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On finite deformation dynamic analysis of saturated soils

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A GENERAL FORMULATION is proposed to treat the dynamic response of saturated soils in finite deformation regime. Considering the soil as a saturated porous medium, the formulation for finite deformation analysis was established by extending Biot's classical theory to incorporate finite deformation effects. Particular attention was given to the flow of water through the soil while the soil skeleton undergoes a finite deformation. The derived formulation constitutes the theoretical basis for analysis of the liquefaction induced flow failure in soil embankments. Due to the integral form of the governing equations, they are specially suitable for application of numerical methods such as the finite element method.

Notations

$({}^{t}\beta), ({}^{t+\Delta t}\beta)$	body configurations at time t and $t + \Delta t$, respectively,
δ_{ij}	Kronecker delta,
${}^{0}\varrho, {}^{t}\varrho, {}^{t+\Delta t}\varrho$	mass densities of soil per unit volume in the configurations at time 0, t, $t + \Delta t$, respectively,
ϱ_s, ϱ_f	mass density of solid particles and pore water, respectively,
${}^{t+\Delta t}\sigma_{ij}$	Cartesian components of the Cauchy total stress tensor measured at time $t + \Delta t$,
$\sigma_{ij}, \overline{\sigma}_{ij}$	total stress and effective stress tensors, respectively,
$\overset{\nabla}{\sigma}, \overset{\nabla}{\overline{\sigma}}$	corotational rates of the total stress and effective stress tensors, respectively,
Ω	material spin tensor,
${}_{t}\Lambda_{ijkl}$	finite deformation tensor of tangent stiffness moduli,
${}^{t+\Delta t}b_i$	i-th component of body force per unit mass measured at time $t + \Delta t$,
${}^{t+\Delta t}_{t}b_i$	body force in the configuration at time $t + \Delta t$ $({}^{t+\Delta t}\beta)$ and measured in the configuration at time t $({}^{t}\beta)$,
${}^{0}dV, {}^{t+\Delta t}dV, {}^{t}dV$	volume of an infinitesimal element in the configuration at time 0, t, $t + \Delta t$,
${}^{t+\Delta t}e_{ij}$	Cartesian components of infinitesimal strain tensor measured at time $t + \Delta t$,
${}^{t+\Delta t}_{t}\varepsilon_{ij}$	the Green–Lagrange strain tensor,
${}^{t+\Delta t}f_i^B, {}^{t+\Delta t}f_i^S$	components of the applied body forces and surface traction, respectively, measured at time $t + \Delta t$,
${}^{t+\Delta t}_{t}F_i^S$	surface traction in the configuration at time $t + \Delta t$ $({}^{t+\Delta t}\beta)$ and measured in the configuration at time t $({}^{t}\beta)$,
$\overset{*}{k}_{ij}$	permeability tensor,
K_S, K_f	bulk moduli for solid particles and pore fluid, respectively,
n	porosity,
p	pore water pressure,

${}^{t+\Delta t}S_u$, ${}^{t+\Delta t}S_T$, ${}^{t+\Delta t}S_p$, ${}^{t+\Delta t}S_q$	different portions of body surface, respectively related to prescribed displacement, traction, pore pressure, and flow, measured at time $t + \Delta t$,
${}^{t+\Delta t}_{t}S_{ij}$	the second Piola–Kirchhoff stress tensor,
u_i	components of incremental displacement at time t,
${}^{t+\Delta t}u_i$, ${}^{t}u_i$	components of displacements at time $t + \Delta t$, and t, respectively,
$\dot{U}_i$	components of the absolute velocity of pore water in the direction of x_i,
$\dot{w}_i$	relative surface velocity of pore water with respect to the soil skeleton,
$({}^{t}x_1, {}^{t}x_2, {}^{t}x_3)$	coordinates of a generic particle of the body in Cartesian coordinate system at time t.

1. Introduction

ANALYSIS OF SOIL liquefaction and its consequences, such as permanent deformations in constructed facilities or earthen structures, requires a rational analytical procedure. Such a procedure should be based on a proper understanding of the physics and mechanics of soil as a particulate medium composed of three phases, i.e. solid particles, water, and air. Due to discontinuous nature of granular soils, it appears that the best approach to study the mechanics of soil is a *micro-mechanical* approach. In principle, if the behaviour of saturated granular soils on the microscopic scale was known, it would be possible to calculate the behaviour of granular soils on the macroscopic scale by applying appropriate statistical methods. In practice, however, such calculations are extremely difficult and, at the present time, limited to some simple cases. On the other hand, our knowledge of mechanical behaviour of soils is mainly based on observations and experimental studies of the samples of soils whose dimensions are large compared to those of an individual particle. In particular, most of the experimental results available in the field of soil mechanics are expressed in terms of the overall macroscopic quantities, such as confining pressure, axial stress, axial strain, etc., which indicate a wide acceptance of *continuum* approach in the study of soil behaviour. In a continuum approach, the particulate nature of soil is ignored and it is assumed that material is uniformly distributed throughout the regions of space. For dry soils or in the case of drainage processes for saturated soils, the regular equations of continuum mechanics may be used to formulate the problem. But in the case of saturated soils which are subjected to disturbances of transient nature, the effect of pore water pressure should be considered by a proper regularization of soil as a two-phase medium [4, 5] or a mixture of two different materials [23, 24, 43].

Both of the aforementioned approaches, i.e. the micro-mechanical and continuum approach, have received much attention during the past three decades. Micro-mechanical approaches have been continuously used to study some of the important features of granular soils, such as dilatancy, shear strength, and anisotropy. However, their application to boundary value problems has been started only recently by introduction of the distinct element method [10, 11, 12].

The distinct element method considers an assembly of large number of particles representing the soil mass and solves the dynamic equilibrium equations for each particle, subject to body forces and boundary interaction forces. The method can potentially handle nonlinearities which may arise from large displacements, rotation, slip, separation and material behaviour, but its performance is highly dependent upon the constitutive laws selected to represent the inter-particle forces. In addition to application of the distinct element method to the dry soils [2, 11], a few attempts have been reported [42] to utilize the method in a simulation of soil liquefaction. However, these developments are in the initial stages and the micro-mechanical approach is far from application to the real boundary value problems.

In contrast to micro-mechanical approach, the continuum approach has been successfully used in the analysis of geotechnical problems during the past few decades. Following the introduction of a coupled stress-flow formulation for dynamics of porous media by BIOT [4, 5], many investigators employed the new formulation to solve some practically significant boundary value problems using the finite element method [38, 48, 18, 19, 21, 40, 36, 37, 49, 50]. A historical review of such applications for liquefaction analysis is given in [33]. Recently ADVANI, *et al.* [1] have used a generalized form of the Biot's formulation for hygrothermo-mechanical evaluation of porous media under finite deformation regime. CHOPRA and DARGUSH [9] have also utilized the Biot's formulation for large deformation analysis of time-dependent problems.

In this paper, a generalized form of Biot's formulation for dynamics of porous media [5, 50] is derived by taking into account the finite deformation effects. The developed formulation serves as the basis for the numerical procedure proposed in a companion paper on the analysis of soil liquefaction and deformations in a finite deformation regime.

2. Statement of the problem

For a saturated earthen structure which occupies an initial volume of ${}^{0}V$ with the boundary surface ${}^{0}S$ at time 0, we seek to establish the governing field equations necessary to evaluate its equilibrium positions and entire time histories of responses during a quasi-static or transient process of deformation.

It is assumed that specified displacements, surface traction, pore water pressure, or water flow boundary conditions are defined on different portions of the boundary surface ${}^{t+\Delta t}S$ at a generic time $t + \Delta t$. These portions of the boundary surface are named ${}^{t+\Delta t}S_u$, ${}^{t+\Delta t}S_T$, ${}^{t+\Delta t}S_p$, and ${}^{t+\Delta t}S_q$, respectively. It is attempted to establish the governing equation without imposing any restriction on the magnitude of strains and displacements which the soil body may experience in the course of deformation. In order to deal with nonlinearities involved in the problem, an incremental analysis is adopted and the equilibrium position at time $t + \Delta t$ is searched for, assuming that the solutions for all time steps from time 0 to time t are known.

We adopt a Lagrangian (material) formulation and follow the material points in their motion. Therefore, in a generic time step from time t to time $t + \Delta t$, it is assumed that the initial configuration of the soil body (${}^0\beta$) and the configuration at time t (${}^t\beta$) are known and we are searching for the configuration of structure at time $t + \Delta t$ (${}^{t+\Delta t}\beta$). In the following development, an updated Lagrangian formulation is followed.

3. The principle of virtual work

Let us consider the motion of a generic point P of a saturated earth structure (Fig. 1). In the process of deformation from the initial configuration at time 0 to the configuration at time t, its coordinates with respect to a fixed Cartesian coordinate system change from $({}^0x_1, {}^0x_2, {}^0x_3)$ to $({}^tx_1, {}^tx_2, {}^tx_3)$, where the left-hand

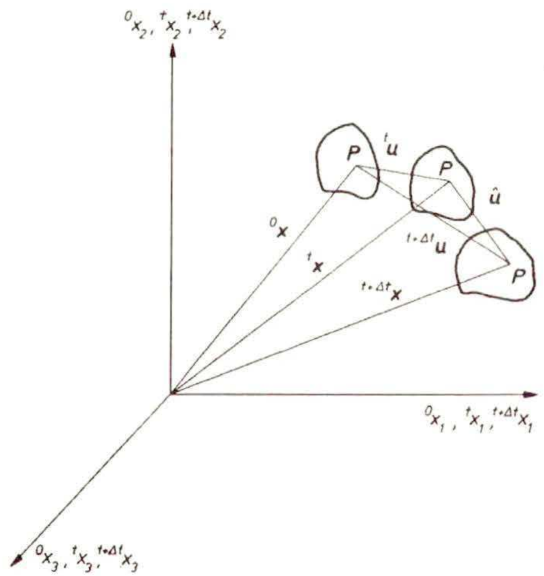

FIG. 1. Three different configuration of the soil body during its motion.

superscripts refer to the configuration of body, and the subscripts refer to different axes of the Cartesian coordinate system. In our analysis, we seek to find the position of each material point in the next configuration, i.e. at time $t + \Delta t$. Let us suppose that the soil body, in the configuration at time $t + \Delta t$, is subjected to a virtual displacement field $\delta\mathbf{u}$ which satisfies all the boundary conditions (Sec. 6). The principle of virtual work requires that the virtual work performed, when the soil body undergoes a virtual displacement $\delta\mathbf{u}$, is equal to the external work done by the body forces and surface traction, i.e.

$$(3.1)\qquad {}^{t+\Delta t}W_v^{\text{int}} = \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}\sigma_{ij}\delta\,{}_{t+\Delta t}e_{ij}\,{}^{t+\Delta t}dV$$
$$= {}^{t+\Delta t}\,W_v^{\text{ext}} = \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}f_i^B\delta u_i\,{}^{t+\Delta t}dV + \int\limits_{{}^{t+\Delta t}S_T} {}^{t+\Delta t}f_i^S\delta u_i\,{}^{t+\Delta t}dS,$$

where the ${}^{t+\Delta t}\sigma_{ij}$ are Cartesian components of the Cauchy total stress tensor, the ${}^{t+\Delta t}e_{ij}$ are Cartesian components of infinitesimal strain tensor, ${}^{t+\Delta t}f_i^B$ and ${}^{t+\Delta t}f_i^S$ are the components of the applied body forces and surface traction, respectively, and δu_i represents the components of virtual displacement field in the direction of axis i of the Cartesian coordinate system. The ${}^{t+\Delta t}S_T$ is a part of soil body surface on which a specified traction ${}^{t+\Delta t}f_i^S$ is applied. The $\delta\,{}_{t+\Delta t}e_{ij}$ is the variation in the small strain tensor defined as follows:

$$(3.2)\qquad \delta_{t+\Delta t}e_{ij} = \delta\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial^{t+\Delta t}x_j} + \frac{\partial u_j}{\partial^{t+\Delta t}x_i}\right)\right] = \frac{1}{2}\left(\frac{\partial(\delta u_i)}{\partial^{t+\Delta t}x_j} + \frac{\partial(\delta u_j)}{\partial^{t+\Delta t}x_i}\right),$$

where u_i is the incremental displacement at time t defined as

$$u_i = {}^{t+\Delta t}u_i - {}^{t}u_i$$

in which ${}^{t+\Delta t}u_i$ and ${}^{t}u_i$ denote the displacements at time $t+\Delta t$ and t, respectively. Note that the first term on the right-hand side of Eq. (3.2) implies the partial derivative of the variation u_i with respect to ${}^{t+\Delta t}x_j$.

In a dynamic loading of saturated soil systems, there are three contributions to the body forces ${}^{t+\Delta t}f_i^B$ in Eq. (3.1):

- ${}^{t+\Delta t}(\varrho b_i)$ body force due to gravity or centrifugal acceleration, where ${}^{t+\Delta t}\varrho$ is the mass density of the soil and ${}^{t+\Delta t}b_i$ is the i-th component of body force per unit mass, both measured at time $t + \Delta t$,
- ${}^{t+\Delta t}(\varrho\,\ddot{u}_i)$ body force due to acceleration of the soil skeleton ${}^{t+\Delta t}\,\ddot{u}_i$; negative sign is used because this force is in opposite direction to ${}^{t+\Delta t}\,\ddot{u}_i$,
- ${}^{t+\Delta t}f_i^{Bw}$ body force due to relative acceleration of the pore water with respect to the soil skeleton.

The first two terms are common in any structural dynamics problem, but the third term ${}^{t+\Delta t}f_i^{Bw}$ is due to the presence of water and its relative motion with respect to the soil skeleton. To account for ${}^{t+\Delta t}f_i^{Bw}$, we note that in a differential volume ${}^{t+\Delta t}dV$ of the soil with porosity n, only $(n\,{}^{t+\Delta t}dV)$ is occupied by the pore water, therefore $\varrho_f(n\,{}^{t+\Delta t}dV)$ is mass of the pore water available in the differential volume of the soil. Here ϱ_f is the mass density of pore water and the following relation holds:

$$\varrho = n\varrho_f + (1-n)\varrho_s, \tag{3.3}$$

where ϱ_s is the mass density of solid particles.

Now if we define a relative average or superficial displacement, w_i, so that $\dot{w}_i$ is the relative superficial velocity [1] of the pore water with respect to soil skeleton (in the direction of axis i, $i = 1,2,3$), the actual displacement of water in the pores is w_i/n. The body force due to the relative acceleration of pore water with respect to the soil skeleton is expressed by

$$^{t+\Delta t}f_i^{Bw} = -\frac{\left[\varrho_f\left(n\,^{t+\Delta t}dV\right)\right]\left(\dfrac{1}{n}\dfrac{D\dot{w}_i}{Dt}\right)}{^{t+\Delta t}dV} = -\varrho_f\frac{D\dot{w}_i}{Dt}, \tag{3.4}$$

where D/Dt is the symbol of total derivative with respect to time [2]. Here we must use a total time derivative, because $\dot{w}_i$ is measured with respect to the soil skeleton that itself is moving and makes a moving coordinate system for measuring $\dot{w}_i$. The negative sign in Eq. (3.4) is used because the $^{t+\Delta t}f_i^{Bw}$ applies in the opposite direction of water flow. It must also be noted that the effect of change of porosity has been ignored in the acceleration term in Eq. (3.4). This effect will be very small during a usual time step.

Considering the above mentioned contributions to the body force $^{t+\Delta t}f_i^{Bw}$, we can now write Eq. (3.1) as

$$\begin{aligned} ^{t+\Delta t}W_v^{\text{int}} &= \int\limits_{^{t+\Delta t}V} {^{t+\Delta t}\sigma_{ij}}\,\delta\,{_{t+\Delta t}e_{ij}}\,{^{t+\Delta t}dV} \\ &= {^{t+\Delta t}W_v^{\text{ext}}} = \int\limits_{^{t+\Delta t}S_T} {^{t+\Delta t}f_i^S}\,\delta u_i^S\,{^{t+\Delta t}dS} \\ &+ \int\limits_{^{t+\Delta t}V}\left[{^{t+\Delta t}\varrho}\,{^{t+\Delta t}b_i} - {^{t+\Delta t}\varrho}\,{^{t+\Delta t}\ddot{u}_i} - {^{t+\Delta t}\varrho_f}\,{^{t+\Delta t}\left(\frac{D\dot{w}_i}{Dt}\right)}\right]\delta u_i\,{^{t+\Delta t}dV}. \end{aligned} \tag{3.5}$$

There are two major difficulties in application of Eq. (3.5) to a finite deformation problem involving saturated soils. First, the configuration at time $t+\Delta t$ is

(1) This is the superficial velocity of water used in Darcy's law for seepage of water through a porous medium, i.e. $\dot{w}_i = v_i = k_{ij}(\partial h/\partial x_j)$, where k_{ij} is the hydraulic conductivity of the soil in the direction i due to a unit flow in the direction j and h is the hydraulic potential at the point of interest.

(2) The material time derivative or the rate of a quantity, $A = A(x(t), t)$ is defined as

$$\frac{DA}{Dt} = \dot{A} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x_i}\dot{x}_i,$$

where A is a scalar quantity and it is a function of time and space.

unknown and the integration over the ${}^{t+\Delta t}V$ and ${}^{t+\Delta t}S_T$ cannot be performed before calculating the equilibrium position at time $t + \Delta t$. The second difficulty is the presence of total stress tensor, ${}^{t+\Delta t}\sigma_{ij}$ in Eq. (3.5) which does not have any direct influence on the mechanical behaviour of the soil and cannot be used in a realistic constitutive equation relating a proper measure of stress to a measure of strain. To resolve the first difficulty, we can rewrite Eq. (3.5) by referring the applied forces, stresses, and strains to a known equilibrium configuration, such as the initial configuration at time 0 (Total Lagrangian Formulation) or the configuration at time t (Updated Lagrangian Formulation). The second problem can be resolved by applying the principle of effective stress and introducing effective stresses in Eq. (3.5). The aforementioned measures are adopted in the following sections.

4. The principle of effective stress

Terzaghi's principle of effective stress can be written in the following form:

$$\sigma_{ij} = \overline{\sigma}_{ij} - p\delta_{ij}, \tag{4.1}$$

where σ_{ij} and $\overline{\sigma}_{ij}$ are the total stress and effective stress tensors, respectively, and p stands for the pore water pressure. The δ_{ij} is the Kronecker delta defined as

$$\begin{aligned} \delta_{ij} &= 1 \qquad \text{for} \quad i = j, \\ \delta_{ij} &= 0 \qquad \text{for} \quad i \neq j. \end{aligned}$$

In direct notation, Eq. (4.1) can be written as

$$\sigma = \overline{\sigma} - p\mathbf{1}, \tag{4.2}$$

where $\mathbf{1}$ is the symbolic form of the Kronecker delta.

Here the conventional sign convention of solid mechanics is used which considers tensile stresses as positive values and compressive stresses as negative values. The negative sign of p in Eqs. (4.1) or (4.2) is associated with the fact that pore pressure is considered as a compressive stress.

Since the effective stress principle is defined in terms of the Cauchy stress tensor which is not an objective measure of stress, it is important to establish a suitable rate form for Eq. (4.2). Taking the time derivative of Eq. (4.2), we find

$$\frac{D}{Dt}(\boldsymbol{\sigma}) = \frac{D}{Dt}(\overline{\boldsymbol{\sigma}}) - \frac{D}{Dt}(p\mathbf{1}) \tag{4.3}$$

or

$$\begin{aligned} \overset{\triangledown}{\boldsymbol{\sigma}} + \sigma_{ij}(\dot{\mathbf{e}}_i \otimes \mathbf{e}_j) + \sigma_{ij}(\mathbf{e}_i \otimes \dot{\mathbf{e}}_j) = \overset{\triangledown}{\overline{\boldsymbol{\sigma}}} + \overline{\sigma}_{ij}(\dot{\mathbf{e}}_i \otimes \mathbf{e}_j) + \overline{\sigma}_{ij}(\mathbf{e}_i \otimes \dot{\mathbf{e}}_j) \\ - \dot{p}\,\mathbf{1} - p\delta_{ij}(\dot{\mathbf{e}}_i \otimes \mathbf{e}_j) - p\delta_{ij}(\mathbf{e}_i \otimes \dot{\mathbf{e}}_j), \end{aligned} \tag{4.4}$$

where $\mathbf{e}_i$ and $\mathbf{e}_j$ are the unit vectors in a Cartesian coordinate system. The $\overset{\triangledown}{\boldsymbol{\sigma}}$ and $\overset{\triangledown}{\overline{\boldsymbol{\sigma}}}$ are corotational rates of the total stress and effective stress tensors, respectively. Using Eq. (4.2), we can write the above equation as

$$\overset{\triangledown}{\boldsymbol{\sigma}} = \overset{\triangledown}{\overline{\boldsymbol{\sigma}}} - \dot{p}\,\mathbf{1}\,. \tag{4.5}$$

Equation (4.5) is of paramount importance in our subsequent developments. We will use this equation in development of the incremental equations governing the dynamics of saturated soils.

As previously mentioned, Eqs. (4.2) and (4.5) enable us to formulate the governing equations of motion (Eq. $(3.5)_1$) in terms of effective stresses. However, substitution of Eq. (4.2) in Eq. (3.5) leads to the appearance of a pore pressure related term which prevents a direct application of Eq. (3.5) as a sole field equation in the solution of boundary value problems in soil dynamics. The additional unknown, p, requires an additional field equation which governs the flow of water through the soil. Derivation of this equation is the subject of the next section.

5. Equations governing the flow of water through a saturated soil

In Sec. 3, we derived an integral equation governing the motion of the soil mass by making use of the principle of virtual work for the bulk mass of the soil body. In this section, we consider the equations of motion and mass balance for the pore fluid (water) alone in order to establish a complementary equation to Eq. (3.5). To this end, let us consider a unit volume of the soil in the current configuration at time $t + \Delta t$ as a control volume for the flow of the pore water. We assume that the coordinate system is attached to the soil skeleton and is moving with it. The flow of water in this control volume is affected by inertial forces and by a viscous (velocity-dependent) drag force due to interaction of the pore water and solid grains. In the following consideration, it is assumed that the viscous drag force can be determined by application of Darcy's law. In a quasi-static flow of the pore water, Darcy's equation is written as

$$\dot{w}_i = -k_{ij}\frac{\partial p}{\partial x_j} \tag{5.1}$$

in which

$$k_{ij} = \frac{1}{\gamma_w}\overset{*}{k}_{ij}\,, \tag{5.2}$$

where $\overset{*}{k}_{ij}$ is a component of the permeability tensor. The $\dot{w}_i$ in Eq. (5.1) is the superficial velocity of water, i.e. the volume of water flowing per unit time and per unit gross area through the face of the control volume perpendicular to the

x_i axis. The negative sign in Eq. (5.1) emphasizes that the water flow occurs in the direction of decreasing potential.

Now if we define the resistivity tensor r_{ij} as the inverse of the specific permeability tensor,

$$r_{ij}k_{jk} = \delta_{ik}\,, \tag{5.3}$$

Eq. (5.1) can be written as:

$$\frac{\partial p}{\partial x_i} = -r_{ij}\,\dot{w}_j = \Re_i\,, \tag{5.4}$$

where $\Re_i$ is the viscous drag force in the direction of x_i axis applied to the pore water flowing through a unit control volume of the soil. Considering the effects of the inertial and body forces (Fig. 2), Eq. (5.4) can be generalized,

$$-\frac{\partial p}{\partial x_i} - r_{ij}\,\dot{w}_j + \varrho_f\left(b_i - \ddot{u}_i - \frac{D\,\dot{w}_i}{Dt}\right) = 0, \tag{5.5}$$

where

$$\ddot{u}_i + \frac{D\,\dot{w}_i}{Dt} = \ddot{u}_i^{\,\mathrm{tot}}$$

represents the total acceleration of pore water.

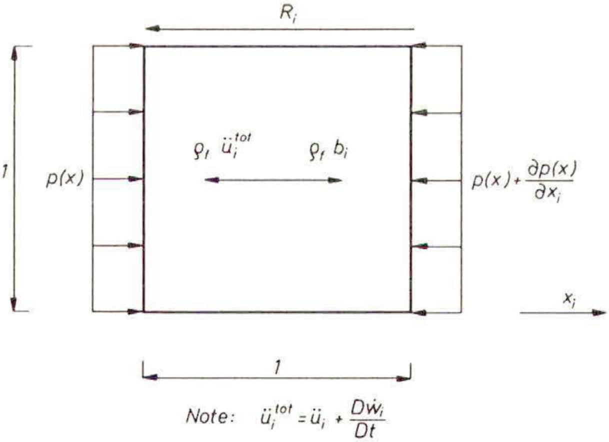

FIG. 2. Free body diagram for the pore fluid in a control volume.

In order to reduce Eq. (5.5) to a form containing only the displacements of soil skeleton (**u**) and pore water pressure (p), we first use the axiom of mass balance to establish a relationship between the rate of change of pore pressure $\dot{p}$ and the rates of volumetric strains for the pore water $\dot{w}_{i,i}$ and the soil skeleton $\dot{u}_{i,i}$. Such a relationship can be used to remove the relative displacement of the pore water w from Eq. (5.5).

Let us consider a unit volume of the soil in which the masses of the pore water and solid grains are respectively $n\varrho_f$ and $(1-n)\varrho_S$. The axiom of mass balance requires that in the process of flow of the water through the soil, these two masses must be conserved, i.e.

$$\frac{D}{Dt}\int_V (n\varrho_f)\,dV = 0, \tag{5.6}$$

$$\frac{D}{Dt}\left[\int_V (1-n)\varrho_S\,dV\right] = 0. \tag{5.7}$$

Equations (5.6) and (5.7) lead to:

$$\dot{n}\varrho_f + n\dot{\varrho}_f + (n\varrho_f)\dot{U}_{i,i} = 0, \tag{5.8}$$

$$-\dot{n}\varrho_S + (1-n)\dot{\varrho}_S + (1-n)\varrho_S\,\dot{u}_{i,i} = 0, \tag{5.9}$$

where $\dot{U}_i$ is the component of the absolute velocity of pore water in the direction of x_i axis, i.e.

$$\dot{w}_i = n(\dot{U}_i - \dot{u}_i). \tag{5.10}$$

Dividing Eqs. (5.8) and (5.9) by ϱ_f and ϱ_S, respectively, and adding up these two equations, we find:

$$\frac{n\dot{\varrho}_f}{\varrho_f} + (1-n)\frac{\dot{\varrho}_S}{\varrho_S} + \left[n(\dot{U}_{i,i} - \dot{u}_{i,i})\right] + \dot{u}_{i,i} = 0, \tag{5.11}$$

or by using Eq. (5.10), we have:

$$\frac{n\dot{\varrho}_f}{\varrho_f} + (1-n)\frac{\dot{\varrho}_S}{\varrho_S} + \dot{w}_{i,i} + \dot{u}_{i,i} = 0. \tag{5.12}$$

The first term in the above equation represents the compressibility of the pore fluid (water) which is of cardinal importance in dynamic analysis of saturated soils. In order to stress the importance of this term, it suffices to mention that the compressibility of pore water (fluid) is highly dependent on the degree of saturation of the soil, and a small fraction of percentage of air in the pore water may significantly increase its compressibility [31]. The second term in Eq. (5.12) accounts for the compressibility of solid grains and, in general, is much smaller than the first term. In the following considerations, we seek to substitute the first two terms in Eq. (5.12) by means of simple constitutive equations. To this end, we note that a change of effective stress will result in a change of volume of solid particles, while a pore pressure change will induce a change of volume in both the solid particles and the pore water. Thus

$$(5.13)\qquad \begin{aligned} (\dot{\varrho}_f) &= \frac{\partial(\varrho_f)}{\partial p}\dot{p}\,, \\ (\dot{\varrho}_S) &= \frac{\partial(\varrho_S)}{\partial p}\dot{p} + \frac{\partial(\varrho_S)}{\partial(\overline{\sigma}_{ii})}\dot{\overline{\sigma}}_{ii}\,. \end{aligned}$$

In practice, the $\dot{\varrho}_S$ is very small and negligible as compared to the $\dot{\varrho}_f$. Thus it can be ignored in the subsequent procedure. However, it is kept in the formulation for the comparison purposes. It is noted that the constitutive law representing the change of ϱ_S is similar for the change of hydrostatic pressure or the change of pore water pressure. Therefore, the terms on the right-hand side in Eq. $(5.13)_2$ can be described in terms of the change of hydrostatic total stress (σ_{ii}), i.e.

$$(5.13)_3\qquad (\dot{\varrho}_S) = \frac{\partial(\varrho_S)}{\partial\sigma_{ii}}\dot{\sigma}_{ii}\,.$$

It is also assumed that the following linear relationships exist between the change of pore water pressure (or any hydrostatic pressure) and the changes of volumes of the pore water and solid grains:

$$(5.14)\qquad \frac{\dfrac{\partial V_S}{\partial p}}{V_S} = \frac{1}{K_S}\,,$$

$$(5.15)\qquad \frac{\dfrac{\partial V_w}{\partial p}}{V_w} = \frac{1}{K_f}\,,$$

where V_S and V_w are the volumes of solid grains and the pore water in a unit volume of the soil mixture, respectively, while K_S and K_f indicate the compressibility of the above constituents. In general, K_S is by several orders of magnitude larger than K_f. Considering $V_S = (1-n)\varrho_S$ and $V_w = n\varrho_f$, and ignoring the change of soil porosity due to the change of p, we can rewrite Eqs. (5.14) and (5.15) as

$$(5.16)\qquad \frac{\dfrac{\partial \varrho_S}{\partial p}}{\varrho_S} = \frac{1}{K_S}\,,$$

$$(5.17)\qquad \frac{\dfrac{\partial \varrho_f}{\partial p}}{\varrho_f} = \frac{1}{K_f}\,.$$

Substituting Eqs. (5.16) and (5.17) in Eq. (5.12), we finally find the equation of mass balance in a desired form:

$$\left(\frac{n}{K_f} + \frac{1-n}{K_S}\right)\dot{p} + \dot{w}_{i,i} + \dot{u}_{i,i} = 0. \tag{5.18}$$

Denoting:

$$\frac{1}{\Gamma} = \frac{n}{K_f} + \frac{1-n}{K_S} \tag{5.19}$$

Eq. (5.18) is written as

$$\frac{1}{\Gamma}\dot{p} + \dot{w}_{i,i} + \dot{u}_{i,i} = 0. \tag{5.20}$$

Equations (5.20) and (5.5) yield the following relations:

$$\dot{w}_{i,i} = \frac{\partial}{\partial x_i}\left\{k_{ij}\left[-\frac{\partial p}{\partial x_j} + \varrho_f\left(b_j - \ddot{u}_j - \frac{D\dot{w}_j}{Dt}\right)\right]\right\} = -\frac{1}{\Gamma}\dot{p} - \dot{u}_{i,i} \tag{5.21}$$

or

$$\frac{1}{\Gamma}\dot{p} + \dot{u}_{i,i} - \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial p}{\partial x_j}\right) + \frac{\partial}{\partial x_i}(k_{ij}\varrho_f b_j) - \frac{\partial}{\partial x_i}\left[k_{ij}\varrho_f\left(\ddot{u}_j + \frac{D\dot{w}_j}{Dt}\right)\right] = 0. \tag{5.22}$$

This is the final equation governing the flow of the pore fluid (water) through the soil and combines the axiom of mass conservation and equation of motion for the pore fluid. Presence of the term $D\dot{w}_j/Dt$ in the above equation is still an undesirable feature which inhibits a direct coupling of Eq. (5.22) with Eq. (3.5) in order to get a coupled systems of equations in terms of **u** and p. However, it has been shown [49] that for the range of frequencies encountered in the earthquake loading, the relative acceleration of the pore water with respect to the soil skeleton is negligible. Therefore by ignoring $D\dot{w}_j/Dt$ in Eq. (5.22), we find:

$$\frac{1}{\Gamma}\dot{p} + \dot{u}_{i,i} - \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial p}{\partial x_j}\right) + \frac{\partial}{\partial x_i}(k_{ij}\varrho_f b_j) - \frac{\partial}{\partial x_i}(k_{ij}\varrho_f\,\ddot{u}_j) = 0. \tag{5.23}$$

Equation (5.23) is written in terms of u and p and is suitable to be solved in combination with Eq. (3.5), for which we also neglect the $D\dot{w}_j/Dt$ term for the foregoing reasons.

6. Boundary conditions

As it was mentioned in Sec. 2, we assume that four types of boundary conditions are specified on different portions of the boundary surface ${}^{t+\Delta t}S$ of the soil body at a generic time $t + \Delta t$. These boundary conditions are defined in the following sub-sections.

6.1. Displacement boundary condition

It is assumed that on a portion of the boundary surface ${}^{t+\Delta t}S$ of the soil body, displacements of soil skeleton are specified as follows:

$$(6.1) \qquad {}^{t+\Delta t}u_i = {}^{t+\Delta t}\overline{u}_i \qquad \text{on} \quad {}^{t+\Delta t}S_u \,,$$

where ${}^{t+\Delta t}\overline{u}_i$ is the specified value of displacement on the boundary surface ${}^{t+\Delta t}S_u$ at time $t + \Delta t$.

6.2. Pore pressure boundary condition

The pore water pressure boundary condition is defined on ${}^{t+\Delta t}S_p$ as follows:

$$(6.2) \qquad {}^{t+\Delta t}p = {}^{t+\Delta t}\overline{p} \qquad \text{on} \quad {}^{t+\Delta t}S_p \,,$$

where ${}^{t+\Delta t}\overline{p}$ is the specified pressure on the surface ${}^{t+\Delta t}S_p$ at time $t + \Delta t$.

6.3. Traction boundary condition

We assume that on a portion of the boundary surface, there is a specified traction which must be in equilibrium with the internal total stresses, i.e.

$$(6.3) \qquad {}^{t+\Delta t}\sigma_{ij} n_j = {}^{t+\Delta t}f_i^S \qquad \text{on} \quad {}^{t+\Delta t}S_T \,,$$

where the ${}^{t+\Delta t}f_i^S$ is the specified traction on the surface ${}^{t+\Delta t}S_T$ with a unit normal of $\mathbf{n}$, and ${}^{t+\Delta t}\sigma_{ij}$ is the total Cauchy stress tensor acting on the neighborhood of the ${}^{t+\Delta t}f_i^S$.

6.4. Water flow boundary condition

It is assumed that on some portion of the boundary surface, the water flow boundary conditions are specified. One of the typical examples of such boundary conditions is the impervious boundary. The water flow boundary condition follows from Eq. (5.8) and is expressed as a flux condition, i.e.

$$(6.4) \qquad \dot{w}_i n_i = \left[-k_{ij} \frac{\partial p}{\partial x_j} + k_{ij} \varrho_f b_j - k_{ij} \varrho_f \left(\ddot{u}_j + \frac{D\dot{w}_j}{Dt} \right) \right] n_i = {}^{t+\Delta t}\overline{q}_S \quad \text{on} \quad {}^{t+\Delta t}S_q,$$

where n_i denotes the i-th component of the outward unit normal to the surface ${}^{t+\Delta t}S_q$, and ${}^{t+\Delta t}\overline{q}_S$ is the prescribed fluid flow on the ${}^{t+\Delta t}S_q$.

7. Constitutive equations for the soil skeleton

The governing field equations developed in Sec. 3 (Eq. (3.5)) and in Sec. 5 (Eq. (5.23)) along with the boundary conditions defined in Sec. 6 are not sufficient to solve a boundary value problem in soil dynamics. For ten unknowns (3 displacements of soil skeleton, pore pressure, and six components of stress tensor) in a boundary value problem, we have established only four governing equations. Thus six constitutive equations are necessary to make the problem well-posed. Due to nonlinearity of soil behaviour, it is desirable to define the constitutive equations in a rate form relating an appropriate measure of stress to the rate of deformation. In a finite deformation analysis, an objective stress rate must be used to ensure that the effects of rigid body rotation are correctly considered. This criterion, however, does not determine completely which stress rate should be used. There are different forms of stress rates which satisfy the objectivity requirement. The most commonly used objective stress rate is the JAUMANN [27, 28] corotational rate of the Cauchy stress tensor, $\overset{\triangledown}{\sigma}_{ij}$, defined as follows:

$$\overset{\triangledown}{\sigma}_{ij} = \dot{\sigma}_{ij} + \sigma_{ik}\Omega_{kj} + \sigma_{jk}\Omega_{ki}\,, \tag{7.1}$$

where $\dot{\sigma}_{ij}$ is a Cartesian component of the material (time) derivative of the Cauchy stress tensor, and Ω_{ij} is a Cartesian component of the spin tensor, i.e.

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} - \frac{\partial \dot{u}_j}{\partial x_i}\right). \tag{7.2}$$

Numerous application of the Jaumann stress rate have been reported in the finite deformation analysis of crystalline solids in the crystal plasticity context (e.g. 22, 35). In a crystal plasticity application, the material spin tensor $\mathbf{\Omega}$ is replaced by the rate of rotation or spin of the crystal lattice. However for non-crystalline solids, a proper choice of the spin tensor is not clear. Previous study by NAGTEGAAL and DE JOND [34] has shown that a direct application of Eq. (7.2) in the large strain simple shear analysis of a material obeying a Mises-type kinematic hardening plasticity results in an oscillatory response during monotonic shearing. Such an unrealistic result has motivated several investigators (e.g. reference [13]) to explore the possibility of removing the stress oscillation by using different spin tensors. Later the original suggestion by MANDEL [32] and KRATOCHVIL [29] for a decomposition of the spin tensor to an "elastic" or "rigid" part and a plastic part, and Mandel's concept of material underlying substructure, motivated DAFALIAS [13, 14, 15] and LORET [30] to propose some constitutive equations for the plastic spin in the case of anisotropic materials. These studies suggested that the "elastic" part of the spin tensor must be used in a Jaumann-type corotational rate.

The concept of plastic spin has received increasing attention in the recent years and many investigators have studied the effect of plastic spin on the large deformation of solid materials (e.g. [44, 45]). One interesting point shown in the closed form analytical solutions presented by DAFALIAS [13, 14, 15] is that unless strong initial anisotropy preexists, the difference in the material response between using the substructure and material spin for a material which is initially isotropic becomes important only after very large strains (of the order of 100%) are developed.

In the light of the above discussion and due to the lack of experimental data necessary for calibration of the constitutive equations for the plastic spin, we will use a corotational stress rate without restricting the formulation to particular choices of the spin tensor.

Assuming an inelastic behaviour for the soil skeleton, we choose the following incremental form to relate the corotational rate of the effective stress tensor $\overset{\nabla}{\overline{\sigma}}_{ij}$ to the rate of deformation tensor $d_{kl} = 1/2(\partial \dot{u}_k/\partial x_l + \partial \dot{u}_l/\partial x_k)$,

$$\overset{\nabla}{\overline{\sigma}}_{ij} = D_{ijkl} d_{kl}, \tag{7.3}$$

where $\mathbf{D}$ is the *tangential stiffness tensor* which may be a function of the current state of effective stresses, strains and some internal variables.

The specific form of the tangential stiffness tensor will depend upon the type of mathematical framework (e.g., elasticity, plasticity, viscoplasticity, etc.) that we choose to model the behaviour of the soil skeleton. Equation (7.3) is general enough to enclose a wide variety of existing frameworks for the soil constitutive modeling.

8. Expression of the virtual work equation in terms of the coordinates of the configuration at time t

As mentioned in Sec. 3, all the integrals appeared in Eq. (3.5) must be written in terms of a known configuration, such as the initial configuration of the soil body (${}^0\beta$) or its converged equilibrium position at the end of the previous time step (${}^t\beta$). Here we choose the latter option and our aim in this section is to rewrite Eq. (3.5) in terms of the coordinates of the configuration at time t.

Let us consider an infinitesimal cubic element of the soil body (Fig. 3) whose volume in the configuration at time t can be expressed as ${}^t dV = \prod_{i=1}^{3} dx_i$. During the motion of soil from time t to time $t + \Delta t$, the material enclosed in the cubic element ${}^t dV$ will occupy a new volume of ${}^{t+\Delta t}dV$ and the initial shape of the element will be distorted. Considering the axiom of mass balance, we can relate ${}^{t+\Delta t}dV$ to ${}^t dV$ by the following equation:

$$ {}^0\varrho\, {}^0 dV = {}^t\varrho\, {}^t dV = {}^{t+\Delta t}\varrho\, {}^{t+\Delta t}dV, \tag{8.1}$$

where ${}^{0}\varrho$, ${}^{t}\varrho$, and ${}^{t+\Delta t}\varrho$ are the mass densities per unit volume in the configurations at time 0, t, $t+\Delta t$, respectively. The ${}^{0}dV$ is the volume of the infinitesimal element in the initial configuration at time 0 (${}^{0}\beta$).

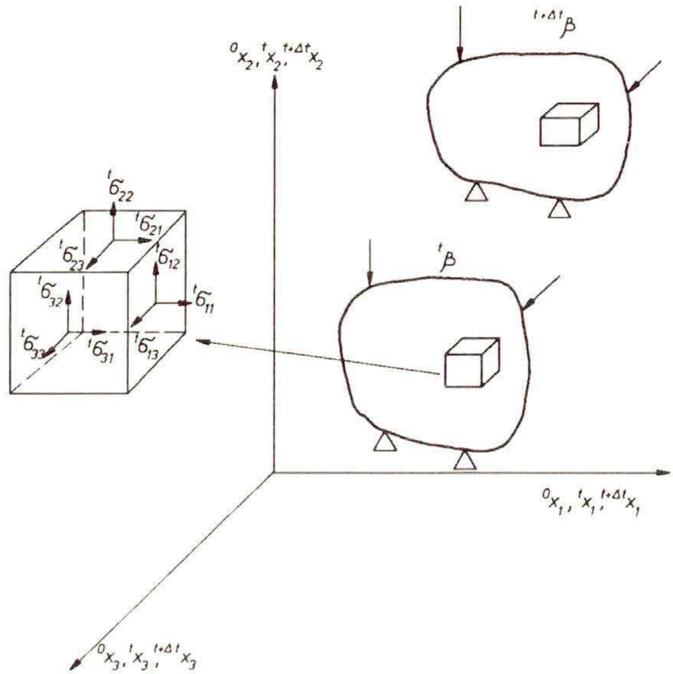

FIG. 3. The soil body at two subsequent configurations.

In general, the external loading, such as surface traction, external water pressure, gravitational and centrifugal loading are deformation-dependent. However, in most geotechnical structures, the aforementioned loading does not induce such a large displacement, large strain, or large rotation which would require a finite deformation analysis. Therefore, it is reasonable to assume that the magnitude and direction of surface force and body forces are independent of the current configuration of the soil body, i.e. [3]

$$
\begin{aligned}
{}^{t+\Delta t}b_i &= {}^{t+\Delta t}_{t}b_i\,, \\
{}^{t+\Delta t}f_i^S\;{}^{t+\Delta t}dS &= {}^{t+\Delta t}_{t}f_i^S\;{}^{t}dS\,,
\end{aligned}
\tag{8.2}
$$

where ${}^{t+\Delta t}_{t}b_i$ and ${}^{t+\Delta t}_{t}f_i^S$ are respectively the body force and surface traction in the configuration at time $t+\Delta t$ (${}^{t+\Delta t}\beta$), and measured in the configuration at time t (${}^{t}\beta$). Combining Eqs. (8.1) and $(8.2)_1$, we have:

$$
{}^{t+\Delta t}\varrho\;{}^{t+\Delta t}b_i\;{}^{t+\Delta t}dV = {}^{t}\varrho\;{}^{t+\Delta t}_{t}b_i\;{}^{t}dV\,. \tag{8.3}
$$

If we further assume that the effect of the pore water relative acceleration $D\dot{w}_j/Dt$ with respect to the soil skeleton is negligible as compared to the inertial effect of the soil bulk mass, we can rewrite Eq. (3.5) by using Eqs. (8.1), (8.2) and (8.3):

$$
(8.4)\qquad {}^{t+\Delta t}W_{\nu}^{\text{ext}} = \int\limits_{{}^tV} {}^t\varrho \; {}^{t+\Delta t}_{t}b_i \; \delta u_i \; {}^t dV - \int\limits_{{}^0V} {}^0\varrho \; {}^{t+\Delta t}\ddot{u}_i \; \delta u_i \; {}^0dV + \int\limits_{{}^tS_T} {}^{t+\Delta t}_{t}f_i^S \; \delta u_i^S \; {}^t dS.
$$

The second integral on the r.h.s. in Eq. (8.4) is evaluated using the initial configuration (${}^0\beta$) and hence its contribution can be calculated prior to the incremental step-by-step analysis.

As to the internal virtual work (Eq. $(3.5)_1$), we first use the principle of effective stress (Eq. (4.1)) to rewrite $(3.5)_1$ in terms of effective stresses. Thus, substituting Eq. (4.1) in Eq. $(3.5)_1$ leads to

$$
{}^{t+\Delta t}W_{\nu}^{\text{int}} = \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}\sigma_{ij} \; \delta \; {}_{t+\Delta t}e_{ij} \; {}^{t+\Delta t}dV = \int\limits_{{}^{t+\Delta t}V} \left({}^{t+\Delta t}\overline{\sigma}_{ij} - {}^{t+\Delta t}p \; \delta_{ij} \right) \delta \; {}_{t+\Delta t}e_{ij} \; {}^{t+\Delta t}dV
$$

or

$$
(8.5)\qquad {}^{t+\Delta t}W_{\nu}^{\text{int}} = \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}\overline{\sigma}_{ij} \; \delta \; {}_{t+\Delta t}e_{ij} \; {}^{t+\Delta t}dV - \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}p \; \delta_{ij}\delta \; {}_{t+\Delta t}e_{ij} \; {}^{t+\Delta t}dV.
$$

We now need to refer the Cauchy effective stress tensor ${}^{t+\Delta t}\overline{\sigma}_{ij}$ and the infinitesimal strain tensor ${}_{t+\Delta t}e_{ij}$ to the configuration at time t (${}^t\beta$). It is well known that the *second Piola–Kirchhoff stress tensor* ${}^{t+\Delta t}_{t}S_{ij}$ and the *Green–Lagrange strain tensor* ${}^{t+\Delta t}_{t}\varepsilon_{ij}$ are a work-conjugate pair of stress and strain measures which relate the ${}^{t+\Delta t}\overline{\sigma}_{ij}$ and ${}_{t+\Delta t}e_{ij}$ to the configuration at time t. The second Piola - Kirchhoff stress tensor ${}^{t+\Delta t}_{t}S_{ij}$ is defined as [8]:

$$
(8.6)\qquad {}^{t+\Delta t}_{t}S_{ij} = \frac{{}^t\varrho}{{}^{t+\Delta t}\varrho} \frac{\partial^t x_i}{\partial^{t+\Delta t}x_m} \overline{\sigma}_{mn} \frac{\partial^t x_j}{\partial^{t+\Delta t}x_n} .
$$

The Green - Lagrange strain tensor can be defined by considering the deformation of a generic line segment of the soil body whose lengths are denoted by ${}^t ds$ and ${}^{t+\Delta t}ds$ in the configurations at time t and $t + \Delta t$, respectively. Without giving the details of this derivation, we find [8]:

$$
(8.7)\qquad {}^{t+\Delta t}_{t}\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial^t x_j} + \frac{\partial u_j}{\partial^t x_i} + \frac{\partial u_k}{\partial^t x_i} \frac{\partial u_k}{\partial^t x_j} \right) .
$$

Taking a variation of the both sides of Eq. (8.7), we have

$$\delta\, {}^{t+\Delta t}_{t}\varepsilon = \frac{1}{2}\left[\frac{\partial(\delta u_i)}{\partial^t x_j} + \frac{\partial(\delta u_j)}{\partial^t x_i} + \frac{\partial(\delta u_k)}{\partial^t x_i}\frac{\partial u_k}{\partial^t x_j} + \frac{\partial u_k}{\partial^t x_i}\frac{\partial(\delta u_k)}{\partial^t x_j}\right], \tag{8.8}$$

where δu_i is the variation (virtual displacement) in the displacement ${}^{t+\Delta t}u_i$. We also note that:

$$\frac{\partial^{t+\Delta t}x_m}{\partial^t x_i} = \delta_{mi} + \frac{\partial u_m}{\partial^t x_i}. \tag{8.9}$$

Combining Eq. (3.2) and the above equation, we can relate the variation $\delta\, {}^{t+\Delta t}_{t}\varepsilon_{ij}$ to $\delta\, {}_{t+\Delta t}e_{mn}$ in the following manner:

$$\delta\, {}^{t+\Delta t}_{t}\varepsilon_{ij} = \frac{\partial^{t+\Delta t}x_m}{\partial^t x_i}\,\delta\, {}_{t+\Delta t}e_{mn}\,\frac{\partial^{t+\Delta t}x_n}{\partial^t x_j}. \tag{8.10}$$

Finally by using Eqs. (8.1), (8.6), and (8.10), we can write Eq. (8.5) as

$${}^{t+\Delta t}W_v^{\text{int}} = \int\limits_V {}^{t+\Delta t}_{t}S_{ij}\;\delta\, {}^{t+\Delta t}_{t}\varepsilon_{ij}\; {}^t dV - \int\limits_{{}^tV}\left({}^{t+\Delta t}_{t}h_{ij}\right)\delta\, {}^{t+\Delta t}_{t}\varepsilon_{ij}\; {}^t dV, \tag{8.11}$$

where

$${}^{t+\Delta t}_{t}h_{mn} = \frac{{}^t\varrho}{{}^{t+\Delta t}\varrho}\,\frac{\partial^t x_i}{\partial^{t+\Delta t}x_m}\,{}^{t+\Delta t}(p\delta_{ij})\,\frac{\partial^t x_j}{\partial^{t+\Delta t}x_n}. \tag{8.12}$$

Equation (8.11) together with Eq. (8.12) complete the virtual work expression in terms of the coordinates of the configuration at time t (${}^t\boldsymbol{\beta}$). However, in order to use this equation in an incremental analysis, it is necessary to establish its equivalent incremental form. Derivation of such incremental form will be discussed in the next section.

9. Incremental form of the virtual work equation

An incremental form of the internal virtual work equation (8.11) can be established by introducing truncated Taylor series expansions of the second Piola–Kirchhoff stress tensor and the **h** tensor, i.e.

$$\begin{aligned} {}^{t+\Delta t}_{t}S_{ij} &= {}^t_tS_{ij} + \left[\frac{d}{dt}(S_{ij})\right]_t \Delta t + \text{higher order terms},\\ {}^{t+\Delta t}_{t}h_{ij} &= {}^t_th_{ij} + \left[\frac{d}{dt}(h_{ij})\right]_t \Delta t + \text{higher order terms}, \end{aligned} \tag{9.1}$$

where

$$ {}^{t}_{t}S_{ij} = {}^{t}\overline{\sigma}_{ij} , \tag{9.2}$$

$$ {}^{t}_{t}h_{ij} = {}^{t}p\delta_{ij} . \tag{9.3}$$

Ignoring the higher order terms in Eqs. (9.1) and using Eqs. (9.2) and (9.3), we have

$$ \begin{aligned} {}^{t+\Delta t}_{t}S_{ij} &= {}^{t}\overline{\sigma}_{ij} + \left[\frac{d}{dt}(S_{ij})\right]_t \Delta t , \\ {}^{t+\Delta t}_{t}h_{ij} &= {}^{t}p\delta_{ij} + \left[\frac{d}{dt}(h_{ij})\right]_t \Delta t. \end{aligned} \tag{9.4}$$

In order to evaluate the second terms on the r.h.s in Eqs. (9.4), we make use of the following kinematic relationships [8]

$$ \frac{d}{dt}\left(\frac{{}^{t}\varrho}{{}^{t+\Delta t}\varrho}\right) = \frac{{}^{t}\varrho}{{}^{t+\Delta t}\varrho} \frac{\partial^{t+\Delta t}\nu_i}{\partial^{t+\Delta t}x_i} , \tag{9.5}$$

$$ \frac{d}{dt}\left(\frac{\partial^{t}x_i}{\partial^{t+\Delta t}x_j}\right) = -\frac{\partial^{t+\Delta t}\nu_k}{\partial^{t+\Delta t}x_j} \frac{\partial^{t}x_i}{\partial^{t+\Delta t}x_k} , \tag{9.6}$$

where ${}^{t+\Delta t}\nu_k$ denotes the velocity of the soil mass in the direction of axis k. Utilizing Eqs. (9.5), (9.6) and (8.6), we find:

$$ \frac{d}{dt}(S_{ij}) = \frac{{}^{t}\varrho}{{}^{t+\Delta t}\varrho} \frac{\partial^{t}x_i}{\partial^{t+\Delta t}x_k} {}^{t+\Delta t}\dot{\overline{\sigma}}{}^{T}_{kl} \frac{\partial^{t}x_j}{\partial^{t+\Delta t}x_l} , \tag{9.7}$$

in which ${}^{t+\Delta t}\dot{\overline{\sigma}}{}^{T}_{kl}$ is the Truesdell rate of the effective stress tensor $\overline{\sigma}_{kl}$ and defined as

$$ \begin{aligned} {}^{t+\Delta t}\dot{\overline{\sigma}}{}^{T}_{kl} = {}^{t+\Delta t}\dot{\overline{\sigma}}_{kl} + {}^{t+\Delta t}\nu_{m,m} \, {}^{t+\Delta t}\overline{\sigma}_{kl} - {}^{t+\Delta t}\nu_{l,m} \, {}^{t+\Delta t}\overline{\sigma}_{km} \\ - {}^{t+\Delta t}\nu_{k,m} \, {}^{t+\Delta t}\overline{\sigma}_{ml} . \end{aligned} \tag{9.8}$$

Since we seek to find $\left[\frac{d}{dt}(S_{ij})\right]_t$, Eq. (9.7) should be evaluated at time t, i.e.

$$ \left[\frac{d}{dt}(S_{ij})\right]_t = {}^{t}\dot{\overline{\sigma}}{}^{T}_{ij} = {}^{t}\dot{\overline{\sigma}}_{ij} + {}^{t}\nu_{m,m} \, {}^{t}\overline{\sigma}_{ij} - {}^{t}\nu_{j,m} \, {}^{t}\overline{\sigma}_{im} - {}^{t}\nu_{i,m} \, {}^{t}\overline{\sigma}_{mj} . \tag{9.9}$$

The Truesdell stress rate appearing in Eq. (9.9) can be related to the Jaumann stress rate by decomposing the velocity gradient $\nu_{i,m}$ to the sum of the rate of deformation tensor d_{im} and the spin tensor Ω_{im}, i.e.

$$ \nu_{i,m} = d_{im} + \Omega_{im} . \tag{9.10}$$

Substituting (9.10) in Eq. (9.9), leads to

$$\left[\frac{d}{dt}(S_{ij})\right]_t = {}^t\dot{\overline{\sigma}}_{ij} + {}^t\nu_{m,m}\,{}^t\overline{\sigma}_{ij} - \left({}^t d_{jm} + {}^t\Omega_{jm}\right){}^t\overline{\sigma}_{im} - \left({}^t d_{im} + {}^t\Omega_{im}\right){}^t\overline{\sigma}_{mj}\,, \tag{9.11}$$

or by using Eq. (7.1), we find:

$$\left[\frac{d}{dt}(S_{ij})\right]_t = {}^t\overset{\nabla}{\overline{\sigma}}_{ij} + {}^t\nu_{m,m}\,{}^t\overline{\sigma}_{ij} - {}^t d_{jm}\,{}^t\overline{\sigma}_{im} - {}^t d_{im}\,{}^t\overline{\sigma}_{mj}\,, \tag{9.12}$$

where ${}^t\overset{\nabla}{\overline{\sigma}}_{ij}$ is the Jaumann rate of the effective stress tensor.

Considering the general form of the constitutive equation (7.3) applied to ${}^t\overset{\nabla}{\overline{\sigma}}_{ij}$ and substituting (9.12) in Eq. (9.4), we have:

$${}^{t+\Delta t}_{\ \ t}S_{ij} = {}^t\overline{\sigma}_{ij} + \Delta t\left({}_tD_{ijkl}\,{}^t d_{kl} + {}^t\nu_{m,m}\,{}^t\overline{\sigma}_{ij} - {}^t d_{jm}\,{}^t\overline{\sigma}_{im} - {}^t d_{im}\,{}^t\overline{\sigma}_{mj}\right). \tag{9.13}$$

It must be noted that the ${}_tD_{ijkl}$ appearing from now on in the subsequent equations is the one which relates the rate of deformation tensor to the Jaumann rate of effective stress. However, if the initial formulation of the constitutive law calls for the use of a corotational rate with respect to a different spin than ${}^t\Omega_{ij}$, then one must perform a subsequent transformation to a Jaumann rate for the effective stress with simultaneous change of the constitutive moduli which will be again defined by ${}_tD_{ijkl}$ after the transformation.

Equation (9.13) can be written in a compact form by using the following relations:

$$\begin{aligned} \Delta t\,{}^t d_{kl} &= \frac{1}{2}\Delta t\left(\frac{\partial\,{}^t\dot{u}_k}{\partial^t x_l} + \frac{\partial\,{}^t\dot{u}_l}{\partial^t x_k}\right) = \frac{1}{2}\left[\frac{\partial(\Delta u_k)}{\partial^t x_l} + \frac{\partial(\Delta u_l)}{\partial^t x_k}\right] = {}_te_{kl}\,, \\ \Delta t\,{}^t\nu_{m,m} &= \frac{\partial(\Delta u_m)}{\partial^t x_m} = {}_te_{mm}\,, \end{aligned} \tag{9.14}$$

where Δu_m is the m-th component of the incremental displacement at a generic point of the soil body. Thus Eq. (9.13) can be written as:

$${}^{t+\Delta t}_{\ \ t}S_{ij} = {}^t\overline{\sigma}_{ij} + {}_t\Lambda_{ijkl}\,{}_te_{kl}\,, \tag{9.15}$$

where

$${}_t\Lambda_{ijkl} = {}_tD_{ijkl} + {}^t\overline{\sigma}_{ij}\delta_{kl} - {}^t\overline{\sigma}_{il}\delta_{kj} - \delta_{il}\,{}^t\overline{\sigma}_{kj}\,. \tag{9.16}$$

The ${}_t\Lambda_{ijkl}$ is the finite deformation tensor of tangent stiffness moduli and it includes the regular tangent stiffness moduli tensor and the effect of stresses at the beginning of the step. It must be noted that if the components of the effective stress tensor are of the same order of magnitude as the ${}_tD_{ijkl}$, contribution of the initial stresses to the ${}_t\Lambda_{ijkl}$ tensor can be significant.

Similarly to Eq. (9.9) for the rate of the second Piola - Kirchhoff stress tensor, one can write the following equation for the rate of the h_{mn},

$$\left[\frac{d}{dt}(h_{mn})\right]_t = \frac{d}{dt}(p\delta_{mn}) + {}^t\nu_{k,k}\,{}^tp\,\delta_{mn} - {}^t\nu_{n,m}\,{}^tp - {}^t\nu_{m,n}\,{}^tp \tag{9.17}$$

recalling that

$${}^td_{mn} = \frac{1}{2}\left({}^t\nu_{m,n} + {}^t\nu_{n,m}\right), \tag{9.18}$$

Eq. (9.17) can be written as

$$\left[\frac{d}{dt}(h_{mn})\right]_t = {}^t\dot{p}\,\delta_{mn} + {}^t\nu_{k,k}\,{}^tp\,\delta_{mn} - 2\,{}^tp\,{}^td_{mn}\,. \tag{9.19}$$

Using Eqs. (9.20) and (9.14), we can now write $(9.1)_2$ as

$${}^{t+\Delta t}_{t}h_{mn} = {}^tp\,\delta_{mn} + ({}^t\Delta p\,\delta_{mn}) + {}_te_{kk}\,{}^tp\,\delta_{mn} - 2\,{}^tp\,{}_te_{mn}\,, \tag{9.20}$$

where

$${}^t(\Delta p) = {}^t\dot{p}\,\Delta t. \tag{9.21}$$

Regarding the variation in the Green - Lagrange strain tensor, i.e. $\delta\,{}^{t+\Delta t}_{t}\varepsilon_{ij}$, we note that:

$$\delta\,{}^{t+\Delta t}_{t}\varepsilon_{ij} = \delta\,{}_te_{ij} + \delta\,{}_t\eta_{ij}\,, \tag{9.22}$$

where

$${}_te_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial^t x_j} + \frac{\partial u_j}{\partial^t x_i}\right) \tag{9.23}$$

and

$${}_t\eta_{ij} = \frac{1}{2}\left(\frac{\partial u_k}{\partial^t x_i}\,\frac{\partial u_k}{\partial^t x_j}\right) \tag{9.24}$$

in which u_k is the incremental displacement.

Substituting Eqs. (9.15), (9.20), and (9.22) in Eq. (8.13), we find

$$\begin{aligned}{}^{t+\Delta t}W_v^{\text{int}} = &\int_{{}^tV}\left({}^t\overline{\sigma}_{ij} + {}_t\Lambda_{ijkl}\,{}_te_{kl}\right)(\delta\,{}_te_{ij} + \delta\,{}_t\eta_{ij})\,{}^tdV \\ &- \int_{{}^tV}\left({}^tp\delta_{mn} + {}^t\Delta p\delta_{mn} + {}_te_{kk}\,{}^tp\delta_{mn} - 2\,{}^tp\,{}_te_{mn}\right)(\delta\,{}_te_{mn} + \delta\,{}_t\eta_{mn})\,{}^tdV.\end{aligned} \tag{9.25}$$

This is the final form of the internal virtual work expression in terms of the coordinates of the configuration at time t. A more compact and computatonally useful form of Eq. (9.25) can be obtained by utilizing Eq. (4.1) and combining the effective stress-related terms implicit in the ${}_t\Lambda_{ijkl}$ tensor (Eq. (9.16)) with the pore pressure-related terms in Eq. (9.25). We finally find:

$$(9.26)\qquad {}^{t+\Delta t}W_v^{\text{int}} = \int\limits_{{}^tV} \left({}^t\sigma_{ij} + {}_tL_{ijkl}\,{}_te_{kl}\right)\left(\delta\,{}_te_{ij} + \delta\,{}_t\eta_{ij}\right){}^tdV - \int\limits_{{}^tV} \left({}^t\Delta p\right)\left(\delta\,{}_te_{ii} + \delta\,{}_t\eta_{ii}\right){}^tdV,$$

where

$$(9.27)\qquad {}_tL_{ijkl} = {}_tD_{ijkl} + {}^t\sigma_{ij}\,\delta_{kl} - {}^t\sigma_{il}\,\delta_{kj} - {}^t\delta_{il}\,{}^t\sigma_{kj}$$

or by using Eqs. (3.5), we have:

$$(9.28)\qquad \int\limits_{{}^tV} {}_tL_{ijkl}\,{}_te_{kl}\,\delta\,{}_te_{ij}\,{}^tdV - \int\limits_{{}^tV} {}^t\Delta p\,\delta\,{}_te_{ii}\,{}^tdV + \int\limits_{{}^tV} {}_tL_{ijkl}\,{}_te_{kl}\,\delta\,{}_t\eta_{ij}\,{}^tdV + \int\limits_{{}^tV} {}^t\sigma_{ij}\,\delta\,{}_t\eta_{ij}\,{}^tdV - \int\limits_{{}^tV} {}^t(\Delta p)\,\delta\,{}_t\eta_{ii}\,{}^tdV = {}^{t+\Delta t}W_v^{\text{ext}} - \int\limits_{{}^tV} {}^t\sigma_{ij}\,\delta\,{}_te_{ij}\,{}^tdV.$$

The last three terms on the left-hand side of Eq. (9.28) are due to finite deformation effects, and they may be omitted in a small deformation analysis. In the case of infinitesimal strains and small rotations, the ${}_tL_{ijkl}$ tensor will also reduce to the ${}_tD_{ijkl}$, tensor of tangent stiffness moduli. It should be mentioned that in an incremental numerical solution, Eq. (9.28) is normally linearized by ignoring the third and fifth terms on the left-hand side in this equation. This linearization is justified due to small effects of these higher order terms in a regular earthquake engineering problem, where the time steps are generally small if a plasticity-based constitutive model is to be used.

Here it must be noted that Eqs. (9.25) and (9.28) are incremental approximations of the internal virtual work at time $t + \Delta t$. These equations, along with equations governing the motion of the pore water (Eq. (5.23)), are used to calculate an incremental displacement and pore water pressure. The calculated incremental values are then used to evaluate approximations to the displacements of soil skeleton, strains, stresses, and pore water pressure at time $t + \Delta t$. The calculated values of displacements can be employed to establish an approximation to the configuration at time $t + \Delta t$ (${}^{t+\Delta t}\beta$, ${}^{t+\Delta t}V$, ${}^{t+\Delta t}S$). Therefore it is possible to calculate the difference between the internal virtual work evaluated with the calculated static and kinematic quantities at time $t + \Delta t$, and the external virtual work. In general, linearization of Eq. (9.28) introduces some errors and the aforementioned difference may not be negligible. Thus, in order to reduce the difference between the estimated internal work and the external work, an iterative solution strategy is necessary. Different schemes may be used for an iterative analysis. A full Newton-Raphson iteration scheme leads to the following form:

$$\begin{aligned}(9.29)\qquad &\int\limits_{^tV} {}_tL_{ijkl}{}^{(m)}\Delta\, {}_te_{kl}{}^{(m)}\,\delta\, {}_te_{ij}{}^{(m)}\, {}^tdV - \int\limits_{^tV} {}^t\Delta p^{(m)}\,\delta\, {}_te_{ii}{}^{(m)}\, {}^tdV \\ &+ \int\limits_{^tV} {}_tL_{ijkl}{}^{(m)}\,\Delta_t e_{kl}{}^{(m)}\,\delta\, {}_t\eta_{ij}{}^{(m)}\, {}^tdV + \int\limits_{^tV} {}^t\sigma_{ij}\,\delta\Delta\, {}_t\eta_{ij}{}^{(m)}\, {}^tdV \\ &- \int\limits_{^tV} {}^t(\Delta p)^{(m)}\,\delta\, {}_t\eta_{ii}{}^{(m)}\, {}^tdV = {}^{t+\Delta t}W_v^{\mathrm{ext}} - \int\limits_{^tV^{(m-1)}} {}^t\sigma_{ij}{}^{(m-1)}\,\delta\, {}_te_{ij}{}^{(m-1)}\, {}^tdV,\end{aligned}$$

where m is the iteration number and the first iteration ($m = 1$) corresponds to Eq. (9.28). The $\Delta_t e_{kl}{}^{(m)}$ in Eq. (9.28) is a component of the incremental strain tensor for iteration m, i.e.

$$(9.30)\qquad \Delta\, {}_te_{kl}{}^{(m)} = \frac{1}{2}\left(\frac{\partial(\Delta u_k{}^{(m)})}{\partial^t x_l} + \frac{\partial(\Delta u_l{}^{(m)})}{\partial^t x_k}\right).$$

Similarly, the $\delta\Delta\, {}_t\eta_{ij}{}^{(m)}$ is defined as

$$(9.31)\qquad \Delta\, {}_t\eta_{kl}{}^{(m)} = \frac{1}{2}\left(\frac{\partial(\Delta u_i{}^{(m)})}{\partial^t x_k}\cdot\frac{\partial(\Delta u_i{}^{(m)})}{\partial^t x_l}\right).$$

Iterations are repeated until the r.h.s. in Eq. (9.29) is negligible within a certain convergence tolerance. After each iteration, the displacements and pore water pressure are updated.

The full Newton scheme adopted in Eq. (9.29) is obviously expensive due to the necessity of evaluation of the constitutive tensor ${}_tL_{ijkl}$ at each iteration. A modified Newton scheme can be achieved by keeping the constitutive tensor ${}_tL_{ijkl}$ constant during each step of incremental solution, i.e.

$$\begin{aligned}(9.32)\qquad &\int\limits_{^tV} {}_tL_{ijkl}\,\Delta\, {}_te_{kl}{}^{(m)}\,\delta\, {}_te_{ij}{}^{(m)}\, {}^tdV - \int\limits_{^tV} {}^t\Delta p^{(m)}\,\delta\, {}_te_{ii}{}^{(m)}\, {}^tdV \\ &+ \int\limits_{^tV} {}_tL_{ijkl}\Delta_t e_{kl}{}^{(m)}\,\delta\, {}_t\eta_{ij}{}^{(m)}\, {}^tdV + \int\limits_{^tV} {}^t\sigma_{ij}\,\delta\Delta\, {}_t\eta_{ij}{}^{(m)}\, {}^tdV \\ &- \int\limits_{^tV} {}^t(\Delta p)^{(m)}\,\delta\, {}_t\eta_{ii}{}^{(m)}\, {}^tdV = {}^{t+\Delta t}W_v^{\mathrm{ext}} - \int\limits_{^tV^{(m-1)}} {}^t\sigma_{ij}{}^{(m-1)}\,\delta\, {}_te_{ij}{}^{(m-1)}\, {}^tdV.\end{aligned}$$

Such a solution strategy has been successfully used in some of the applications reported in [33].

As a final note in this section, it should be mentioned that the case of a deformation-dependent external loading can be conveniently handled by applying an iterative incremental procedure as described for Eq. (9.29). For example, in the case of centrifugal loading, the body force applied to an infinitesimal volume of the soil is a function of its current position, i.e.

$$ {}^{t+\Delta t}b_i = {}^{t+\Delta t}b_i({}^{t+\Delta t}\mathbf{x}). \tag{9.33}$$

In such a case, the corresponding term in Eq. $(2.5)_1$ is approximated as follows:

$$\int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}\varrho\, {}^{t+\Delta t}b_i\, \delta u_i\, {}^{t+\Delta t}dV \approx \int\limits_{{}^{t+\Delta t}V^{(m-1)}} {}^{t+\Delta t}\varrho^{(m-1)}\, {}^{t+\Delta t}b_i\left({}^{t+\Delta t}\mathbf{x}^{(m-1)}\right) \delta u_i\, {}^{t+\Delta t}dV, \tag{9.34}$$

where

$$ {}^{t+\Delta t}\varrho^{(m-1)} = {}^{t}\varrho \det\left|\frac{\partial\, {}^{t+\Delta t}x_i{}^{(m-1)}}{\partial\, {}^{t}x_j}\right|. \tag{9.35}$$

The approximation introduced in Eq. (9.35) is only accurate for a small load increment. Evidently, a better approximation for the finite load increments can be achieved by linearizing ${}^{t+\Delta t}b_i$, as it was done for the second Piola–Kirchhoff stress tensor and the Green–Lagrange strain tensor. Such a linearization, however, introduces a new contribution to the stiffness matrix and reduces the computational efficiency of the formulation, as mentioned in [3].

10. Integral form of the equation governing the flow of the pore water

In Sec. 5 we have established a differential equation (Eq. (5.23)) governing the flow of the pore water through the soil. For the purpose of numerical solutions, however, it is appropriate to develop an integral form of this equation which complements the virtual work equation developed in the previous section.

In order to establish an integral form of Eq. (5.23), we recall that this equation is basically an expression of the axiom of mass balance implying that a tendency of volumetric strain in the soil skeleton (first term in Eq. (5.23)) is counteracted by a change in pore pressure (second term), and by the flow of the pore water through the soil (the last three terms). Therefore, a weak form of Eq. (5.23) can be generated by using the Galerkin weighted residual method and recognizing that the pore water pressure is the appropriate weighting function on the volumetric strain rate, i.e.

$$(10.1)\qquad \int\limits_{{}^{t+\Delta t}V} \left\{ \frac{1}{\Gamma}\, {}^{t+\Delta t}\dot{p} + {}^{t+\Delta t}\dot{u}_{i,i} - \frac{\partial}{\partial\, {}^{t+\Delta t}x_i} \left({}^{t+\Delta t}k_{ij} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j} \right) + \frac{\partial}{\partial\, {}^{t+\Delta t}x_i} \left({}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}b_j \right) - \frac{\partial}{\partial\, {}^{t+\Delta t}x_i} \left({}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}\ddot{u}_j \right) \right\} \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV = 0,$$

where $\delta\, {}^{t+\Delta t}p$ is a virtual pore water pressure analogous to the virtual displacement $\delta\mathbf{u}$ previously used in the virtual work expression (Eq. (5.1)).

An expanded form of Eq. (10.1) can be written as

$$(10.2)\qquad \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}\dot{u}_{i,i}\, \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV + \int\limits_{{}^{t+\Delta t}V} \frac{1}{\Gamma}\, {}^{t+\Delta t}\dot{p}\, \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV + \int\limits_{{}^{t+\Delta t}V} \left[\frac{\partial}{\partial\, {}^{t+\Delta t}x_i} \left(-\, {}^{t+\Delta t}k_{ij} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j} + {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}b_j - {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}\ddot{u}_j \right) \right] \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV = 0.$$

Applying the Green's theorem to the last integral, we have:

$$(10.3)\qquad \int\limits_{{}^{t+\Delta t}V} {}^{t+\Delta t}\dot{u}_{i,i}\, \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV + \int\limits_{{}^{t+\Delta t}V} \frac{1}{\Gamma}\, {}^{t+\Delta t}\dot{p}\, \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dV + \int\limits_{{}^{t+\Delta t}V} \left({}^{t+\Delta t}k_{ij} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j} - {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}b_j + {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}\ddot{u}_j \right) \frac{\partial}{\partial\, {}^{t+\Delta t}x_i} (\delta\, {}^{t+\Delta t}p)\, {}^{t+\Delta t}dV + \int\limits_{{}^{t+\Delta t}S} \left(-\, {}^{t+\Delta t}k_{ij} \frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j} + {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}b_j - {}^{t+\Delta t}k_{ij}\, {}^{t+\Delta t}\varrho_f\, {}^{t+\Delta t}\ddot{u}_j \right) . n_i\, \delta\, {}^{t+\Delta t}p\, {}^{t+\Delta t}dS = 0,$$

where n_i is the component of outward unit normal vector to the surface ${}^{t+\Delta t}S$ in the direction of x_i axis.

Utilizing the water flow boundary condition (Eq. (6.4)), we can write Eq. (10.3) in the form

$$(10.4) \qquad \int\limits_{^{t+\Delta t}V} {}^{t+\Delta t}\dot{u}_{i,i}\,\delta\,{}^{t+\Delta t}p\,{}^{t+\Delta t}dV + \int\limits_{^{t+\Delta t}V} \frac{1}{\Gamma}\,{}^{t+\Delta t}\dot{p}\,\delta\,{}^{t+\Delta t}p\,{}^{t+\Delta t}dV$$
$$+ \int\limits_{^{t+\Delta t}V} \left({}^{t+\Delta t}k_{ij} \frac{\partial\,{}^{t+\Delta t}p}{\partial\,{}^{t+\Delta t}x_j} - {}^{t+\Delta t}k_{ij}\,{}^{t+\Delta t}\varrho_f\,{}^{t+\Delta t}b_j \right.$$
$$\left. + {}^{t+\Delta t}k_{ij}\,{}^{t+\Delta t}\varrho_f\,{}^{t+\Delta t}\ddot{u}_j \right) \frac{\partial}{\partial\,{}^{t+\Delta t}x_i}(\delta\,{}^{t+\Delta t}p)\,{}^{t+\Delta t}dV$$
$$+ \int\limits_{^{t+\Delta t}S_q} {}^{t+\Delta t}\bar{q}_S\,\delta\,{}^{t+\Delta t}p\,{}^{t+\Delta t}dS = 0.$$

Equation (10.4) is written in terms of the coordinates of the current configuration ${}^{t+\Delta t}\beta$, whose equilibrium position is to be calculated while proceeding from time t to $t+\Delta t$ in the incremental solution. By applying the chain rule and using Eq. (8.1), we can write Eq. (10.2) in terms of the coordinates of the configuration at time t, ${}^{t}\beta$, i.e.

$$(10.5) \qquad \int\limits_{^{t}V} \frac{{}^{t}\varrho}{{}^{t+\Delta t}\varrho} \left(\frac{\partial\,{}^{t+\Delta t}\dot{u}_i}{\partial\,{}^{t}x_j}\,\delta\,{}^{t+\Delta t}p \right) \frac{\partial\,{}^{t}x_j}{\partial\,{}^{t+\Delta t}x_i}\,{}^{t}dV + \int\limits_{^{t+\Delta t}V} \frac{{}^{t}\varrho}{{}^{t+\Delta t}\varrho} \left(\frac{{}^{t+\Delta t}\dot{p}}{\Gamma}\,\delta\,{}^{t+\Delta t}p \right) {}^{t}dV$$
$$+ \int\limits_{^{t}V} \frac{{}^{t}\varrho}{{}^{t+\Delta t}\varrho} \frac{\partial\,{}^{t}x_k}{\partial\,{}^{t+\Delta t}x_j} \left[{}^{t+\Delta t}k_{ij} \frac{\partial\,{}^{t+\Delta t}p}{\partial\,{}^{t}x_k} \frac{\partial(\delta\,{}^{t+\Delta t}p)}{\partial\,{}^{t}x_l} \right] \frac{\partial\,{}^{t}x_l}{\partial\,{}^{t+\Delta t}x_i}\,{}^{t}dV$$
$$- \int\limits_{^{t}V} \frac{{}^{t}\varrho}{{}^{t+\Delta t}\varrho} \left[{}^{t+\Delta t}k_{ij}\,{}^{t+\Delta t}\varrho_f\,{}^{t+\Delta t}b_j \frac{\partial(\delta\,{}^{t+\Delta t}p)}{\partial\,{}^{t}x_l} \right] \frac{\partial\,{}^{t}x_l}{\partial\,{}^{t+\Delta t}x_i}\,{}^{t}dV$$
$$+ \int\limits_{^{t+\Delta t}V} \frac{{}^{t}\varrho}{{}^{t+\Delta t}\varrho} \left[{}^{t+\Delta t}k_{ij}\,{}^{t+\Delta t}\varrho_f\,{}^{t+\Delta t}\ddot{u}_j \frac{\partial(\delta\,{}^{t+\Delta t}p)}{\partial\,{}^{t}x_l} \right] \frac{\partial\,{}^{t}x_l}{\partial\,{}^{t+\Delta t}x_i}\,{}^{t}dV$$
$$+ \int\limits_{^{t}S_q} {}^{t+\Delta t}_{\ \ t}\bar{q}_S\,\delta\,{}^{t+\Delta t}p\,{}^{t}dS = 0,$$

where ${}^{t+\Delta t}_{\ \ t}\bar{q}_S$ was assumed to be a deformation-independent flow on ${}^{t+\Delta t}S_q$, so that

$$(10.6) \qquad {}^{t+\Delta t}\bar{q}_S\,{}^{t+\Delta t}dS = {}^{t+\Delta t}_{\ \ t}\bar{q}_S\,{}^{t}dS.$$

It is noted that due to the presence of ${}^{t+\Delta t}\varrho_f$, ${}^{t+\Delta t}k_{ij}$, and the inverse of the deformation gradient tensor, $(\partial\,{}^{t}x_l)/(\partial\,{}^{t+\Delta t}x_i)$, in Eq. (10.5), most of the integrals in this equation cannot be evaluated without further simplifying assumptions. Similar to the procedure adopted in Sec. 8, one may utilize linearization technique to reduce Eq. (10.5) to a suitable form for incremental iterative analysis. Linearization of Eq. (10.5), however, leads to a very complicated equation which significantly reduces the computational efficiency of the formulation. Therefore, in order to avoid such a difficulty, it is suggested to use some reasonable approximation for the aforementioned redundant terms (${}^{t+\Delta t}\varrho_f$, ${}^{t+\Delta t}k_{ij}$, ...). For example, in the first iteration, one may utilize the values obtained from the previous time step. Corrections to the results of the first iteration can be achieved by establishing approximate position of the configuration at time $t + \Delta t$. The new configuration can be used in iterative solution of Eq. (10.4) written in the following form:

$$(10.7)\quad \int\limits_{{}^{t+\Delta t}V^{(m-1)}} \frac{\partial\, {}^{t+\Delta t}\dot{u}_i}{\partial\, {}^{t+\Delta t}x_i^{(m-1)}} \delta\, {}^{t+\Delta t}p\; {}^{t+\Delta t}dV + \int\limits_{{}^{t+\Delta t}V^{(m-1)}} \frac{1}{\Gamma}\, {}^{t+\Delta t}\dot{p}\; \delta\, {}^{t+\Delta t}p\; {}^{t+\Delta t}dV$$
$$+ \int\limits_{{}^{t+\Delta t}V^{(m-1)}} {}^{t+\Delta t}k_{ij}^{(m-1)} \left[\frac{\partial\, {}^{t+\Delta t}p}{\partial\, {}^{t+\Delta t}x_j^{(m-1)}} - {}^{t+\Delta t}\varrho_f^{(m-1)}\, {}^{t+\Delta t}b_j^{(m-1)} \right.$$
$$\left. + {}^{t+\Delta t}\varrho_f^{(m-1)}\, {}^{t+\Delta t}\ddot{u}_j \right] \frac{\partial(\delta\, {}^{t+\Delta t}p)}{\partial\, {}^{t+\Delta t}x_i^{(m-1)}}\, {}^{t+\Delta t}dV$$
$$+ \int\limits_{{}^{t+\Delta t}S_q^{(m-1)}} {}^{t+\Delta t}\bar{q}_S\, \delta\, {}^{t+\Delta t}p\; {}^{t+\Delta t}dS = 0,$$

where the right superscript $(m - 1)$ refers to the iteration number $(m - 1)$, and the case $m = 1$ is defined as

$$(10.8)\qquad {}^{t+\Delta t}(\;)^{(0)} = {}^{t}(\;),$$

and ${}^{t+\Delta t}\varrho_f^{(m-1)}$ is calculated by using the following equation:

$$(10.9)\qquad J^{(m-1)} = \left| \frac{\partial\, {}^{t+\Delta t}\mathbf{x}^{(m-1)}}{\partial\, {}^{t}\mathbf{x}} \right| = \frac{{}^{t+\Delta t}\varrho^{(m-1)}}{{}^{t}\varrho} = \frac{(1-n)\, {}^{t+\Delta t}\varrho_f^{(m-1)} + n\varrho_S}{(1-n)\, {}^{t}\varrho_f + n\varrho_S}.$$

In Eq. (10.9), we assumed that the change of soil porosity during the time increment was negligible. This assumption is used to prevent the need for iteration over porosity.

Equation (10.7) has a number of special characteristics which distinguish it from the virtual work expression, i.e. Eq. (9.25). First, the dependence of the acceleration term on ${}^{t+\Delta t}\varrho_f$ and ${}^{t+\Delta t}k_{ij}$ requires that the corresponding "mass" matrix in a discretized solution procedure should be calculated in every iteration. Similar situation renders the body force contribution in Eq. (10.7) a deformation-dependent loading. It is also noted that components of the effective permeability tensor are variable quantities which may change due to the change of fabric in the soil skeleton. Unfortunately, experimental data in order to characterize such a change in the soil fabric is very limited.

In so far as the permeability tensor is concerned, it is important to note that in a finite deformation regime, a generic element of the soil system may undergo large rotations. Therefore, special care is necessary to define the coefficients of the permeability tensor in terms of the coordinates of the Cartesian reference system used in the Lagrangian formulation. Assuming that the permeability coefficients are intrinsic to the soil element, it can easily be shown [7] that the matrix of the permeability coefficients obeys the following transformation:

$$\mathbf{k} = \mathbf{R}^T \mathbf{k}_0 \mathbf{R}, \tag{10.10}$$

where $\mathbf{k}_0$ is the matrix of permeability coefficients in the initial position and $\mathbf{R}$ is the matrix characterizing the rotation of the soil element with respect to the reference Cartesian coordinates.

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