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Stokes flow past a composite porous spherical shell with a solid core

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A GENERAL SOLUTION of the Brinkman equations in the form of an infinite series is presented. A representation for the solution of Brinkman's equations is also proposed and its equivalence to the infinite series is established. The usefulness of the representation is demonstrated by applying it to design a general method of solving an arbitrary Stokes flow past a composite porous spherical shell with a rigid core. Some physical properties, such as the drag and torque exerted on the composite sphere are calculated. Several illustrative examples are discussed.

1. Introduction

IN THE STUDY of flow and heat transfer problems in porous media, two models which have been extensively used are those due to DARCY [1] and BRINKMAN [2]. However, the Brinkman model seems to be favoured in some problems in porous media, owing to the limitations of Darcy's law. The inadequacy of Darcy's law in the formulation of problems in bounded porous media is primarily due to the order of Darcy's equations being lower than the second order Navier–Stokes equations. A variety of flow and heat transfer problems in porous media were solved using the Brinkman's equations. In this paper, we give a general solution of the Brinkman equations in the form of an infinite series by using a procedure followed by LAMB [3] in the case of Stokes equations. We also propose a representation for the solution of Brinkman equations in terms of two scalar functions and establish its equivalence to the series solution. We shall use this representation to study the problem of an arbitrary Stokes flow of an incompressible, viscous fluid past a composite porous sphere with a rigid core, using the Brinkman model in the porous region. The results obtained by MASLIYAH *et al.* [4] who considered a uniform flow past a composite porous sphere with a rigid core can be recovered as a special case. Some illustrative examples are discussed.

2. Structure of the general solution of Brinkman's equations

We consider Brinkman's equations

$$-\nabla p + \mu\nabla^2\mathbf{V} = \frac{\mu}{k}\mathbf{V}, \tag{2.1}$$

and the equation of continuity

$$\nabla\cdot\mathbf{V} = 0, \tag{2.2}$$

where $\mathbf{V}$ is the velocity, p is the pressure, μ is the coefficient of dynamic viscosity, and $k > 0$ is the permeability coefficient of the porous medium. Equation (2.1) can be rewritten as

$$\mu(\nabla^2 - \lambda^2)\mathbf{V} = \nabla p, \tag{2.3}$$

where $\lambda^2 = 1/k$.

The general solution of the equation

$$(\nabla^2 - \lambda^2)\Psi = 0, \tag{2.4}$$

is as follows:

$$\Psi = \sum_{-\infty}^{\infty} (X_n F_n(\lambda r) + Y_n H_n(\lambda r))\chi_n\,, \tag{2.5}$$

where X_n, Y_n are arbitrary constants, $\chi_n = r^n S_n(\theta, \phi)$ is a solid harmonic of degree n, and

$$S_n(\theta, \phi) = \sum_{m=0}^{n} P_n^m(\zeta)(A_{nm} \cos m\phi + B_{nm} \sin m\phi), \qquad \zeta = \cos\theta.$$

The functions $F_n(z)$ and $H_n(z)$ $(z = \lambda r)$ are defined as follows,

$$z^n F_n(z) = \sqrt{\frac{\pi}{2z}} I_{n+\frac{1}{2}}(z), \qquad z^n H_n(z) = \sqrt{\frac{\pi}{2z}} K_{n+\frac{1}{2}}(z),$$

where $\sqrt{\frac{\pi}{2z}} I_{n+\frac{1}{2}}(z)$ and $\sqrt{\frac{\pi}{2z}} K_{n+\frac{1}{2}}(z)$ are the modified spherical Bessel functions which are finite at the origin and infinity, respectively. The functions $F_n(\lambda r)$ or $H_n(\lambda r)$ are retained in the solution depending on whether the motion is finite at the origin or at infinity, respectively. Suppose we assume the condition of finiteness of the motion at the origin $r = 0$, then the general solution of Eqs. (2.2) and (2.3) is

$$p = \sum_{-\infty}^{\infty} p_n,$$

$$\mathbf{V} = \sum_{-\infty}^{\infty} \Bigg(\left[(n+1)F_{n-1}(\lambda r) + nF_{n+1}(\lambda r)\lambda^2 r^2\right] \nabla\phi_n - n(2n+1)F_{n+1}(\lambda r)\lambda^2 \mathbf{r}\phi_n - F_n(\lambda r)\nabla \times (\mathbf{r}\chi_n) - \frac{1}{\lambda^2\mu}\nabla p_n \Bigg), \tag{2.6}$$

where χ_n, ϕ_n and p_n are solid harmonics of positive degree n. When the condition of finiteness at the origin is not imposed, we have an additional system of solutions in which the functions $F_n(\lambda r)$ are replaced by $H_n(\lambda r)$.

3. A representation for the solution of Brinkman's equations

We now propose a representation for the velocity and pressure in Brinkman's equations (2.2) and (2.3) in terms of two scalar functions A and B and establish its equivalence to the series solution given in (2.6). We assume the following form for the velocity $\mathbf{V}$,

$$\begin{aligned} \mathbf{V} &= \operatorname{curl}\operatorname{curl}(\mathbf{r}A) + \operatorname{curl}(\mathbf{r}B), \\ &= \operatorname{grad}\operatorname{div}(\mathbf{r}A) - \nabla^2(\mathbf{r}A) + \operatorname{curl}(\mathbf{r}B). \end{aligned} \tag{3.1}$$

Equation (2.2) is satisfied identically and substitution of $(3.1)_2$ in Eq. (2.3) results in

$$\begin{aligned} &\operatorname{grad}\left(p - \mu\frac{\partial}{\partial r}\left[r(\nabla^2 - \lambda^2)A\right]\right) \\ &= \mu\left(-\hat{e}_r r(\nabla^4 - \lambda^2\nabla^2)A + \hat{e}_\theta \csc\theta\frac{\partial}{\partial\phi}(\nabla^2 - \lambda^2)B - \hat{e}_\phi\frac{\partial}{\partial\theta}(\nabla^2 - \lambda^2)B\right), \end{aligned} \tag{3.2}$$

where $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ are the unit vectors along the radial, transverse and azimuthal directions, respectively. Equations (2.2) and (2.3) are satisfied if

$$\begin{aligned} p &= p_0 + \mu\frac{\partial}{\partial r}[r(\nabla^2 - \lambda^2)A], \\ \nabla^2(\nabla^2 - \lambda^2)A &= 0, \\ (\nabla^2 - \lambda^2)B &= 0. \end{aligned} \tag{3.3}$$

A general solution of $(3.3)_2$ is given by $A = A_1 + A_2$, where A_1 and A_2 are, respectively, the solutions of

$$\begin{aligned} \nabla^2 A_1 &= 0, \\ (\nabla^2 - \lambda^2)A_2 &= 0. \end{aligned} \tag{3.4}$$

Equation $(3.1)_1$ can also be written as

$$\mathbf{V} = 2\operatorname{grad} A + r\frac{\partial}{\partial r}\operatorname{grad} A - \mathbf{r}\nabla^2 A + \operatorname{curl}(\mathbf{r}B). \tag{3.5}$$

From the above equation, we recover the solution given in Eqs. (2.6) by assuming

$$\begin{aligned} B &= -\sum_{-\infty}^{\infty} F_n(\lambda r)\chi_n, \\ A_1 &= -\sum_{-\infty}^{\infty}\frac{1}{\lambda^2\mu}\frac{p_n}{(n+1)}, \\ A_2 &= \sum_{-\infty}^{\infty}(2n+1)F_n(\lambda r)\phi_n. \end{aligned} \tag{3.6}$$

It is observed that such B, A_1 and A_2 satisfy Eqs. $(3.3)_3$ and (3.4), respectively. It may be noted that when the condition of finiteness at the origin is not imposed, the functions $H_n(\lambda r)$ also have to be considered along with the functions $F_n(\lambda r)$. Thus $(3.1)_1$ and $(3.3)_1$ give a general solution of the Brinkman's equations. Similar representations have been considered earlier in the literature and, more recently, in connection with the solution of Stokes equations by PALANIAPPAN *et al.* [5]. However, the application of the representation proposed here to the Brinkman's equations is new and this representation lends itself to useful applications in problems of flows through porous media; in particular, in problems involving spherical boundaries, owing to the simplicity of its form. This fact is exemplified in the next section in the discussion of a general, non-axisymmetric Stokes flow past a composite porous spherical shell with a rigid core, using the Brinkman model in the porous region.

4. Stokes flow over a composite sphere: Solid core with a porous shell

Consider a stationary, solid, impermeable sphere of radius b surrounded by a porous shell of permeability k and thickness $(a - b)$. We shall consider a non-axisymmetric, Stokes flow of an incompressible, viscous fluid over the composite sphere. The Stokes equations are

$$\begin{aligned} \mu\nabla^2\mathbf{V} &= \nabla p, \\ \nabla\cdot\mathbf{V} &= 0. \end{aligned} \tag{4.1}$$

We find it advantageous to use the representation, proposed by PALANIAPPAN *et al.* [5] for the solution of the Stokes equations (4.1), given below in the form

$$\begin{aligned} \mathbf{V} &= \text{curl curl}(\mathbf{r}A) + \text{curl}(\mathbf{r}B), \\ p &= p_0 + \mu\frac{\partial}{\partial r}\left[r\nabla^2 A\right], \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \nabla^4 A &= 0, \\ \nabla^2 B &= 0. \end{aligned} \tag{4.3}$$

Suppose now that the basic, unperturbed velocity is given by

$$\mathbf{V}_0 = \text{curl curl}(\mathbf{r}A_0) + \text{curl}(\mathbf{r}B_0), \tag{4.4}$$

where

$$\begin{aligned} A_0 &= \sum_{n=1}^{\infty}\left(\alpha_n r^n + \alpha'_n r^{n+2}\right) S_n(\theta,\phi), \\ B_0 &= \sum_{n=1}^{\infty}\xi_n r^n T_n(\theta,\phi), \end{aligned} \tag{4.5}$$

where

$$
\begin{aligned}
S_n(\theta,\phi) &= \sum_{m=0}^{n} P_n^m(\zeta)(A_{nm}\cos m\phi + B_{nm}\sin m\phi), \qquad \zeta = \cos\theta,\\
T_n(\theta,\phi) &= \sum_{m=0}^{n} P_n^m(\zeta)(C_{nm}\cos m\phi + D_{nm}\sin m\phi),
\end{aligned}
\tag{4.6}
$$

$\alpha_n, \alpha_n', \xi_n, A_{nm}, B_{nm}, C_{nm}$ and D_{nm} are known constants and $P_n^m(\zeta)$ is the Legendre polynomial. For the flow quantities in the region $a < r < \infty$ we shall use the superscript e. Therefore in the presence of the sphere, we shall assume the modified flow in this region to be given by $(\mathbf{V}^e, p^e)$ in terms of two scalar functions A^e and B^e, where

$$
\begin{aligned}
\nabla^4 A^e &= 0,\\
\nabla^2 B^e &= 0.
\end{aligned}
\tag{4.7}
$$

The equations which describe the flow field in the porous region $b < r < a$ are assumed to be the Brinkman equations (2.1) and (2.2). We make use of the representation $(3.1)_1$ and $(3.3)_1$ proposed for the Brinkman's equations, to find the modified flow $(\mathbf{V}^i, p^i)$ in this region in terms of two scalar functions A^i and B^i, where

$$
\begin{aligned}
\nabla^2(\nabla^2 - \lambda^2)A^i &= 0,\\
(\nabla^2 - \lambda^2)B^i &= 0.
\end{aligned}
\tag{4.8}
$$

We assume the following forms for these scalar functions as

$$
\begin{aligned}
A^e(r,\theta,\phi) &= \sum_{n=1}^{\infty}\left(\alpha_n r^n + \alpha_n' r^{n+2} + \frac{\beta_n}{r^{n+1}} + \frac{\beta_n'}{r^{n-1}}\right) S_n(\theta,\phi),\\
B^e(r,\theta,\phi) &= \sum_{n=1}^{\infty}\left(\xi_n r^n + \frac{\sigma_n}{r^{n+1}}\right) T_n(\theta,\phi),\\
A^i(r,\theta,\phi) &= A_1^i(r,\theta,\phi) + A_2^i(r,\theta,\phi),\\
B^i(r,\theta,\phi) &= \sum_{n=1}^{\infty}(\gamma_n f_n(\lambda r) + \gamma_n' g_n(\lambda r))\, T_n(\theta,\phi),
\end{aligned}
\tag{4.9}
$$

where

$$
\begin{aligned}
A_1^i(r,\theta,\phi) &= \sum_{n=1}^{\infty}\left(\varepsilon_n r^n + \frac{\varepsilon_n'}{r^{n+1}}\right) S_n(\theta,\phi),\\
A_2^i(r,\theta,\phi) &= \sum_{n=1}^{\infty}(\delta_n f_n(\lambda r) + \delta_n' g_n(\lambda r))\, S_n(\theta,\phi),
\end{aligned}
\tag{4.9'}
$$

where $f_n(z) = \sqrt{\frac{\pi}{2z}} I_{n+\frac{1}{2}}(z)$ and $g_n(z) = \sqrt{\frac{\pi}{2z}} K_{n+\frac{1}{2}}(z)$. The boundary conditions to be satisfied at $r = a$ and $r = b$ are

1) continuity of velocity components on the surface $r = a$

$$\begin{aligned} q_r^e(a,\theta,\phi) &= q_r^i(a,\theta,\phi), \\ q_\theta^e(a,\theta,\phi) &= q_\theta^i(a,\theta,\phi), \\ q_\phi^e(a,\theta,\phi) &= q_\phi^i(a,\theta,\phi); \end{aligned} \tag{4.10}$$

2) continuity of stresses on the surface $r = a$

$$\begin{aligned} T_{rr}^e(a,\theta,\phi) &= T_{rr}^i(a,\theta,\phi), \\ T_{r\theta}^e(a,\theta,\phi) &= T_{r\theta}^i(a,\theta,\phi), \\ T_{r\phi}^e(a,\theta,\phi) &= T_{r\phi}^i(a,\theta,\phi); \end{aligned} \tag{4.11}$$

3) no-slip conditions on the surface $r = b$

$$\begin{aligned} q_r^i(b,\theta,\phi) &= 0, \\ q_\theta^i(b,\theta,\phi) &= 0, \\ q_\phi^i(b,\theta,\phi) &= 0, \end{aligned} \tag{4.12}$$

where q_r^i, q_θ^i and q_ϕ^i are the radial, transverse and azimuthal velocities, T_{rr}^i is the normal stress and $T_{r\theta}^i$ and $T_{r\phi}^i$ are the tangential stresses in the region $b < r < a$. The corresponding velocities and stresses in the region $a < r < \infty$ are defined in a similar manner using the superscript e.

In terms of the scalar functions which appear in (4.8)–(4.9′), the boundary conditions (4.10)–(4.12) can be restated as follows

$$\begin{aligned} A^e(a,\theta,\phi) &= A^i(a,\theta,\phi), \\ A_r^e(a,\theta,\phi) &= A_r^i(a,\theta,\phi), \\ A_{rr}^e(a,\theta,\phi) &= A_{rr}^i(a,\theta,\phi), \\ a(A_{rrr}^e(a,\theta,\phi) - A_{rrr}^i(a,\theta,\phi)) &= -\lambda^2 \frac{\partial}{\partial r}(rA^i)(a,\theta,\phi), \\ B^e(a,\theta,\phi) &= B^i(a,\theta,\phi), \\ B_r^e(a,\theta,\phi) &= B_r^i(a,\theta,\phi), \\ A^i(b,\theta,\phi) &= 0, \\ A_r^i(b,\theta,\phi) &= 0, \\ B^i(b,\theta,\phi) &= 0. \end{aligned} \tag{4.13}$$

The functions A^e, B^e, A^i and B^i which correspond to the modified flow can be determined by determining the nine unknown constants $\beta_n, \beta_n', \sigma_n, \varepsilon_n, \varepsilon_n', \delta_n, \delta_n',$

γ_n, and γ_n' from the nine equations (4.13) in terms of $\alpha_n, \alpha_n', \xi_n, A_{nm}, B_{nm}, C_{nm}$ and D_{nm}. The nine unknown constants are determined to be as follows:

$$
\begin{aligned}
\beta_n &= \frac{\text{numb}}{\text{deno}}, \\
\beta_n' &= \frac{\text{numb}'}{\text{deno}}, \\
\varepsilon_n &= \frac{\text{nume}}{\text{deno}}, \\
\varepsilon_n' &= \frac{\text{nume}'}{\text{deno}}, \\
\delta_n &= \frac{\text{numd}}{\text{deno}}, \\
\delta_n' &= \frac{\text{numd}'}{\text{deno}}, \\
\sigma_n &= -\left[a + \frac{s_n(1+2n)}{\lambda t_n}\right] a^{2n}\xi_n, \\
\gamma_n &= \frac{(1+2n)a^{n-1}g_n(\lambda b)}{\lambda t_n}\xi_n, \\
\gamma_n' &= -\frac{(1+2n)a^{n-1}f_n(\lambda b)}{\lambda t_n}\xi_n,
\end{aligned}
\tag{4.14}
$$

where

$$
\begin{aligned}
\text{deno} = 2a^{n+5}b^{n+1}\lambda^3\{&[(1+n)a^{2n+3}b\lambda^2 + na^2b^{2+2n}\lambda^2 - n(1-4n^2)a^{1+2n}b]a_n \\
&+(1-2n)\lambda[(1+n)a^{2+2n}b + nab^{2+2n}]b_n \\
&+n(1-4n^2)a^nb^{2+n}c_n + n(1-4n^2)a^{n+2}b^nr_n \\
&-n(1+2n)\lambda a^2b^{1+2n}s_n - n(1-4n^2)ab^{1+2n}t_n\},
\end{aligned}
$$

$$
\begin{aligned}
\text{numb} = a^{3n+6}b^{n+1}\lambda^3(2n-1)\{&[a^{2n+3}b\lambda^2(1+n) + 2a^{2n+1}b(1+n)(1+2n) \\
&+a^2b^{2n+2}\lambda^2n]a_n + 2ab\lambda[a^{2n+1}(1+n) + b^{2n+1}n]b_n \\
&-2a^nb^{n+2}(1+n)(1+2n)c_n + 2a^{2+n}b^nn(1+2n)r_n \\
&-a^2b^{1+2n}\lambda n(1+2n)s_n - 2ab^{1+2n}n(1+2n)t_n\}\alpha_n \\
&+a^{3n+6}b^{n+1}\lambda(1+2n)\{ab[a^{2n+4}\lambda^4(1+n) \\
&+a^3b^{2n+1}\lambda^4n + 4a^{2n+2}\lambda^2(2n+5) \\
&-4a^{2n}(1-4n^2)(2n+3) + 4\lambda^2ab^{2n+1}(2n+3)]a_n \\
&+ab\lambda[4a^{2n+3}\lambda^2(1+n) + 4a^2b^{2n+1}\lambda^2n \\
&-4a^{2n+1}(1-2n)(2n+3) + 4b^{2n+1}(1-2n)(2n+3)]b_n \\
&+2a^nb^{n+2}(1+2n)[2(1-2n)(2n+3) - 3\lambda^2a^2]c_n
\end{aligned}
$$

[cont.]
$$-2a^{n+2}b^n(1+2n)[3a^2\lambda^2-2(1-2n)(2n+3)]r_n$$
$$-a^2b^{2n+1}\lambda(1+2n)[a^2\lambda^2 n+4(2n+3)]s_n$$
$$-4ab^{2n+1}(1+2n)[a^2\lambda^2 n+(1-2n)(2n+3)]t_n\}\alpha'_n,$$

$$\text{numb}' = -a^{3n+6}b^{n+2}(1+2n)\lambda^4\{[a^{2n+1}(1+n)+b^{2n+1}n]\lambda a_n$$
$$-b^{2n}n(1+2n)s_n\}\alpha_n$$
$$-a^{3n+6}b^{n+1}(3+2n)\lambda^3\{[a^{2n+3}b\lambda^2(1+n)+a^2b^{2n+2}\lambda^2 n$$
$$+2a^{2n+1}b(1+n)(1+2n)]a_n+2\lambda[a^{2n+2}b(1+n)+ab^{2n+2}n]b_n$$
$$+2a^nb^{n+2}n(1+2n)c_n-2a^{n+2}b^n(1+n)(1+2n)r_n$$
$$-a^2b^{2n+1}\lambda n(1+2n)s_n-2ab^{2n+1}n(1+2n)t_n\}\alpha'_n,$$

$$\text{numd} = 2a^{2n+5}b^{n+1}\lambda^2(1-4n^2)\{a^{n+1}b^n n(1+2n)g_n(\lambda a)$$
$$-\lambda[b^{2n+2}n+a^{2n+1}b(1+n)]g_{n-1}(\lambda b)-b^{2n+1}n(1+2n)g_n(\lambda b)\}\alpha_n$$
$$+2a^{2n+6}b^{n+1}\lambda(1+2n)(3+2n)\{[a^{2n+2}b\lambda^2(1+n)$$
$$+\lambda^2ab^{2n+2}n-2a^{2n}b(1-4n^2)]g_{n-1}(\lambda b)$$
$$-a^{n+2}b^n\lambda(n-2)(1+2n)g_n(\lambda a)+2a^{n+1}b^n(1-4n^2)g_{n-1}(\lambda a)$$
$$+ab^{2n+1}\lambda n(1+2n)g_n(\lambda b)\}\alpha'_n,$$

$$\text{numd}' = -2\lambda^2(1-4n^2)a^{2n+5}b^{n+1}\{a^{n+1}b^n n(1+2n)f_n(\lambda a)$$
$$+\lambda(b^{2n+2}n+a^{2n+1}b(1+n))f_{n-1}(\lambda b)$$
$$-b^{2n+1}n(1+2n)f_n(\lambda b)\}\alpha_n$$
$$+2\lambda(1+2n)(3+2n)a^{2n+6}b^{n+1}\{a^{2n+2}b\lambda^2(1+n)f_{n-1}(\lambda b)$$
$$+a^{n+2}b^n\lambda(n-2)(1+2n)f_n(\lambda a)$$
$$+2a^{n+1}b^n(1-4n^2)f_{n-1}(\lambda a)+ab^{2n+2}n\lambda^2f_{n-1}(\lambda b)$$
$$-ab^{2n+1}n(1+2n)\lambda f_n(\lambda b)-2a^{2n}b(1-4n^2)f_{n-1}(\lambda b)\}\alpha'_n,$$

$$\text{nume} = -2\lambda^3a^{2n+5}b^{n+2}n(1-4n^2)[a^{n+1}a_n-b^{n+1}c_n]\alpha_n$$
$$+2\lambda^2a^{2n+7}b^{n+2}(1+2n)(3+2n)\{a^{n+1}(n-2)\lambda a_n$$
$$-2a^n(1-2n)b_n-b^{n+1}n\lambda c_n\}\alpha'_n,$$

$$\text{nume}' = 2\lambda^2a^{3n+6}b^{2n+2}(1-4n^2)\{\lambda b^{n+1}na_n-b^nn(1+2n)s_n$$
$$+\lambda a^nb(1+n)c_n\}\alpha_n$$
$$+\lambda a^{3n+6}b^{2n+2}(1+2n)(3+2n)\{-2\lambda^2a^2b^{n+1}(n-2)a_n$$
$$+4\lambda ab^{n+1}(1-2n)b_n+(4a^nb(1-4n^2)-2\lambda^2a^{n+2}b(1+n))c_n$$
$$+2\lambda a^2b^n(n-2)(1+2n)s_n+4ab^n(4n^2-1)t_n\}\alpha'_n,$$

and

$$\begin{aligned}
a_n &= g_n(\lambda a)f_{n-1}(\lambda b) + f_n(\lambda a)g_{n-1}(\lambda b),\\
b_n &= g_{n-1}(\lambda a)f_{n-1}(\lambda b) - f_{n-1}(\lambda a)g_{n-1}(\lambda b),\\
c_n &= g_n(\lambda b)f_{n-1}(\lambda b) + f_n(\lambda b)g_{n-1}(\lambda b),\\
r_n &= g_n(\lambda a)f_{n-1}(\lambda a) + f_n(\lambda a)g_{n-1}(\lambda a),\\
s_n &= g_n(\lambda a)f_n(\lambda b) - f_n(\lambda a)g_n(\lambda b),\\
t_n &= g_n(\lambda b)f_{n-1}(\lambda a) + f_n(\lambda b)g_{n-1}(\lambda a).
\end{aligned}$$

5. Drag and torque

The force exerted by the fluid on the composite sphere is given by

$$\mathbf{D} = \mathbf{X}/Y, \tag{5.1}$$

where

$$\begin{aligned}
\mathbf{X} = &\Big\{12\pi\mu\lambda a^2\{(2a^3 + b^3)\lambda a_1 - 3b^2 s_1\}\alpha_1\\
&+20\pi\mu a^3\{(2a^4\lambda^2 + ab^3\lambda^2 + 12a^2)a_1 + 2\lambda(2a^3 + b^3)b_1\\
&+6b^2 c_1 - 12a^2 r_1 - 3ab^2\lambda s_1 - 6b^2 t_1\}\alpha_1'\Big\}(A_{11}\hat{i} + B_{11}\hat{j} + A_{10}\hat{k}),\\
= &\ 6\pi\mu\lambda a^2\{(2a^3 + b^3)\lambda a_1 - 3b^2 s_1\}[\mathbf{V}_0]_0\\
&+\pi\mu a^3\{(2a^4\lambda^2 + ab^3\lambda^2 + 12a^2)a_1 + 2\lambda(2a^3 + b^3)b_1\\
&+6b^2 c_1 - 12a^2 r_1 - 3ab^2\lambda s_1 - 6b^2 t_1\}[\nabla^2\mathbf{V}_0]_0,
\end{aligned} \tag{5.2}$$

(see Appendix)

$$\begin{aligned}
Y = &\{(2a^4\lambda^2 + ab^3\lambda^2 + 3a^2)a_1 - \lambda(2a^3 + b^3)b_1\\
&-3b^2 c_1 - 3a^2 r_1 - 3ab^2\lambda s_1 + 3b^2 t_1\},
\end{aligned} \tag{5.3}$$

and where $\mathbf{V}_0$ is the velocity corresponding to the basic flow, and $[\ ]_0$ denotes the evaluation at the origin $r = 0$.

Similarly, the torque $\mathbf{T}$ is given by

$$\begin{aligned}
\mathbf{T} &= 8\pi\mu\left\{a^3 + \frac{3a^2 s_1}{\lambda t_1}\right\}\xi_1(C_{11}\hat{i} + D_{11}\hat{j} + C_{10}\hat{k}),\\
&= 4\pi\mu\left\{a^3 + \frac{3a^2 s_1}{\lambda t_1}\right\}[\nabla\times\mathbf{V}_0]_0,
\end{aligned} \tag{5.4}$$

(see Appendix).

It is found that when $a = b$, in the limit $k \to 0$, i.e., $\lambda \to \infty$, we recover the well known Faxen's laws [6] for drag and torque acting on a rigid sphere of radius a, i.e.,

$$\begin{aligned} \mathbf{D} &= 6\pi\mu a[\mathbf{V}_0]_0 + \pi\mu a^3[\nabla^2\mathbf{V}_0]_0, \\ \mathbf{T} &= 4\pi\mu a^3[\nabla \times \mathbf{V}_0]_0. \end{aligned} \tag{5.5}$$

Similarly, when $b = 0$, we recover the expressions for drag and torque obtained by PADMAVATHI and AMARANATH [7] for the Stokes flow past a porous sphere, i.e.,

$$\begin{aligned} \mathbf{D} &= \frac{12\pi\mu a^3\lambda^2 f_1(\lambda a)[\mathbf{V}_0]_0}{((2a^2\lambda^2+3)f_1(\lambda a)+2a\lambda f_0(\lambda a))} \\ &\quad + \frac{2\pi\mu[(a^5\lambda^2+6a^3)f_1(\lambda a)-2a^4\lambda f_0(\lambda a)][\nabla^2\mathbf{V}_0]_0}{((2a^2\lambda^2+3)f_1(\lambda a)+2a\lambda f_0(\lambda a))}, \\ \mathbf{T} &= 4\pi\mu\left(\frac{a^3\lambda f_0(\lambda a)-3a^2 f_1(\lambda a)}{\lambda f_0(\lambda a)}\right)[\nabla \times \mathbf{V}_0]_0. \end{aligned} \tag{5.6}$$

6. Effective viscosity

The effective viscosity μ^* of a dilute suspension of composite porous spheres with rigid cores, each of outer radius a is found (as in [7]) to be

$$\mu^* = \mu\left\{1 + \frac{5R}{2S}\Phi\right\}, \tag{6.1}$$

where

$$\begin{aligned} R &= a\lambda[(3a^5+2b^5)\lambda a_2 - 10b^4 s_2], \\ S &= 2[(a\lambda^2(3a^5+2b^5)+30a^4)a_2 - 3\lambda(3a^5+2b^5)b_2 \\ &\quad -10ab(3b^2c_2+3a^2r_2+\lambda b^3 s_2)+30b^4 t_2], \end{aligned} \tag{6.2}$$

where Φ denotes the concentration by volume of the fluid containing the spheres.

When $a = b$, in the limit $k \to 0$, we obtain the well known formula due to EINSTEIN [8] for the effective viscosity of a dilute suspension of rigid spheres

$$\mu^* = \mu\left\{1 + \frac{5}{2}\Phi\right\}. \tag{6.3}$$

When $b \to 0$, we recover the formula obtained by PADMAVATHI and AMARANATH [7] for a dilute suspension of porous spheres of radius a

$$\mu^* = \mu\left\{1 + \frac{5(a^3\lambda^3 f_0(\lambda a)-3a^2\lambda^2 f_1(\lambda a))}{2[(a^3\lambda^3+10a\lambda)f_0(\lambda a)-30f_1(\lambda a)]}\Phi\right\}. \tag{6.4}$$

7. Examples

7.1. Stokeslet

Consider a Stokeslet of strength $F_1/8\pi\mu$ located at $(0, 0, c)$, $c > a$, its axis extending along the positive direction of the x-axis. The corresponding expressions for A_0 and B_0 due to the Stokeslet are [5]

$$
\begin{aligned}
A_0(r,\theta,\phi) &= \frac{F_1 R_1}{8\pi\mu c}(r\cos\theta - c + R_1)\frac{\cos\phi}{r\sin\theta}, \\
B_0(r,\theta,\phi) &= \frac{F_1}{4\pi\mu c}(r\cos\theta - c + R_1)\frac{\sin\phi}{r\sin\theta},
\end{aligned} \tag{7.1}
$$

where

$$
R_1^2 = r^2 + c^2 - 2cr\cos\theta. \tag{7.2}
$$

For $r < c$,

$$
\begin{aligned}
A_0(r,\theta,\phi) = \frac{F_1}{8\pi\mu}\sum_{n=1}^{\infty}\Bigg[&\frac{r^{n+2}}{(n+1)(2n+3)c^{n+2}} \\
&-\frac{(n-2)r^n}{n(n+1)(2n-1)c^n}\Bigg] P_n^1(\zeta)\cos\phi, \\
B_0(r,\theta,\phi) = \frac{F_1}{4\pi\mu}\sum_{n=1}^{\infty}&\left[\frac{r^n}{n(n+1)c^{n+1}}\right] P_n^1(\zeta)\sin\phi.
\end{aligned} \tag{7.3}
$$

The drag **D** and torque **T** are given by

$$
\mathbf{D} = \frac{M}{N}F_1\hat{i}, \tag{7.4}
$$

where

$$
\begin{aligned}
M = \Big(&3\lambda a^2c^2\{(2a^3+b^3)\lambda a_1 - 3b^2s_1\} \\
&+a^3\{(2a^4\lambda^2 + ab^3\lambda^2 + 12a^2)a_1 + 2\lambda(2a^3+b^3)b_1 \\
&+6b^2c_1 - 12a^2r_1 - 3ab^2\lambda s_1 - 6b^2t_1\}\Big), \\
N = 4c^3\{&(2a^4\lambda^2 + ab^3\lambda^2 + 3a^2)a_1 - \lambda(2a^3+b^3)b_1 \\
&-3b^2c_1 - 3a^2r_1 - 3ab^2\lambda s_1 + 3b^2t_1\},
\end{aligned} \tag{7.5}
$$

$$
\mathbf{T} = \left(a^3 + \frac{3a^2s_1}{\lambda t_1}\right)\frac{F_1}{c^2}\hat{j}.
$$

As before, the results for the rigid case [5] are recovered by putting $k \to 0$ i.e., $\lambda \to \infty$ and $a = b$.

$$\begin{aligned} \mathbf{D} &= \left(\frac{3a}{4c} + \frac{a^3}{4c^3}\right) F_1 \hat{i}, \\ \mathbf{T} &= \frac{a^3}{c^2} F_1 \hat{j}. \end{aligned} \tag{7.6}$$

Similarly when $b \to 0$, we recover the results obtained for the case of a porous sphere [7]

$$\begin{aligned} \mathbf{D} &= \left(\frac{(3a^3c^2\lambda^2 + a^5\lambda^2 + 6a^3)f_1(\lambda a) - 2a^4\lambda f_0(\lambda a)}{2c^3[(2a^2\lambda^2 + 3)f_1(\lambda a) + 2a\lambda f_0(\lambda a)]}\right) F_1 \hat{i}, \\ \mathbf{T} &= \frac{a^3\lambda f_0(\lambda a) - 3a^2 f_1(\lambda a)}{c^2\lambda f_0(\lambda a)} F_1 \hat{j}. \end{aligned} \tag{7.7}$$

7.2. Uniform flow

The basic, undisturbed flow is given by

$$\begin{aligned} A_0 &= \frac{U}{2} r\cos\theta, \\ B_0 &= 0, \\ \mathbf{D} &= \frac{6\pi\mu\lambda a^2\{(2a^3 + b^3)\lambda a_1 - 3b^2 s_1\}U}{\{Za_1 - \lambda(2a^3 + b^3)b_1 - 3b^2c_1 - 3a^2r_1 - 3ab^2\lambda s_1 + 3b^2t_1\}} \hat{k}, \\ \mathbf{T} &= 0, \end{aligned} \tag{7.8}$$

where

$$Z = 2a^4\lambda^2 + ab^3\lambda^2 + 3a^2.$$

This result agrees with that of MASLIYAH *et al.* [4] who solved the uniform flow past a composite porous sphere with a rigid core.

8. Conclusions

An infinite series solution and a representation for the solution of Brinkman's equations are presented. They are shown to be equivalent. It is found that this representation is very useful for discussing an arbitrary Stokes flow past a composite porous sphere with a rigid core, and a general method is suggested for finding the solution. The formulae to calculate drag and torque are given. The effective viscosity of a dilute suspension of composite porous spheres with rigid cores is calculated. The previous results pertaining to Stokes flow past rigid and porous spheres are recovered as special cases. It may be noted that the method suggested in this paper can also be used effectively to discuss the problem of Stokes flow past a porous spherical shell, where the rigid core in the present problem is replaced by a region filled with a viscous fluid.

Appendix

$$[\mathbf{V}_0]_0 = [2\,\mathrm{grad}\,A_0]_0 = 2\alpha_1[A_{11}\hat{i} + B_{11}\hat{j} + A_{10}\hat{k}],$$
$$[\nabla^2\mathbf{V}_0]_0 = 20\alpha_1'[A_{11}\hat{i} + B_{11}\hat{j} + A_{10}\hat{k}],$$
$$[\nabla \times \mathbf{V}_0]_0 = 2\xi_1[C_{11}\hat{i} + D_{11}\hat{j} + C_{10}\hat{k}].$$

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