T = -D$	$\varsigma = -(\det F)T$	$\overset{\nabla}{\varsigma} = \dot{\varsigma} + L^T\varsigma + \varsigma L$
R	$\bar{k} = \frac{1}{2}(B^{-1} - 1)$	$\overset{\triangle}{\bar{k}} = \dot{\bar{k}} - \dot{R}R^T\bar{k} + \bar{k}\dot{R}R^T$	$\underline{\varsigma} = R\tilde{\tau}R^T$	$\overset{\nabla}{\underline{\varsigma}} = \underline{\varsigma} - \dot{R}R^T\underline{\varsigma} + \underline{\varsigma}\dot{R}R^T$
U	$\underline{k} = \frac{1}{2}(1 - C)$	$\overset{\triangle}{\underline{k}} = \dot{\underline{k}} - \dot{U}U^{-1}\underline{k}$ $-\underline{k}(\dot{U}U^{-1})^T$	$\bar{\varsigma} = U^{-1}\tilde{\tau}U^{-1}$ $= R^T\varsigma R$	$\overset{\nabla}{\bar{\varsigma}} = \dot{\bar{\varsigma}} + (\dot{U}U^{-1})^T\bar{\varsigma}$ $+\bar{\varsigma}\dot{U}U^{-1}$
P	$\pi_W = P\varepsilon P^T$	$\overset{\triangle}{\pi}_W = \dot{\pi}_W - W\pi_W + \pi_W W$	$\sigma_W = P\tilde{\tau}P^T$	$\overset{\nabla}{\sigma}_W = \dot{\sigma}_W - W\sigma_W$ $+\sigma_W W$

Next, we give the equivalent representations of hyperelastic constitutive equations using generalized dual variables. By definition, an elastic material is hyperelastic if and only if the work done by the the actual surface tractions in every closed homogeneous deformation process is non-negative (see e.g. TRUESDELL and NOLL [16, Sect. 82 & 83]). This is equivalent to the existence of scalar-valued functions $H_{(m)}$ and $h_{(m)}$,

$$H_{(m)} = \overline{H}_{(m)}(\mathbf{E}_{(m)}), \tag{6.1}$$

$$h_{(m)} = \overline{h}_{(m)}(\boldsymbol{\varepsilon}_{(m)}), \tag{6.2}$$

satisfying the relations

$$W = \dot{H}_{(m)} = \dot{h}_{(m)} \tag{6.3}$$

and therefore

$$\tilde{\mathbf{T}}_{(m)} = \frac{\partial \overline{H}_{(m)}}{\partial \mathbf{E}_{(m)}}, \qquad \tilde{\boldsymbol{\tau}}_{(m)} = \frac{\partial \overline{h}_{(m)}}{\partial \boldsymbol{\varepsilon}_{(m)}}, \tag{6.4}$$

respectively. Taking into account the relations (5.44)–(5.45), $H_{(m)}$ and $h_{(m)}$ can also be written in the form

$$H_{(m)} = \overline{H}_{(m)}(\mathbf{E}_{(m)}) = \overline{H}_{(m)}(\boldsymbol{\Psi}^T \boldsymbol{\Pi}^{(\Psi)}_{(m)} \boldsymbol{\Psi}) =: \hat{H}_{(m)}(\boldsymbol{\Pi}^{(\Psi)}_{(m)}, \boldsymbol{\Psi}), \tag{6.5}$$

$$h_{(m)} = \overline{h}_{(m)}(\boldsymbol{\varepsilon}_{(m)}) = \overline{h}_{(m)}(\boldsymbol{\Psi}^{-1} \boldsymbol{\pi}^{(\Psi)}_{(m)} \boldsymbol{\Psi}^{T-1}) =: \hat{h}_{(m)}(\boldsymbol{\pi}^{(\Psi)}_{(m)}, \boldsymbol{\Psi}), \tag{6.6}$$

respectively. From these equations, the stress relations $(6.4)_{1,2}$ as well as the transformation formulas (5.67) and (5.68), we conclude that

$$\boldsymbol{\Sigma}^{(\Psi)}_{(m)} = \frac{\partial \hat{H}_{(m)}}{\partial \boldsymbol{\Pi}^{(\Psi)}_{(m)}}, \tag{6.7}$$

$$\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \frac{\partial \hat{h}_{(m)}}{\partial \boldsymbol{\pi}^{(\Psi)}_{(m)}}, \tag{6.8}$$

which are the spatial counterparts of $(6.4)_1$ and $(6.4)_2$, respectively. In view of (5.50)–(5.57), also the representations

$$\boldsymbol{\Sigma}^{(\Psi)}_{(m)} = \frac{\partial \overline{\overline{H}}_{(m)}}{\partial E_{(m)kl}} \mathbf{g}^{(\Psi)}_k \otimes \mathbf{g}^{(\Psi)}_l, \tag{6.9}$$

$$\boldsymbol{\sigma}^{(\Psi)}_{(m)} = \frac{\partial \overline{\overline{h}}_{(m)}}{\partial \varepsilon^{kl}_{(m)}} \mathbf{g}^{(\Psi)k} \otimes \mathbf{g}^{(\Psi)l}, \tag{6.10}$$

apply, where the functions $\overline{\overline{H}}_{(m)}$ and $\overline{\overline{h}}_{(m)}$ are given by

$$\overline{H}_{(m)}(\mathbf{E}_{(m)}) = \overline{H}_{(m)}(E_{(m)kl}\mathbf{G}^k \otimes \mathbf{G}^l) =: \overline{\overline{H}}_{(m)}(E_{(m)kl}), \tag{6.11}$$

$$\overline{h}_{(m)}(\boldsymbol{\varepsilon}_{(m)}) = \overline{h}_{(m)}(\varepsilon^{kl}_{(m)}\mathbf{G}_k \otimes \mathbf{G}_l) =: \overline{\overline{h}}_{(m)}(\varepsilon^{kl}_{(m)}), \tag{6.12}$$

respectively.

For $m = 2$ we write $E_{(2)kl} \equiv E_{kl}$ and $\varepsilon^{kl}_{(2)} \equiv \varepsilon^{kl}$. Then, for $\mathbf{\Psi} = \mathbf{F}$,

$$\overline{H}_{(2)}(\mathbf{E}) = \widehat{H}_{(2)}(\mathbf{A}, \mathbf{F}) = \overline{\overline{H}}_{(2)}(E_{kl}), \tag{6.13}$$

$$\overline{h}_{(2)}(\boldsymbol{\varepsilon}) = \widehat{h}_{(2)}\boldsymbol{\alpha}, \mathbf{F}) = \overline{\overline{h}}(\varepsilon^{kl}). \tag{6.14}$$

In this case, (6.7)–(6.10) reduce to

$$\mathbf{S} = \frac{\partial \widehat{H}_{(2)}}{\partial \mathbf{A}} = \frac{\partial \overline{\overline{H}}_{(2)}}{\partial E_{kl}} \mathbf{g}_k^{(F)} \otimes \mathbf{g}_l^{(F)}, \tag{6.15}$$

$$\boldsymbol{\varsigma} = \frac{\partial \widehat{h}_{(2)}}{\partial \boldsymbol{\alpha}} = \frac{\partial \overline{\overline{h}}_{(2)}}{\partial \varepsilon_{kl}} \mathbf{g}^{(F)k} \otimes \mathbf{g}^{(F)l}, \tag{6.16}$$

respectively. Furthermore, setting $G_{ij} := \mathbf{G}_i \cdot \mathbf{G}_j$, $\gamma_{ij} := \mathbf{g}_i^{(F)} \cdot \mathbf{g}_j^{(F)}$, as well as $G^{ij} := \mathbf{G}^i \cdot \mathbf{G}^j$, $\gamma^{ij} := \mathbf{g}^{(F)i} \cdot \mathbf{g}^{(F)j}$, we arrive at the identities

$$E_{kl} = \frac{1}{2}(\gamma_{kl} - G_{kl}), \tag{6.17}$$

$$\varepsilon_{kl} = \frac{1}{2}(\gamma^{kl} - G^{kl}). \tag{6.18}$$

Hence,

$$\mathbf{S} = 2\frac{\partial \widetilde{H}_{(2)}}{\partial \gamma_{kl}} \mathbf{g}_k^{(F)} \otimes \mathbf{g}_l^{(F)} \tag{6.19}$$

and

$$\boldsymbol{\varsigma} = 2\frac{\partial \widetilde{h}_{(2)}}{\partial \gamma^{kl}} \mathbf{g}^{(F)k} \otimes \mathbf{g}^{(F)l}, \tag{6.20}$$

where the functions $\widetilde{H}_{(2)}(\cdot)$ and $\widetilde{h}_{(2)}(\cdot)$ are defined by

$$H_{(2)} = \overline{\overline{H}}_{(2)}(E_{kl}) = \overline{\overline{H}}_{(2)}\left(\frac{1}{2}(\gamma_{kl} - G_{kl})\right) =: \widetilde{H}_{(2)}(\gamma_{kl}), \tag{6.21}$$

$$h_{(2)} = \overline{\overline{h}}_{(2)}(\varepsilon^{kl}) = \overline{\overline{h}}_{(2)}\left(\frac{1}{2}(\gamma^{kl} - G^{kl})\right) =: \widetilde{h}_{(2)}(\gamma^{kl}). \tag{6.22}$$

Equation (6.19) corresponds to the well-known Doyle–Ericksen formula (see DOYLE and ERICKSEN [13]).

Further examples for the application of dual variables and their associated rates in Continuum Mechanics are provided in HAUPT and TSAKMAKIS [3].

7. Duality for two-point tensors

The concept of dual variables developed can be extended to two-point tensors as well. For example, formula (3.41) shows that $\mathbf{F}$ is conjugate in Hill's sense to $\mathbf{T}_R$. However, $\dot{\mathbf{F}}$ does not indicate the same properties as $\mathbf{F}$ under an observer transformation. Proceeding to define an associated rate $\overset{\triangle}{\mathbf{F}}$ for $\mathbf{F}$ which behaves like $\mathbf{F}$ under the observer transformations, we consider a skew-symmetric tensor $\mathbf{\Omega}$, so that (3.41) is rewritten as

$$W = \mathbf{T}_R \cdot (\dot{\mathbf{F}} - \mathbf{\Omega}\mathbf{F}). \tag{7.1}$$

This is possible, since $\mathbf{T}_R\mathbf{F}^T$ is symmetric. Note that by the polar decomposition $\mathbf{F}$ is related to the Lagrangean tensor $\mathbf{U}$ by means of $(3.5)_1$. Therefore, it appears natural to define $\overset{\triangle}{\mathbf{F}}$ in such a way, that $\overset{\triangle}{\mathbf{F}}$ is related to $\dot{\mathbf{U}}$ in the same manner as $\mathbf{F}$ to $\mathbf{U}$:

$$\frac{\overset{\triangle}{D}}{Dt}\mathbf{F} \equiv \overset{\triangle}{\mathbf{F}} = \mathbf{R}\dot{\mathbf{U}}\,. \tag{7.2}$$

From this, as well as $(3.5)_1$ and (7.1), it follows that

$$\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T\,, \tag{7.3}$$

and therefore

$$\overset{\triangle}{\mathbf{F}} = \dot{\mathbf{F}} - \dot{\mathbf{R}}\mathbf{R}^T\mathbf{F}. \tag{7.4}$$

It is not difficult now to show that $\mathbf{F}$ and $\overset{\triangle}{\mathbf{F}}$ behave similarly if the observer transformations are regarded.

Following steps similar as in the case of symmetric tensors, we define higher associated derivatives of $\mathbf{F}$ by

$$\frac{\overset{\triangle}{D^n}}{Dt^n}\mathbf{F} = \left(\frac{\overset{\triangle}{D^{n-1}}}{Dt^{n-1}}\mathbf{F}\right)^{\bullet} - \dot{\mathbf{R}}\mathbf{R}^T\left(\frac{\overset{\triangle}{D^{n-1}}}{Dt^{n-1}}\mathbf{F}\right) = \mathbf{R}\frac{d^n}{dt^n}\mathbf{U}\,. \tag{7.5}$$

Next, we note that

$$\mathbf{T}_R \cdot \mathbf{F} = \tilde{\mathbf{T}}_{(1)} \cdot \mathbf{U}\,, \tag{7.6}$$

where

$$\tilde{\mathbf{T}}_{(1)} = \mathbf{T}_{(\mathrm{BS})} = \frac{1}{2}\left(\mathbf{T}_R^T\mathbf{R} + \mathbf{R}^T\mathbf{T}_R\right) \tag{7.7}$$

was referred to in (4.2) as the symmetrized Biot stress tensor ([14]). This motivates us to define the associated time derivatives for $\mathbf{T}_R$ in the form

$$\frac{\overset{\nabla}{D}{}^n}{Dt^n}\mathbf{T}_R = \left(\frac{\overset{\nabla}{D}{}^{n-1}}{Dt^{n-1}}\mathbf{T}_R\right)^{\bullet} - \dot{\mathbf{R}}\mathbf{R}^T\left(\frac{\overset{\nabla}{D}{}^{n-1}}{Dt^{n-1}}\mathbf{T}_R\right), \tag{7.8}$$

having the property

$$\frac{d^n}{dt^n}\widetilde{\mathbf{T}}_{(1)} = \frac{1}{2}\left(\left(\frac{\overset{\nabla}{D}{}^n}{Dt^n}\mathbf{T}_R\right)^T\mathbf{R} + \mathbf{R}^T\left(\frac{\overset{\nabla}{D}{}^n}{Dt^n}\mathbf{T}_R\right)\right). \tag{7.9}$$

As a result, we have then ($\overset{\nabla}{\mathbf{T}}_R := \overset{\nabla}{D}\,\mathbf{T}_R/Dt$):

$$(\mathbf{T}_R\cdot\mathbf{F})^{\bullet} = \overset{\nabla}{\mathbf{T}}_R\cdot\mathbf{F} + \mathbf{T}_R\cdot\overset{\triangle}{\mathbf{F}}. \tag{7.10}$$

The results derived above, concerning the pair $(\mathbf{F}, \mathbf{T}_R)$, can be extended, in exactly the same way, to the pair $(\mathbf{F}^{T-1}, \boldsymbol{\tau}_R)$, where

$$\boldsymbol{\tau}_R := \boldsymbol{\varsigma}\mathbf{F} = -(\det\mathbf{F})\mathbf{T}\mathbf{F}. \tag{7.11}$$

We recall, from the polar decomposition $(3.5)_1$, that

$$\mathbf{F}^{T-1} = \mathbf{R}\mathbf{U}^{-1}. \tag{7.12}$$

This motivates us to define the associated time derivatives of $\mathbf{F}^{T-1}$ as follows:

$$\frac{\overset{\triangle}{D}{}^n}{Dt^n}\mathbf{F}^{T-1} = \left(\frac{\overset{\triangle}{D}{}^{n-1}}{Dt^{n-1}}\mathbf{F}^{T-1}\right)^{\bullet} - \dot{\mathbf{R}}\mathbf{R}^T\left(\frac{\overset{\triangle}{D}{}^{n-1}}{Dt^{n-1}}\mathbf{F}^{T-1}\right) = \mathbf{R}\left(\frac{d^n}{dt^n}\mathbf{U}^{-1}\right). \tag{7.13}$$

Thus, the stress power W becomes

$$W = \boldsymbol{\tau}_R\cdot\left(\mathbf{F}^{T-1}\right)^{\bullet} = \boldsymbol{\tau}_R\cdot\left(\mathbf{F}^{T-1}\right)^{\triangle}, \tag{7.14}$$

where

$$\left(\mathbf{F}^{T-1}\right)^{\triangle} = \frac{\overset{\triangle}{D}}{Dt}\mathbf{F}^{T-1}. \tag{7.15}$$

([14]) The analysis in the present paper is based on the relation between $\mathbf{T}_R$ and $\mathbf{T}_{(BS)}$. However, the results remain valid, if the analysis is referred to the relation between $\mathbf{T}_R$ and the Biot stress tensor $\mathbf{T}_{(B)} = \mathbf{R}^T\mathbf{T}_R$, defined in (4.3).

Furthermore, the identity

$$\boldsymbol{\tau}_R \cdot \mathbf{F}^{T-1} = \tilde{\boldsymbol{\tau}}_{(1)} \cdot \mathbf{U}^{-1} \tag{7.16}$$

holds, where

$$\tilde{\boldsymbol{\tau}}_{(1)} = \frac{1}{2}\left(\boldsymbol{\tau}_R^T \mathbf{R} + \mathbf{R}^T \boldsymbol{\tau}_R^T\right). \tag{7.17}$$

The last relation motivates us to define the associated time derivatives of $\boldsymbol{\tau}_R$ by

$$\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\boldsymbol{\tau}_R = \left(\frac{\overset{\triangledown}{D}{}^{n-1}}{Dt^{n-1}}\boldsymbol{\tau}_R\right)^{\bullet} - \dot{\mathbf{R}}\,\mathbf{R}^T\left(\frac{\overset{\triangledown}{D}{}^{n-1}}{Dt^{n-1}}\boldsymbol{\tau}_R\right), \tag{7.18}$$

satisfying the relation

$$\frac{\overset{\triangledown}{d}{}^n}{dt^n}\tilde{\boldsymbol{\tau}}_{(1)} = \frac{1}{2}\left(\left(\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\boldsymbol{\tau}_R\right)^T \mathbf{R} + \mathbf{R}^T\left(\frac{\overset{\triangledown}{D}{}^n}{Dt^n}\boldsymbol{\tau}_R\right)\right). \tag{7.19}$$

Again, a relation of the form

$$\left(\boldsymbol{\tau}_R \cdot \mathbf{F}^{T-1}\right)^{\bullet} = \overset{\triangledown}{\boldsymbol{\tau}}_R \cdot \mathbf{F}^{T-1} + \boldsymbol{\tau}_R \cdot \left(\mathbf{F}^{T-1}\right)^{\vartriangle} \tag{7.20}$$

holds, where

$$\overset{\triangledown}{\boldsymbol{\tau}}_R := \frac{\overset{\triangledown}{D}}{Dt}\boldsymbol{\tau}_R. \tag{7.21}$$

Traditionally, in formulating constitutive equations, we assume $m = 2$. However, if we deal e.g. with problems concerning uniqueness or constitutive inequalities, further pairs of dual variables may be convenient in formulating the theory. As an example, discussing intrinsic stability of the material, Hill (see HILL [4–6]) proposed a class of constitutive inequalities, which must be satisfied for some domain of deformation spaces. In the nomenclature of the present work, Hill's inequalities correspond either to

$$W_{(m)}^{\text{incr}} = \overset{\triangledown}{\boldsymbol{\Sigma}}{}_{(m)}^{(\Psi)} \cdot \overset{\vartriangle}{\boldsymbol{\Pi}}{}_{(m)}^{(\Psi)} > 0, \tag{7.22}$$

or to

$$w_{(m)}^{\text{incr}} = \overset{\triangledown}{\boldsymbol{\Sigma}}{}_{(m)}^{(\Psi)} \cdot \overset{\vartriangle}{\boldsymbol{\pi}}{}_{(m)}^{(\Psi)} > 0. \tag{7.23}$$

As a consequence, for $m = 1$, Eqs. (7.22) and (7.23) reduce to

$$W_{(1)}^{\text{incr}} = \dot{\tilde{\mathbf{T}}}_{(1)} \cdot \dot{\mathbf{E}}_{(1)} = \overset{\triangledown}{\mathbf{T}}_R \cdot \overset{\vartriangle}{\mathbf{F}} > 0, \tag{7.24}$$

and

$$w_{(1)}^{\text{incr}} = \dot{\tilde{\mathbf{T}}}_{(1)} \cdot \dot{\mathbf{E}}_{(1)} = \overset{\nabla}{\mathbf{T}}_R \cdot (\mathbf{F}^{T-1})^{\triangle} > 0 , \tag{7.25}$$

respectively. These relations demonstrate that dual variables, in combination with associated time derivatives, are appropriate terms for formulating objective constitutive inequalities, even in the case of two-point stress and strain tensors (in this context see also OGDEN [8, p. 407]).

Appendix A

Let

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}, \qquad t^* = t - a \tag{A.1}$$

describe an observer transformation in E, where $\mathbf{c}(t)$ denotes some vector-valued function of time and $a \in \mathbb{R}$. For our purposes, it suffices to assume $\mathbf{Q}(t)$ to be a proper orthogonal second-order tensor.

Assuming the reference configuration to be independent of the observer, the observer transformation (A.1) implies for the motion (3.1)

$$\bar{\mathbf{x}}(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{Q}(t)\bar{\mathbf{x}}(\mathbf{X}, t), \qquad t^* = t - a . \tag{A.2}$$

Well-known results obtainable from (A.2) are the transformation rules

$$\mathbf{F}^* = \mathbf{QF}, \qquad \mathbf{R}^* = \mathbf{QR}, \qquad \mathbf{U}^* = \mathbf{U}, \qquad \mathbf{V}^* = \mathbf{QVQ}^T . \tag{A.3}$$

An Eulerian second-order tensor $\mathbf{A}$ is said to be objective if it satisfies the transformation rule

$$\mathbf{A}^* = \mathbf{QAQ}^T \tag{A.4}$$

under the observer transformation (A.1). Commonly, it is assumed that the stress tensor $\mathbf{S}$ is objective, i.e.,

$$\mathbf{S}^* = \mathbf{QSQ}^T . \tag{A.5}$$

Now, let $\mathbf{S}$ be represented by

$$\mathbf{S} = S_{kl}\boldsymbol{\mu}_k \otimes \boldsymbol{\mu}_l , \tag{A.6}$$

so that

$$\mathbf{T}_{(g)} = \sum_{i=1}^{3} \frac{S_{ij}}{\lambda_i g'(\lambda_i)} \mathbf{M}_i \otimes \mathbf{M}_i + 2 \sum_{i \neq j} \alpha_{(g)ij} S_{ij} \mathbf{M}_i \otimes \mathbf{M}_j , \tag{A.7}$$

by $(4.16)_2$, (4.18). On using the relations (A.3), it is a straightforward matter to derive the transformation rules $(i, j = 1, 2, 3)$

$$\begin{aligned} \boldsymbol{\mu}_i^* &= \mathbf{Q}\boldsymbol{\mu}_i , & \mathbf{M}_i^* &= \mathbf{M}_i , & & \\ \lambda_i^* &= \lambda_i , & g(\lambda_i^*) &= g(\lambda_i), & g'(\lambda_i^*) &= g'(\lambda_i), \\ \ell_{(g)ij}^* &= \ell_{(g)ij} , & \alpha_{(g)ij}^* &= \alpha_{(g)ij} , & S_{ij}^* &= S_{ij} . \end{aligned} \tag{A.8}$$

Hence,

(A.9) $$\mathbf{T}^*_{(g)} = \mathbf{T}_{(g)},$$

from (A.7). Thus, we have

(A.10) $$\left(\mathbf{T}^*_{(g)}\right)^{\cdot} = \dot{\mathbf{T}}_{(g)}$$

and therefore $(i, j = 1, 2, 3)$

(A.11) $$\mathbf{M}^*_i \cdot \left(\mathbf{T}^*_{(g)}\right)^{\cdot} \mathbf{M}^*_j = \mathbf{M}_i \cdot \dot{\mathbf{T}}_{(g)} \mathbf{M}_j .$$

Next, we discuss how $D_{(g)}\mathbf{S}/Dt$ is affected under the observer transformation (A.1). To this end, using (4.21), we rewrite $(4.23)_1$ in the form

(A.12) $$\begin{aligned}\frac{D_{(g)}}{Dt}\mathbf{S} &= \mathcal{P}_{(g)}[\dot{\mathbf{T}}_{(g)}] \\ &= \sum_{i=1}^{3} \lambda_i g'(\lambda_i) \left(\mathbf{M}_i \cdot \dot{\mathbf{T}}_{(g)} \mathbf{M}_i\right) \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_i + \frac{1}{2} \sum_{i \neq j} \frac{1}{\alpha^{(g)}_{ij}} \left(\mathbf{M}_i \cdot \dot{\mathbf{T}}_{(g)} \mathbf{M}_j\right) \boldsymbol{\mu}_i \otimes \boldsymbol{\mu}_j .\end{aligned}$$

From this result, as well as from (A.8) and (A.11), we conclude that

$$\frac{D_{(g)}}{Dt^*}\mathbf{S}^* = \mathbf{Q} \left(\frac{D_{(g)}}{Dt}\mathbf{S}\right) \mathbf{Q}^T ,$$

which shows that $D_{(g)}\mathbf{S}/Dt$ represents an objective Eulerian second-order tensor.

Appendix B

In this Appendix we give possible physical interpretations for the stress tensors $\boldsymbol{\Sigma}^{(\Psi)}_{(2)}$ and $\boldsymbol{\sigma}^{(\Psi)}_{(2)}$, which confessedly are somewhat artificial.

By Cauchy's theorem, we have

(B.1) $$\mathbf{t} = \mathbf{T}\mathbf{n} = \mathbf{S}\left[(\det \mathbf{F})^{-1}\mathbf{n}\right] = \varsigma \left[(\det \mathbf{F})^{-1}\mathbf{m}\right],$$

where $\mathbf{t}$ represents the stress vector acting on a surface element in the current configuration

(B.2) $$d\mathbf{a} = \mathbf{n}\, da ,$$

oriented by a unit normal $\mathbf{n}$, and

(B.3) $$\mathbf{m} := -\mathbf{n} .$$

Let now $d\mathbf{a}$ be represented by

(B.4) $$d\mathbf{a} = d\mathbf{x}_{[1]} \times d\mathbf{x}_{[2]} ,$$

where $d\mathbf{x}_{[i]}, i = 1, 2$, are non-collinear line elements in the current configuration. For the corresponding surface element

(B.5) $$d\mathbf{A}_0 = \mathbf{N}_0\, dA_0$$

$(\mathbf{N}_0 \cdot \mathbf{N}_0 = 1)$ in the reference configuration, the well-known formula

(B.6) $$d\mathbf{a} = (\det \mathbf{F})\mathbf{F}^{T-1} d\mathbf{A}_0$$

holds, with

(B.7) $$d\mathbf{A}_0 = d\mathbf{X}_{[1]} \times d\mathbf{X}_{[2]}$$

and

(B.8) $$d\mathbf{X}_{[i]} = \mathbf{F}^{-1} d\mathbf{x}_{[i]} ,$$

by (5.16). Furthermore, assuming that the transformation rule (5.16) applies also to the vector $\mathrm{t}\, da$, we can introduce a transformed "force" $d\widetilde{\mathrm{Q}}$ in the reference configuration by

(B.9) $$\mathrm{t}\, da = \mathbf{F}\, d\widetilde{\mathrm{Q}} .$$

Analogously, further transformed "forces" $d\widetilde{\mathrm{Q}}^{(\Psi)}$ are given by

(B.10) $$d\mathrm{Q}^{(\Psi)} := \mathbf{\Psi}\, d\widetilde{\mathrm{Q}} ,$$

with $d\mathrm{Q}^{(F)} = \mathrm{t}\, da$. In addition, we define the "stress vectors"

(B.11) $$\mathrm{t}^{(\Psi)} := \frac{d\mathrm{Q}^{(\Psi)}}{dA^{(\Psi)}} ,$$

where $dA^{(\Psi)}$ is given by the relation

(B.12) $$d\mathbf{A}^{(\Psi)} = \mathbf{N}^{(\Psi)}\, dA^{(\Psi)} = (\det \mathbf{\Psi})\mathbf{\Psi}^{T-1} d\mathbf{A}_0$$

$(\mathbf{N}^{(\Psi)} \cdot \mathbf{N}^{(\Psi)} = 1)$, which is analogous to (B.6). Finally, on the basis of $(\mathrm{B}.1)_2$, it is not difficult to derive the relation

(B.13) $$\mathrm{t}^{(\Psi)} = \mathbf{\Sigma}^{(\Psi)}_{(2)} \left[(\det \mathbf{\Psi})^{-1}\mathbf{N}^{(\Psi)}\right]$$

with $\left(\mathrm{t}^{(F)}, \mathbf{\Sigma}^{(F)}_{(2)}, \mathbf{N}^{(F)}\right) = (\mathrm{t}, \mathbf{S}, \mathbf{n})$. Thus, the stress tensor $\mathbf{\Sigma}^{(\Psi)}_{(2)}$ acting on the "weighted normal" $(\det \mathbf{\Psi})^{-1}\mathbf{N}^{(\Psi)}$ gives the "stress vector" $\mathrm{t}^{(\Psi)}$.

The physical interpretation of $\boldsymbol{\sigma}^{(F)}_{(2)}$ is similar. We start by considering again the surface element $d\mathbf{a}$ (see Eqs. (B.2) and (B.4)). Besides (B.6), the surface element $d\mathbf{a}$ can be mapped on the reference configuration as follows. Let $d\mathbf{Y}_{[i]}$ be vectors in the reference configuration, which are related to $d\mathbf{x}_{[i]}$ by means of (5.18),

(B.14) $$d\mathbf{Y}_{[i]} = \mathbf{F}^{T}\, d\mathbf{x}_{[i]} .$$

We define the transformed "surface element" in the reference configuration $d\mathbf{a}_0$ by

(B.15) $$d\mathbf{a}_0 = \mathbf{n}_0 da_0 = d\mathbf{Y}_{[1]} \times d\mathbf{Y}_{[2]},$$

with $\mathbf{n}_0 \cdot \mathbf{n}_0 = 1$. It is readily shown that $d\mathbf{a}$ is related to $d\mathbf{a}_0$ through

(B.16) $$d\mathbf{a} = (\det \mathbf{F})^{-1} \mathbf{F}\, d\mathbf{a}_0 .$$

Next, assuming the transformation formula (5.18) (or (B.14)) to apply also to the vector $\mathsf{t}\, da$, we can introduce a transformed "force" in the reference configuration $d\tilde{\mathsf{q}}$ by

(B.17) $$\mathsf{t}\, da = \mathbf{F}^{T-1}\, d\tilde{\mathsf{q}} .$$

Analogously, further transformed "forces" $d\mathsf{q}^{(\Psi)}$ are defined through

(B.18) $$d\mathsf{q}^{(\Psi)} := \mathbf{\Psi}^{T-1}\, d\tilde{\mathsf{q}} .$$

Finally, we introduce the "stress vectors"

(B.19) $$\mathsf{t}^{(\Psi)} := \frac{d\mathsf{q}^{(\Psi)}}{da^{(\Psi)}},$$

where $da^{(\Psi)}$ is given by the relation

(B.20) $$d\mathbf{a}^{(\Psi)} = \mathbf{n}^{(\Psi)}\, da^{(\Psi)} = (\det \mathbf{\Psi})^{-1} \mathbf{\Psi}\, d\mathbf{a}_0 ,$$

$(\mathbf{n}^{(\Psi)} \cdot \mathbf{n}^{(\Psi)} = 1)$, which is analogous to (B.16). Then, on the basis of $(\mathrm{B.1})_3$, it can be seen that

(B.21) $$\mathsf{t}^{(\Psi)} = \boldsymbol{\sigma}^{(\Psi)}_{(2)} \left[\frac{(\det \mathbf{\Psi})}{(\det \mathbf{F})^2} \mathbf{m}^{(\Psi)} \right],$$

where

(B.22) $$\mathbf{m}^{(\Psi)} := -\mathbf{n}^{(\Psi)}$$

and $\left(\mathsf{t}^{(F)}, \boldsymbol{\sigma}^{(F)}_{(2)}, \mathbf{m}^{(F)}\right) = (\mathsf{t}, \boldsymbol{\varsigma}, \mathbf{m})$. That is, the stress tensor $\boldsymbol{\sigma}^{(\Psi)}_{(2)}$ acting on the "weighted normal" $\left((\det \mathbf{\Psi})/(\det \mathbf{F})^2\right) \mathbf{m}^{(\Psi)}$ gives the "stress vector" $\mathsf{t}^{(\Psi)}$.

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