Stiffness loss of laminates with aligned intralaminar cracks Part I. Macroscopic constitutive relations

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THE PAPER deals with analysis of reduction of the in-plane effective elastic moduli of the $[0^\circ_m/90^\circ_n]_s$ laminates weakened by aligned cross-cracks in the 90°-layer. A regular crack pattern is assumed. The case of dense crack distribution is modelled by (h_0, l_0) approach, while the case of arbitrary crack density c_d is described by a more accurate model (h_0, l). Both models have been derived in our paper [10]. Closed-form formulae describing decaying curves $E_1(c_d)$, $E_2(c_d)$, $\nu_{12}(c_d)$, $\nu_{21}(c_d)$, $G_{12}(c_d)$ are found by solution of the local problems for both models.

1. Introduction

CROSS-PLY LAMINATES of the $[0^\circ_m/90^\circ_n]_s$ type incur matrix cracking, interlaminar delamination and fibre breakage. The matrix cracks observed are straight or curved, cf. GROVES et al. [3]. The aim of the present paper is to assess the loss of effective elastic characteristics of the laminates with the straight matrix cracks going transversely through the whole thickness of the 90°-plies. The influence of crack curving as well as the onset of delamination is neglected. The cracks are assumed to be aligned. The present paper is mainly concerned with the case when these cracks are equally spaced. The assumption seems to be non-restrictive, since matrix cracks form usually regular patterns, cf. GARRETT and BAILEY [2], HIGHSMITH and REIFSNIDER [6], GROVES [4] and GROVES et al. [3]. The method (h_0, l) to be used has been proposed by us in [9, 10] and mathematically justified in TELEGA and LEWIŃSKI [15]. This method makes it possible to evaluate reduction of all components of the stiffness matrix of the three-layer balanced (transversely symmetric) laminates with transverse cracks in the internal layer.

The aim of this part of the paper is to find closed-form formulae interrelating effective Young's moduli E^c_1, E^c_2, effective Kirchhoff's modulus G^c_{12} and Poisson's ratios $\nu^c_{\alpha\beta}$ with crack density c_d. The second part of the paper [11] is devoted to placing these results into the available literature of the subject as well as to compare the theoretical predictions of the (h_0, l) model with experimental data.

The following conventions are employed: small Greek indices (except for ε) run over 1, 2, while Latin ones (except for h) take values 1, 2, 3; h labels quantities of the homogenized description. Summation convention concerns repeated indices at different levels. Sometimes the same letter denotes an index and a parameter (e.g. α, β, γ, δ, etc. defined in the Appendix), which should not lead to ambiguities. The system of notations is compatible with that employed in LEWIŃSKI and TELEGA [9, 10]. Some auxiliary quantities are defined in the Appendix.

2. A laminate composed of orthotropic plies. The case of short cracks parallel to the axis x_2

The aim of this section is to exhibit simplifications in the homogenized description of in-plane deformations of the cracked laminate considered in [10] which take place when:

  i) the plies are orthotropic, and

  ii) the cracks weakening the internal layer are aligned.

Assume that the axes x_1, x_2 are axes of orthotropy. Cracks are parallel to the axis x_2, cf. Fig. 1. In view of the orthotropy assumption, the only non-zero components of the stiffness are

(2.1)
$$A_v^{\alpha\alpha\beta\beta}, \ A_v^{\alpha\beta\alpha\beta}; \ A_{vw}^{\alpha\alpha}; \ A_{vu}^{\alpha\alpha\beta\beta}, \ A_{vu}^{\alpha\beta\alpha\beta},$$
$$A_{uw}^{\alpha\alpha}, \ A_u^{\alpha\alpha\beta\beta}, \ A_u^{\alpha\beta\alpha\beta}; \ A_w; \qquad \alpha, \beta = 1, 2.$$
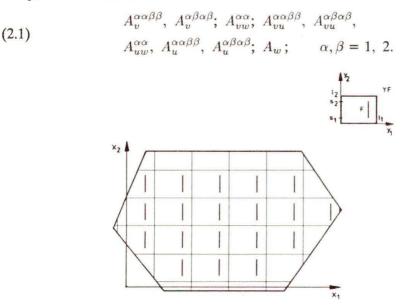

FIG. 1. Laminate with short aligned cracks.

Consequently, the only non-vanishing components of the tensors $A_\sigma^{\alpha\beta\lambda\mu}$, cf.([10], Eqs. (4.9)), are

(2.2)
$$A_1^{\alpha\alpha\beta\beta}, \ A_2^{\alpha\alpha\beta\beta}, \ A_1^{\alpha\beta\alpha\beta} = A_v^{\alpha\beta\alpha\beta}, \ A_2^{\alpha\beta\alpha\beta} = A_{vu}^{\alpha\beta\alpha\beta}.$$

Note that $A_1^{1122} = A_1^{2211}$ but $A_2^{1122} \neq A_2^{2211}$.

The components of the vectors **N**, **T** are $(0, 1)$ and $(1, 0)$, respectively, (cf. [10], Fig. 1). According to the definition (4.7) given in [10] of the tensor of crack deformation measures $\boldsymbol{\varepsilon}^F$, one finds:

(2.3)
$$\varepsilon_{1\lambda}^F = \frac{1}{\lambda l_1 l_2} \int_{s_1}^{s_2} [\![u_\lambda^1]\!] \, dy_2 \,, \qquad \varepsilon_{22}^F = 0, \qquad \lambda = 1, 2.$$

Thus, regardless of the type of the scaling, the homogenized constitutive relations have the form (cf. [10], Eqs. (4.5), (5.36))

$$
\begin{aligned}
N_h^{11} &= A_v^{1111} \left(\alpha_{11} \varepsilon_{11}^h + \alpha_{12} \varepsilon_{22}^h - \beta_{11} \varepsilon_{11}^F \right), \\
N_h^{22} &= A_v^{1111} \left(\alpha_{12} \varepsilon_{11}^h + \alpha_{22} \varepsilon_{22}^h - \beta_{21} \varepsilon_{11}^F \right), \\
N_h^{12} &= 2 A_v^{1212} \left(\varepsilon_{12}^h - \hat{\alpha} \varepsilon_{12}^F \right),
\end{aligned}
$$

(2.4)

where $\varepsilon_{\alpha\beta}^F = \varepsilon_{\alpha\beta}^F(\varepsilon_{11}^h, \varepsilon_{22}^h, \varepsilon_{12}^h)$; the coefficients involved in (2.4) are defined by Eqs. (A.1). We cannot expect that in general $\varepsilon_{\alpha\alpha}^F = \varepsilon_{\alpha\alpha}^F(\varepsilon_{11}^h, \varepsilon_{22}^h)$ and $\varepsilon_{12}^F = \varepsilon_{12}^F(\varepsilon_{12}^h)$, since the cracks considered are of a unilateral type.

3. Parallel cracks: effective characteristics according to the (h_0, l_0) approach

From now onward we shall deal with a laminate composed of orthotropic plies and weakened by straight-line cracks in the internal layer, lying at equal distances l. The crack lines coincide with $x_1 = nl$ lines ($n = 1, 2, \ldots$), cf. Fig. 2. The aim of this section is to find effective stiffnesses of the laminate considered resulting from the (h_0, l_0) method discussed in ref. [10], Sec. 4. This method follows from the in-plane scaling: $h \to h$, $l_\alpha \to \varepsilon l_\alpha$, and hence it will also be called the in-plane scaling approach. Results of this model apply for laminates with cracks of high density. Predictions of the model will be independent of the value of the l/h ratio.
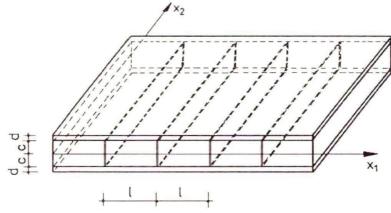

FIG. 2. Laminate with infinite aligned cracks.

The key to homogenization is to find solutions to the local problem (P_{loc}°) formulated in Ref. [10], Sec. 4. Geometry of the periodicity cell is here simple, cf. Fig. 3. For the sake of simplicity, the crack F is located so that it divides the cell into two equal parts.
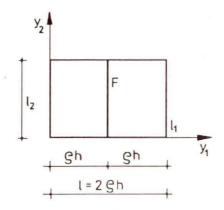
FIG. 3. Basic cell of periodicity.

Since the crack F lies along the axis y_2, one can guess that the solution to the basic cell problem (P_{loc}°) of Sec. 4 in [10] does not depend on y_2. Similar problem has been solved by Lewiński and Telega [8], hence only the outline of the derivation will be given here.

The unknown fields of (P_{loc}°) are $v_1^1(y_1)$, $u_1^1(y_1)$ and $v_2^1(y_1)$, $u_2^1(y_1)$. It turns out that these two pairs of functions are solutions of the independent (decoupled) stretching and shearing problems.

3.1. Solution of the local stretching problem. Stiffnesses $A_c^{\alpha\alpha\beta\beta}$

The unknown fields are $v_1^1(y_1)$ and $u_1^1(y_1)$. Let $\xi = y_1/h$ be a non-dimensional variable; $\xi \in (0, 2\varrho)$; $2\varrho = l/h$. Analysis of the local equations of (P_{loc}°) shows that both unknown fields are piece-wise linear in ξ, i.e.

$$(3.1) \qquad v_1^1 = \begin{cases} c_1\xi + c_2, \\ D_1\xi + D_2, \end{cases} \qquad u_1^1 = \begin{cases} E_1\xi + E_2, & \xi \in (0, \varrho), \\ F_1\xi + F_2, & \xi \in (\varrho, 2\varrho). \end{cases}$$

The stress resultants (cf. [10], Eqs. (4.11)) are given by

$$(3.2) \qquad N_0^{11} = \frac{1}{h}\left[A_1^{1111}\frac{dv_1^1}{d\xi} + A_2^{1111}\frac{du_1^1}{d\xi}\right] + n_0^{11},$$

$$L_0^{11} = \frac{1}{h}\left[A_2^{1111}\frac{dv_1^1}{d\xi} + A_4^{1111}\frac{du_1^1}{d\xi}\right] + l_0^{11},$$

where n_0^{11}, l_0^{11} are defined by Eqs. (4.14) in Ref. [10].

The constants c_α, D_α, E_α, F_α are interrelated according to:
- periodicity conditions

$$(3.3) \qquad \begin{aligned} v_1^1(0) &= v_1^1(2\varrho), & u_1^1(0) &= u_1^1(2\varrho), \\ N_0^{11}(0) &= N_0^{11}(2\varrho), & L_0^{11}(0) &= L_0^{11}(2\varrho); \end{aligned}$$

• switching and contact conditions

$$(3.4) \quad \begin{aligned} v_1^1(\varrho - 0) &= v_1^1(\varrho + 0), \qquad N_0^{11}(\varrho - 0) = N_0^{11}(\varrho + 0), \\ L_0^{11}(\varrho - 0) &= L_0^{11}(\varrho + 0) \le 0, \qquad L_0^{11}(\varrho - 0)[\![u_1^1]\!] = 0, \end{aligned}$$

$$(3.5) \quad [\![u_1^1]\!] = u_1^1(\varrho + 0) - u_1^1(\varrho - 0) \ge 0.$$

Analysis of the above conditions leads to

$$\varepsilon_{11}^y(\mathbf{v}^1) = 0, \qquad \langle \gamma_{11}^y(\mathbf{u}^1) \rangle = \begin{cases} 0 & \text{for } l_0^{11} \le 0, \\[2mm] -\dfrac{l_0^{11}}{A_4^{1111}} & \text{for } l_0^{11} > 0. \end{cases}$$

Here $\langle \cdot \rangle = \dfrac{1}{l} \displaystyle\int_0^l (\cdot) dy_1$. Since $\mathbf{u}^1$ is periodic, cf. Eq. (3.3)$_2$, one can make use of the relation: $\langle \gamma_{11}^y(\mathbf{u}^1) \rangle = -[\![u_1^1]\!]/l$.

Hence we find a non-zero component of the crack deformation tensor (2.3)

$$(3.6) \quad \varepsilon_{11}^F := \frac{[\![u_1^1]\!]}{l} = \begin{cases} 0, & \text{if } E_h \le 0 \text{ (the crack is closed)}, \\ F_{11}^0 E_h, & \text{if } E_h > 0 \text{ (the crack is open)}, \end{cases}$$

where, cf. Eq. (4.14)$_2$ in Ref. [10]

$$(3.7) \quad \begin{aligned} F_{11}^0 &= A_v^{1111}/A_4^{1111}, \qquad E_h = l_0^{11}/A_v^{1111}, \\ l_0^{11} &= A_3^{1111}\varepsilon_{11}^h + A_3^{1122}\varepsilon_{22}^h, \end{aligned}$$

or, using relations (Ref. [10], Eq. (4.12)$_2$, (4.12)$_1$), (A.1) we can write

$$(3.8) \quad F_{11}^0 = [\gamma - (\lambda_1)^2/\mu]^{-1}, \qquad E_h = \beta_{11}\varepsilon_{11}^h + \beta_{21}\varepsilon_{22}^h.$$

According to the formula (4.5) in Ref. [10] and Eqs. (2.4), we arrive at the homogenized constitutive relationships for axial stress resultants

$$(3.9) \quad N_h^{\alpha\alpha} = \begin{cases} A_1^{\alpha\alpha 11}\varepsilon_{11}^h + A_1^{\alpha\alpha 22}\varepsilon_{22}^h, & \text{for } E_h \le 0, \\ A_c^{\alpha\alpha 11}\varepsilon_{11}^h + A_c^{\alpha\alpha 22}\varepsilon_{22}^h, & \text{for } E_h > 0, \end{cases}$$

where

$$(3.10) \quad A_c^{\alpha\alpha\beta\beta} = A_1^{\alpha\alpha\beta\beta} - A_2^{\alpha\alpha 11}A_3^{11\beta\beta}(A_4^{1111})^{-1}.$$

Relations (3.9) are continuous along the line $E_h = 0$.

By virtue of the symmetry relations (cf. [10], Eqs. $(4.9)_1$ and $(4.12)_1$)

(3.11)
$$A_1^{\alpha\beta\lambda\mu} = A_1^{\lambda\mu\alpha\beta}, \qquad A_2^{\alpha\beta\lambda\mu} = A_3^{\lambda\mu\alpha\beta},$$

we have

(3.12)
$$A_c^{\alpha\alpha\beta\beta} = A_c^{\beta\beta\alpha\alpha},$$

hence the symmetry $A_c^{2211} = A_c^{1122}$ holds also when the crack is open.

It turns out to be helpful to write the components $A_c^{\alpha\alpha\beta\beta}$ in the form

(3.13)
$$A_c^{\lambda\lambda\mu\mu}/A_v^{1111} = \alpha_{\lambda\mu} - \beta_{\lambda 1}\beta_{\mu 1}F_{11}^0,$$

the non-dimensional coefficients $\alpha_{\lambda\mu}$ and $\beta_{\lambda\mu}$ being defined in the Appendix.

In the "technical" notation relationships (3.9) should be rewritten as follows:

• in the case of closed cracks ($E_h \leq 0$)

(3.14)
$$\begin{bmatrix} N_h^{11} \\ N_h^{22} \end{bmatrix} = \frac{2h}{1 - \nu_{12}\nu_{21}} \begin{bmatrix} E_1 & \nu_{12}E_1 \\ \nu_{21}E_2 & E_2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^h \\ \varepsilon_{22}^h \end{bmatrix},$$

• in the case of open cracks ($E_h > 0$)

(3.15)
$$\begin{bmatrix} N_h^{11} \\ N_h^{22} \end{bmatrix} = \frac{2h}{1 - \nu_{12}^c\nu_{21}^c} \begin{bmatrix} E_1^c & \nu_{12}^c E_1^c \\ \nu_{21}^c E_2^c & E_2^c \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^h \\ \varepsilon_{22}^h \end{bmatrix}.$$

The orthotropic constants for the case of closed cracks are given by

(3.16)
$$\nu_{12} = \frac{A_1^{1122}}{A_1^{1111}}, \qquad \nu_{21} = \frac{A_1^{1122}}{A_1^{2222}}, \qquad E_\alpha = (1 - \nu_{12}\nu_{21})\frac{A_1^{\alpha\alpha\alpha\alpha}}{2h}.$$

The components ν_{12}^c, ν_{21}^c and E_α^c are defined in terms of $A_c^{\alpha\alpha\beta\beta}$ in a similar manner.

3.2. Solution of the shearing local problem. Stiffness A_c^{1212}

It turns out that the unknown fields v_2^1, u_2^1 are piecewise linear functions:

(3.17)
$$v_2^1 = \begin{cases} c_1\xi + c_2, \\ D_1\xi + D_2, \end{cases} \qquad u_2^1 = \begin{cases} E_1\xi + E_2, & \xi \in (0, \varrho), \\ F_1\xi + F_2, & \xi \in (\varrho, 2\varrho). \end{cases}$$

The stress resultants, cf. ([10], Eqs. (4.11)–(4.14))

(3.18)
$$N_0^{21} = \frac{1}{h}\left[A_1^{2121}\frac{dv_2^1}{d\xi} + A_2^{2121}\frac{du_2^1}{d\xi}\right] + n_0^{21},$$

$$L_0^{21} = \frac{1}{h}\left[A_2^{2121}\frac{dv_2^1}{d\xi} + A_4^{2121}\frac{du_2^1}{d\xi}\right] + l_0^{21},$$

are piecewise constant. The periodicity conditions read:

$$(3.19) \qquad \begin{aligned} v_2^1(0) &= v_2^1(2\varrho), & u_2^1(0) &= u_2^1(2\varrho), \\ N_0^{21}(0) &= N_0^{21}(2\varrho), & L_0^{21}(0) &= L_0^{21}(2\varrho), \end{aligned}$$

while the switching conditions are

$$(3.20) \qquad \begin{aligned} v_2^1(\varrho - 0) &= v_2^1(\varrho + 0), \\ N_0^{21}(\varrho - 0) &= N_0^{21}(\varrho + 0), & L_0^{21}(\varrho - 0) &= L_0^{21}(\varrho + 0). \end{aligned}$$

Using (3.17)–(3.20) one finds

$$\varepsilon_{21}^y(\mathbf{v}^1) = 0, \qquad \langle \gamma_{21}^y(\mathbf{u}^1) \rangle = -\frac{l_0^{21}}{2A_4^{2121}}.$$

Due to the orthotropy we have $A_4^{2121} = A_2^{2121}$. By virtue of the relation $l_0^{21} = 2A_2^{2121}\varepsilon_{12}^h$ and Eq. (4.6) in [10], one finds

$$(3.21) \qquad \varepsilon_{12}^F = \varepsilon_{12}^h.$$

Therefore the homogenized constitutive relation $(2.4)_3$ assumes the form

$$(3.22) \qquad N_h^{12} = 2A_c^{1212}\varepsilon_{12}^h, \qquad A_c^{1212}/A_v^{1212} = 1 - \hat{\alpha},$$

where $\hat{\alpha}$ is defined by $(A.1)_5$. According to (A.2) we have $1 - \hat{\alpha} = d/h$. The effective Kirchhoff moduli of the cracked and uncracked laminate are

$$(3.23) \qquad G_{12} = A_v^{1212}/2h, \qquad G_{12}^c = A_c^{1212}/2h.$$

3.3. The homogenized potential

Having found relationships (3.9) and (3.22), one can express the homogenized constitutive relations in the hyperelastic form, cf. [10], Eq. (4.18)

$$(3.24) \qquad N_h^{\alpha\beta} = \frac{\partial \mathcal{V}_h}{\partial \varepsilon_{\alpha\beta}^h},$$

the potential $\mathcal{V}_h$ being defined by

$$(3.25) \qquad \mathcal{V}_h = \begin{cases} \mathcal{V}_h^0 & \text{for} \quad E_h \leq 0, \\ \mathcal{V}_h^c & \text{for} \quad E_h \geq 0, \end{cases}$$

where, cf. [10], Eq. (4.22)

$$(3.26) \qquad \begin{aligned} 2\mathcal{V}_h^0 &= \sum_{\alpha,\beta} A_1^{\alpha\alpha\beta\beta}\varepsilon_{\alpha\alpha}^h\varepsilon_{\beta\beta}^h + 2A_c^{1212}\left[(\varepsilon_{12}^h)^2 + (\varepsilon_{21}^h)^2\right], \\ 2\mathcal{V}_h^c &= \sum_{\alpha,\beta} A_c^{\alpha\alpha\beta\beta}\varepsilon_{\alpha\alpha}^h\varepsilon_{\beta\beta}^h + 2A_c^{1212}\left[(\varepsilon_{12}^h)^2 + (\varepsilon_{21}^h)^2\right]. \end{aligned}$$

The formula $(3.26)_2$ can be rewritten as follows

(3.27) $$\mathcal{V}_h^c = \mathcal{V}_h^0 - \frac{1}{2}A_v^{1111}F_{11}^0(E_h)^2.$$

By virtue of (3.27) one can readily prove that $\mathcal{V}_h$ is of class C^1 (not C^2), the result already known from Sec. 4 of Ref. [10]. We see that the line $E_h = 0$, cf. $(3.8)_2$,

(3.28) $$\beta_{11}\varepsilon_{11}^h + \beta_{21}\varepsilon_{22}^h = 0$$

is a line of discontinuity of the second order derivatives of the potential $\mathcal{V}_h$, cf. Fig. 4a. This figure characterizes the $[0°/90_3°]_s$ glass/epoxy laminate examined in detail in Sec. 3.1 of the second part of the present paper [11].
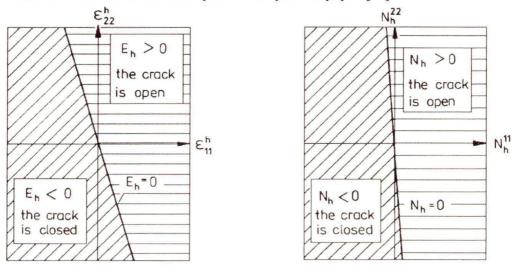
FIG. 4. a. $(\varepsilon_{11}^h, \varepsilon_{22}^h)$-plane; condition $E_h > 0$ for the crack opening in the glass/epoxy $[0°, 90_3°]_s$ laminate tested by HIGHSMITH and REIFSNIDER [6]; $E_h = 0.4298\varepsilon_{11}^h + 0.1286\varepsilon_{22}^h$. b. (N_h^{11}, N_h^{22})-plane. Condition $N_h > 0$ for the crack opening; the same laminate, $N_h = 0.632N_h^{11} + 0.0408N_h^{22}$.

3.4. Inverted form of the homogenized constitutive relations

The constitutive relations (3.24) can be inverted. We shall now find this inverse form. The main problem reduces to inverting relations (3.9). For this purpose we introduce here matrix notation.

Let us define the following vectors and matrices

$$\mathbf{k} = [\beta_{11}, \beta_{21}], \qquad \mathbf{k}_\perp = [-\beta_{21}, \beta_{11}], \qquad \mathbf{C} = \begin{bmatrix} \mathbf{k} \\ \mathbf{k}_\perp \end{bmatrix},$$

(3.29) $$\mathbf{A}_m = \begin{bmatrix} A_m^{1111} & A_m^{1122} \\ A_m^{1122} & A_m^{2222} \end{bmatrix}, \qquad m = 1 \quad \text{or} \quad c,$$

$$\boldsymbol{\varepsilon} = \left[\varepsilon_{11}^h, \varepsilon_{22}^h\right]^T, \qquad \mathbf{N} = \left[N_h^{11}, N_h^{22}\right]^T.$$

The constitutive relations (3.9) can be written as follows

$$(3.30) \qquad \mathbf{N} = \begin{cases} \mathbf{A}_1 \boldsymbol{\varepsilon}, & \text{for} \quad \mathbf{k} \cdot \boldsymbol{\varepsilon} \le 0, \\ \mathbf{A}_c \boldsymbol{\varepsilon}, & \text{for} \quad \mathbf{k} \cdot \boldsymbol{\varepsilon} \ge 0. \end{cases}$$

This relation is continuous because $\mathcal{V}_h$ is of class C^1; hence

$$(3.31) \qquad \mathbf{A}_1 \mathbf{k}_\perp^T = \mathbf{A}_c \mathbf{k}_\perp^T.$$

Let us set $\mathbf{e} = \mathbf{C}\boldsymbol{\varepsilon}$, $\mathbf{e} = (e_1, e_2)$. The inverse relation reads

$$(3.32) \qquad \boldsymbol{\varepsilon} = \mathbf{C}^{-1}\mathbf{e}, \qquad \mathbf{C}^{-1} = \frac{1}{\det \mathbf{C}} \mathbf{C}^T.$$

Hence (3.30) assumes the form

$$(3.33) \qquad \mathbf{N} = \begin{cases} \mathbf{B}_1 \mathbf{e}, & \text{for} \quad e_1 \le 0, \\ \mathbf{B}_c \mathbf{e}, & \text{for} \quad e_1 \ge 0, \end{cases}$$

where

$$(3.34) \qquad \mathbf{B}_1 = \mathbf{A}_1 \mathbf{C}^{-1}, \qquad \mathbf{B}_c = \mathbf{A}_c \mathbf{C}^{-1}.$$

Consequently

$$(3.35) \qquad \mathbf{e} = \begin{cases} \mathbf{D}_1 \mathbf{N}, & \text{for} \quad e_1 \le 0, \\ \mathbf{D}_c \mathbf{N}, & \text{for} \quad e_1 \ge 0, \end{cases}$$

where

$$(3.36) \qquad \mathbf{D}_1 = \mathbf{C}\mathbf{A}_1^{-1}, \qquad \mathbf{D}_c = \mathbf{C}\mathbf{A}_c^{-1}.$$

Our aim is to express conditions $e_1 < 0$ or $e_1 > 0$ in terms of $\mathbf{N}$. To this end let us define a new vector $\mathcal{E} = \mathbf{D}_1 \mathbf{N}$. This definition does not depend on the sign of e_1. We express (3.35) in terms of $\mathcal{E}$:

$$(3.37) \qquad \mathbf{e} = \begin{cases} \mathcal{E}, & \text{for} \quad e_1 \le 0, \\ \mathbf{P}\mathcal{E}, & \text{for} \quad e_1 \ge 0, \end{cases}$$

where

$$(3.38) \qquad \mathbf{P} = \mathbf{D}_c \mathbf{D}_1^{-1} \qquad \text{or} \qquad \mathbf{P} = \frac{1}{\det \mathbf{C}} \mathbf{C}\mathbf{A}_c^{-1} \mathbf{A}_1 \mathbf{C}^T.$$

One can prove that

$$(3.39) \qquad P_{11} = \frac{\det \mathbf{A}_1}{\det \mathbf{A}_c} \qquad \text{and} \qquad P_{12} = 0.$$

The last equality is crucial here. It is a consequence of continuity requirements (3.31). The relation $(3.39)_1$ implies $P_{11} > 0$. Hence we conclude that

$$(3.40) \qquad e_1 = \begin{cases} \mathcal{E}_1, & \text{for} \quad e_1 \leq 0, \\ P_{11}\mathcal{E}_1, & \text{for} \quad e_1 \geq 0, \quad P_{11} > 0. \end{cases}$$

The relations given above show that sign $e_1 = $ sign $\mathcal{E}_1$, which makes it possible to rewrite (3.37) in the form

$$(3.41) \qquad \mathbf{C}\boldsymbol{\varepsilon} = \begin{cases} \mathbf{D}_1\mathbf{N}, & \text{for} \quad \mathcal{E}_1 \leq 0, \\ \mathbf{D}_c\mathbf{N}, & \text{for} \quad \mathcal{E}_1 \geq 0, \end{cases}$$

and, finally, to find

$$(3.42) \qquad \boldsymbol{\varepsilon} = \begin{cases} \mathbf{A}_1^{-1}\mathbf{N}, & \text{for} \quad \mathcal{E}_1 \leq 0, \\ \mathbf{A}_c^{-1}\mathbf{N}, & \text{for} \quad \mathcal{E}_1 \geq 0. \end{cases}$$

The condition $\mathcal{E}_1 \leq 0$ can be expressed as $N_h \leq 0$, where

$$(3.43) \qquad N_h = (\beta_{11}\alpha_{22} - \beta_{21}\alpha_{21})N_h^{11} + (\alpha_{11}\beta_{21} - \beta_{11}\alpha_{12})N_h^{22},$$

and sign $N_h = $ sign E_h.

The inverted form of the homogenized relations (3.9) is

$$(3.44) \qquad \varepsilon_{\alpha\alpha}^h = \begin{cases} \dfrac{1}{2hE_\alpha}(N_h^{\alpha\alpha} - \nu_{\beta\alpha}N_h^{\beta\beta}), & \text{for} \quad N_h \leq 0, \\[2mm] \dfrac{1}{2hE_\alpha^c}(N_h^{\alpha\alpha} - \nu_{\beta\alpha}^c N_h^{\beta\beta}), & \text{for} \quad N_h \geq 0, \end{cases}$$

and $\beta = 3 - \alpha$; do not sum over α and β!

Recalling relations (4.22) of Ref. [10] and (3.22) one can easily express $\mathcal{V}_h$ in terms of $N_h^{\alpha\beta}$. Its line of discontinuity of its second order derivatives is $N_h = 0$, cf. Fig. 4b. The data for this figure were taken for the laminate considered in Sec. 3.1 of Ref. [11].

REMARK 3.1

The considerations of Section 3 may be viewed as a practical procedure for finding the complementary or dual effective potential $\mathcal{V}_h^*$. Detailed study of duality is provided by our mathematical paper (Telega and Lewiński [15]). Nevertheless it is worth noting that $\mathcal{V}_h^*$ may be determined as the Fenchel conjugate of $\mathcal{V}_h$, i.e.

$$(3.45) \qquad \mathcal{V}_h^*(\mathbf{E}^*) = \sup\left\{ E^{*\alpha\beta}E_{\alpha\beta} - \mathcal{V}_h(\mathbf{E}) \mid \mathbf{E} \in \mathbb{E}_s^2 \right\}, \qquad \mathbf{E}^* \in \mathbb{E}_s^2.$$

The complementary potential $\mathcal{V}_h^*$ is strictly convex, of class C^1 and

$$(3.46) \qquad \boldsymbol{\varepsilon}^h = \frac{\partial \mathcal{V}_h^*}{\partial \mathbf{N}^h}, \qquad \boldsymbol{\varepsilon}^h \in \mathbb{E}_s^2, \qquad \mathbf{N}^h \in \mathbb{E}_s^2.$$

4. Parallel cracks: the space-scaling homogenization approach – model (h_0, l)

The aim of this section is to find effective characteristics of the laminate of Fig. 2 according to the (h_0, l) model of Sec. 5, Ref. [10]. The predictions of the loss of effective stiffnesses found in this section involve the l/h ratio and apply for arbitrary values of this ratio.

Similarly as in the in-plane scaling method, the local problem (P^2_{loc}) (formulated in Sec. 5.2 of Ref. [10]) splits up into two: stretching and shearing problems. The unknown functions depend solely on $y_1 = h\xi$.

4.1. Solution of the stretching local problem

The unknown functions are v_1^1, u_1^1 and w^2. The non-vanishing stress resultants are given by Eqs. (5.20) of Ref. [10]; they assume the form

(4.1)
$$\begin{aligned}
N_0^{11} &= A_v^{1111}(v' + \alpha u' + \beta w) + n_0^{11}, \\
N_0^{22} &= A_v^{1111}(\beta_1 v' + \beta_4 u' + \beta_2 w) + n_0^{22}, \\
L_0^{11} &= A_v^{1111}(\alpha v' + \gamma u' + \lambda_1 w) + l_0^{11}, \\
L_0^{22} &= A_v^{1111}(\beta_4 v' + \gamma_3 u' + \beta_3 w) + l_0^{22}, \\
R_0 &= \frac{1}{h^2} A_v^{1111}(\beta v' + \lambda_1 u' + \mu w), \\
Q_0^1 &= \frac{\delta}{h} A_v^{1111}(u - w'), \qquad (\cdot)' = d(\cdot)/d\xi,
\end{aligned}$$

where new unknowns have been introduced

(4.2)
$$v = v_1^1/h, \qquad u = u_1^1/h,$$

(4.3)
$$w = w^2/h^2 + w_0, \qquad w_0 = \frac{1}{\mu}\left(\beta \varepsilon_{11}^h + \beta_2 \varepsilon_{22}^h\right).$$

The quantities $n_0^{\alpha\alpha}$, $l_0^{\alpha\alpha}$ are defined by Eqs. (4.14) of Ref. [10]. The new coefficients involved in (4.1)–(4.3) are defined by (A.1).

The equilibrium equations reduce to the form

(4.4)
$$\frac{dN_0^{11}}{d\xi} = 0, \qquad \frac{dL_0^{11}}{d\xi} = hQ_0^1, \qquad \frac{dQ_0^1}{d\xi} = -hR_0.$$

On expressing the equilibrium equations (4.4) in terms of the unknowns (v, u, w), one arrives at the following system of differential equations

(4.5)
$$\begin{aligned}
v'' + \alpha u'' + \beta w' &= 0, \\
\alpha v'' + (\gamma u'' - \delta u) + \lambda w' &= 0, \\
-\beta v' - \lambda u' + (\delta w'' - \mu w) &= 0.
\end{aligned}$$

The strong formulation of the local problem amounts here to finding the fields (v, u, w) defined on the interval $[0, 2\varrho]$ such that:
- the equations (4.5) are satisfied for each $\xi \in (0, \varrho) \cup (\varrho, 2\varrho)$;
- the periodicity conditions

(4.6)
$$v(0) = v(2\varrho), \qquad u(0) = u(2\varrho), \qquad w(0) = w(2\varrho),$$
$$N_0^{11}(0) = N_0^{11}(2\varrho), \qquad L_0^{11}(0) = L_0^{11}(2\varrho), \qquad Q_0^1(0) = Q_0^1(2\varrho)$$

are satisfied;
- the switching conditions are fulfilled at $\xi = \varrho$

(4.7)
$$v(\varrho - 0) = v(\varrho + 0), \qquad w(\varrho - 0) = w(\varrho + 0),$$
$$N_0^{11}(\varrho - 0) = N_0^{11}(\varrho + 0), \qquad Q_0^1(\varrho - 0) = Q_0^1(\varrho + 0);$$

(4.8)
$$L = L_0^{11}(\varrho - 0) = L_0^{11}(\varrho + 0) \leq 0,$$
$$L[\![u]\!] = 0, \qquad [\![u]\!] = u(\varrho + 0) - u(\varrho - 0) \geq 0.$$

A detailed solution to the problem stated above will be given a little later. Suppose now that this solution is known. Similarly as in Sec. 3, the problem can be reduced to finding the field ε_{11}^F given by (2.3)$_1$. In the case considered here $s_1 = 0$, $s_2 = l_2$, $|Y| = l_1 l_2 = l l_2$; hence

(4.9)
$$\tilde{\varepsilon}_{11}^F := \frac{[\![u_1^1]\!]}{l} = \frac{[\![u]\!]}{2\varrho}.$$

The tilde over ε_{11}^F indicates that this quantity is evaluated by the (h_0, l) approach. Thus the only unknown which is really needed for assessing the loss of stiffnesses is the jump $[\![u]\!]$.

Let us proceed now to the analysis of the local problem. One can note first that the unknown w can be eliminated from Eqs. (4.5). One finds

(4.10)
$$\mu_{11} v'' + (\mu_{12} u'' + \mu_{13} u) = 0,$$
$$(\mu_{21} v'' + \mu_{22} v) + (\mu_{23} u'' + \mu_{24} u) = c_1 \xi + c_2,$$

where $\mu_{\alpha k}$ are defined in the Appendix and c_α are arbitrary constants; $u'' = d^2 u / d\xi^2$. The fields u and v satisfy the following uncoupled equations

(4.11)
$$Lu = 0, \qquad Lv = \mu_{13}(c_1 \xi + c_2),$$

where

(4.12)
$$L = a_1 \frac{d^4}{d\xi^4} + a_2 \frac{d^2}{d\xi^2} + a_3;$$

the coefficients a_1, a_2 and a_3 are defined in the Appendix. Let $\pm\sigma$, $\pm\omega$ be the roots of the characteristic equation

$$(4.13) \qquad a_1 x^4 + a_2 x^2 + a_3 = 0.$$

In general, σ and ω can assume real or complex values. In the latter case $\sigma = \overline{\omega}$; the bar denotes the complex conjugate.

Symmetries characterizing the problem imply that the fields (u, v) are antisymmetric with respect to the point $\xi = \varrho$. Thus we can write

$$(4.14) \qquad u = \begin{cases} u_{\mathrm{I}}, \\ u_{\mathrm{II}}, \end{cases} \qquad v = \begin{cases} v_{\mathrm{I}}, & \xi \in (0, \varrho), \\ v_{\mathrm{II}}, & \xi \in (\varrho, 2\varrho), \end{cases}$$

where

$$(4.15) \qquad \begin{aligned} u_{\mathrm{I}} &= B_1 e^{-\sigma\xi} + B_2 e^{-\sigma(\varrho-\xi)} + B_3 e^{-\omega\xi} + B_4 e^{-\omega(\varrho-\xi)}, \\ u_{\mathrm{II}} &= -B_1 e^{-\sigma(2\varrho-\xi)} - B_2 e^{-\sigma(\xi-\varrho)} - B_3 e^{-\omega(2\varrho-\xi)} - B_4 e^{-\omega(\xi-\varrho)}; \end{aligned}$$

$$(4.16) \qquad \begin{aligned} v_{\mathrm{I}} &= D_1 \xi + D_2 + G_1 e^{-\sigma\xi} + G_2 e^{-\sigma(\varrho-\xi)} + G_3 e^{-\omega\xi} + G_4 e^{-\omega(\varrho-\xi)}, \\ v_{\mathrm{II}} &= F_1 \xi + F_2 - G_1 e^{-\sigma(2\varrho-\xi)} - G_2 e^{-\sigma(\xi-\varrho)} - G_3 e^{-\omega(2\varrho-\xi)} - G_4 e^{-\omega(\xi-\varrho)}, \end{aligned}$$

here B_i, G_i, D_α and F_α are unknown constants. The first equilibrium equation $(4.5)_1$ makes it possible to determine the function w, being equal to w_{I} for $\xi \in (0, \varrho)$ and w_{II} for $\xi \in (\varrho, 2\varrho)$

$$(4.17) \qquad \begin{aligned} w_{\mathrm{I}} &= K_1 + \frac{\sigma}{\beta}(\alpha B_1 + G_1)e^{-\sigma\xi} - \frac{\sigma}{\beta}(\alpha B_2 + G_2)e^{-\sigma(\varrho-\xi)} \\ &\qquad + \frac{\omega}{\beta}(\alpha B_3 + G_3)e^{-\omega\xi} - \frac{\omega}{\beta}(\alpha B_4 + G_4)e^{-\omega(\varrho-\xi)}, \\ w_{\mathrm{II}} &= L_1 + \frac{\sigma}{\beta}(\alpha B_1 + G_1)e^{-\sigma(2\varrho-\xi)} - \frac{\sigma}{\beta}(\alpha B_2 + G_2)e^{-\sigma(\xi-\varrho)} \\ &\qquad + \frac{\omega}{\beta}(\alpha B_3 + G_3)e^{-\omega(2\varrho-\xi)} - \frac{\omega}{\beta}(\alpha B_4 + G_4)e^{-\omega(\xi-\varrho)}. \end{aligned}$$

Having found the formulae (4.14)–(4.17) one can express the stress resultants N_0^{11}, L_0^{11}, Q_0^1 in terms of the functions involved in (4.15)–(4.17) and unknown constants D_α, F_α, K_1, L_1, B_i, G_i; $i = 1, 2, 3, 4$; $\alpha = 1, 2$. We shall omit details of the evaluation of these constants and report only the final results. The relative opening of the crack $\tilde{\varepsilon}_{11}^F$ defined by Eq. (4.9) depends on sign $l_0^{11} = $ sign E_h:

$$(4.18) \qquad \tilde{\varepsilon}_{11}^F = \begin{cases} 0, & \text{for} \quad E_h \leq 0, \\ F_{11}(\varrho)E_h, & \text{for} \quad E_h > 0, \end{cases}$$

where E_h has been defined by Eq. $(3.8)_2$ and the function $F_{11}(\varrho)$ has the form

$$(4.19) \qquad F_{11}(\varrho) = f_{11} \left[\frac{\beta_{11}}{\sigma^2 \omega^2} g_1(\sigma, \omega) + g_2(\sigma, \omega) F(\varrho; \omega, \sigma) \right]^{-1},$$

where

$$(4.20) \qquad g_\alpha(\sigma, \omega) = \gamma_{\alpha 1} + \gamma_{\alpha 2} \sigma^2 \omega^2 + \gamma_{\alpha 3}(\sigma^2 + \omega^2)$$

and

$$(4.21) \qquad F(\varrho; \omega, \sigma) = \frac{\varrho}{\sigma^2 - \omega^2} \left(\frac{\operatorname{cth} \omega \varrho}{\omega} - \frac{\operatorname{cth} \sigma \varrho}{\sigma} \right).$$

Parameters $\gamma_{\alpha k}$, f_{11} and β_{11}, depending on the geometry and material properties of the laminate, are defined in the Appendix.

Note that the function $F_{11}(\varrho)$ preserves its form after the change: $\sigma \to \omega$, $\omega \to \sigma$; moreover, g_α do not depend on whether σ and ω are real- or complex-valued. In fact, in view of (4.13)

$$(4.22) \qquad \sigma^2 + \omega^2 = -a_2/a_1, \qquad \sigma^2 \omega^2 = a_3/a_1.$$

If $\sigma = p - iq$, $\omega = \overline{\sigma} = p + iq$ ($p, q \in \mathbb{R}$) we change the definition

$$(4.23) \qquad F(\varrho; \omega, \sigma) = F_0(\varrho; p, q).$$

After appropriate manipulations we find

$$(4.24) \qquad F_0(\varrho; p, q) = \frac{f(p\varrho, q\varrho)}{2pq(p^2 + q^2)},$$

where the function f is defined by

$$(4.25) \qquad f(x, y) = \frac{y \operatorname{sh} 2x + x \sin 2y}{\operatorname{ch} 2x - \cos 2y}.$$

4.2. Assessing loss of the $\widetilde{A}_c^{\alpha\alpha\beta\beta}$ stiffnesses

Having found the relation $\widetilde{\varepsilon}_{11}^F(\varepsilon^h)$ one can determine the homogenized constitutive relationships via Eqs. (2.4)

$$(4.26) \qquad N_h^{\alpha\alpha} = \begin{cases} A_1^{\alpha\alpha 11} \varepsilon_{11}^h + A_1^{\alpha\alpha 22} \varepsilon_{22}^h, & \text{for} \quad E_h \leq 0, \\ \widetilde{A}_c^{\alpha\alpha 11} \varepsilon_{11}^h + \widetilde{A}_c^{\alpha\alpha 22} \varepsilon_{22}^h,, & \text{for} \quad E_h > 0, \end{cases}$$

where the reduced stiffnesses can be expressed by a single formula

$$(4.27) \qquad \widetilde{A}_c^{\lambda\lambda\mu\mu}/A_v^{1111} = \alpha_{\lambda\mu} - \beta_{\lambda 1} \beta_{\mu 1} F_{11}(\varrho),$$

and the coefficients $\alpha_{\lambda\mu}$ and $\beta_{\lambda\mu}$ are defined by Eqs. (A.1). The relations (4.26) are continuous along the line $E_h = 0$.

The constitutive relationship (4.26) can be expressed in terms of orthotropic constants. For the case of closed cracks ($E_h \leq 0$) these relations have the form (3.14), and for the case of open cracks ($E_h > 0$) they assume the form

(4.28)
$$
\begin{bmatrix} N_h^{11} \\ N_h^{22} \end{bmatrix} = \frac{2h}{1 - \tilde{\nu}_{12}^c \tilde{\nu}_{21}^c} \begin{bmatrix} \tilde{E}_1^c & \tilde{\nu}_{12}^c \tilde{E}_1^c \\ \tilde{\nu}_{21}^c \tilde{E}_2^c & \tilde{E}_2^c \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^h \\ \varepsilon_{22}^h \end{bmatrix},
$$

where

(4.29)
$$
\tilde{\nu}_{12}^c = \frac{\tilde{A}_c^{1122}}{\tilde{A}_c^{1111}}, \qquad \tilde{\nu}_{21}^c = \frac{\tilde{A}_c^{1122}}{\tilde{A}_c^{2222}},
$$
$$
\tilde{E}_\alpha^c = (1 - \tilde{\nu}_{12}^c \tilde{\nu}_{21}^c) \frac{\tilde{A}_c^{\alpha\alpha\alpha\alpha}}{2h}.
$$

The formula for $\tilde{E}_1^c(\varrho)$ does not coincide with the analogous formula found by HASHIN [5], although one can note a similarity between the formulae (2.40) and (2.46) of HASHIN [5] and formulae (4.27) for $\lambda = \mu = 1$, (4.19) and (4.24) derived above.

REMARK 4.1

The constitutive relations (4.26) can be inverted to the form similar to (3.44), where instead of E_α^c, $\nu_{\alpha\beta}^c$ one should put $\tilde{E}_\alpha^c$, $\tilde{\nu}_{\alpha\beta}^c$. The condition $N_h < 0$ or $N_h > 0$ remains unchanged.

4.3. Solution of the shearing local problem

The dimensionless fields

(4.30)
$$
\hat{u} = u_2^1/h, \qquad \hat{v} = v_2^1/h,
$$

will play the role of basic unknowns. The stress resultants that intervene in the shearing deformation are, cf. ([10], Eq. (5.20))

(4.31)
$$
N_0^{21} = A_v^{2121} \left(\frac{d\hat{v}}{d\xi} + \hat{\alpha}\frac{d\hat{u}}{d\xi} + 2\varepsilon_{21}^h \right),
$$
$$
L_0^{21} = \hat{\alpha} A_v^{2121} \left(\frac{d\hat{v}}{d\xi} + \frac{d\hat{u}}{d\xi} + 2\varepsilon_{21}^h \right),
$$
$$
Q_0^2 = \frac{\hat{\delta}}{h} A_v^{2121} \hat{u}.
$$

The equilibrium equations

(4.32)
$$
\frac{dN_0^{21}}{d\xi} = 0, \qquad \frac{dL_0^{21}}{d\xi} = hQ_0^2,
$$

expressed in terms of the unknowns (4.30) assume the form

(4.33)
$$\frac{d^2\widehat{v}}{d\xi^2} + \widehat{\alpha}\frac{d^2\widehat{u}}{d\xi^2} = 0, \qquad \widehat{\alpha}\frac{d^2\widehat{v}}{d\xi^2} + \left(\widehat{\alpha}\frac{d^2}{d\xi^2} - \widehat{\delta}\right)\widehat{u} = 0.$$

Further analysis will be confined to the case when the matrix

(4.34)
$$\begin{bmatrix} A_v^{2121} & A_{vu}^{2121} \\ A_{vu}^{2121} & A_u^{2121} \end{bmatrix}$$

is positive definite, which for real laminates is not a restriction. This means that

(4.35)
$$0 < \widehat{\alpha} < 1,$$

which is readily satisfied since $\widehat{\alpha} = c/h$, cf. (A.2). Let us pass to the strong formulation of the local problem considered. Our goal is to find the fields $(\widehat{v}, \widehat{u})$ defined on $[0, 2\varrho]$ and satisfying:

- the equations (4.33) for $\xi \in (0, \varrho)$ and $\xi \in (\varrho, 2\varrho)$,
- the conditions of periodicity

(4.36)
$$\widehat{v}(0) = \widehat{v}(2\varrho), \qquad \widehat{u}(0) = \widehat{u}(2\varrho),$$
$$N_0^{21}(0) = N_0^{21}(2\varrho), \qquad L_0^{21}(0) = L_0^{21}(2\varrho),$$

- the switching conditions at $\xi = \varrho$

(4.37)
$$\widehat{v}(\varrho - 0) = \widehat{v}(\varrho + 0), \qquad N_0^{21}(\varrho - 0) = N_0^{21}(\varrho + 0),$$
$$L_0^{21}(\varrho - 0) = L_0^{21}(\varrho + 0) = 0.$$

Prior to solving the local problem formulated above let us recall that the only field we need for assessing the loss of G_{12} is the quantity ε_{12}^F, cf. Eq. (2.3). Here

(4.38)
$$2\widetilde{\varepsilon}_{12}^F := \frac{[\![u_2^1]\!]}{l} = \frac{[\![\widehat{u}]\!]}{2\varrho}.$$

The tilde indicates that we use the space-scaling (h_0, l) method. The homogenized constitutive relation has the form $(2.4)_3$ with ε_{12}^F defined by Eq. (4.38).

Bearing in mind that we are now interested only in finding the field $\widetilde{\varepsilon}_{12}^F$, we proceed to analyze the local problem. Equations (4.33) yields the governing equations of the form

(4.39)
$$L\widehat{v} = 0, \qquad L\widehat{u} = 0, \qquad L = \frac{d^4}{d\xi^4} - (\widehat{\lambda})^2\frac{d^2}{d\xi^2}.$$

The parameter

(4.40)
$$\widehat{\lambda} = \left[\frac{\widehat{\delta}}{\widehat{\alpha}(1-\widehat{\alpha})}\right]^{1/2},$$

is positive, cf. Eq. (4.35). Taking into account (A.2) one can express $\widehat{\lambda}$ by the formula

(4.41)
$$\widehat{\lambda} = \frac{h}{c}\left[\frac{3((c/d)+1)}{(dG_A/cG_T)+1}\right]^{1/2}.$$

Thus the fields $(\widehat{u},\widehat{v})$ are spanned over the basis $\{1,\xi,\exp(\widehat{\lambda}\xi),\exp(-\widehat{\lambda}\xi)\}$ on both subintervals $(0,\varrho)$ and $(\varrho,2\varrho)$. For the sake of brevity we omit the derivation and report only the final result:

(4.42)
$$\widehat{\varepsilon}^F_{12} = \frac{h}{c}F_{12}(\widehat{\lambda}\varrho)\varepsilon^h_{12},$$
$$F_{12}(x) = \left(1 + \frac{d}{c}x\,\mathrm{cth}\,x\right)^{-1}.$$

4.4. Assessing the loss of the Kirchhoff modulus

According to the definition $(2.4)_3$ combined with (4.42), one finds

(4.43) $\qquad N^{12}_h = 2\widetilde{A}^{1212}_c\varepsilon^h_{12},\qquad \widetilde{A}^{1212}_c/A^{1212}_v = 1 - F_{12}(\widehat{\lambda}\varrho),$

(4.44) $\qquad \widetilde{G}^c_{12} = \widetilde{A}^{1212}_c/2h,$

where $\widetilde{G}^c_{12}$ is the reduced Kirchhoff modulus of the laminate. One can prove that formulae (4.43) and (4.44) coincide with those of HASHIN [5, Eq. (3.22)], TAN and NUISMER [14] and TSAI and DANIEL [16].

4.5. Homogenized potential

Having derived the homogenized constitutive relations (4.26) and (4.43) we can combine them to form the hyperelastic law, cf. Eq. (5.30) in Ref. [10]

(4.45)
$$N^{\alpha\beta}_h = \frac{\partial W_h}{\partial\varepsilon^h_{\alpha\beta}}.$$

The hyperelastic potential is given by

(4.46) $\qquad\qquad$ ████████████████ $\qquad$ for $E_h \leq 0,$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ for $E_h \geq 0,$

where

(4.47)
$$2W_h^0 = \sum_{\alpha,\beta} A_1^{\alpha\alpha\beta\beta}\varepsilon_{\alpha\alpha}^h\varepsilon_{\beta\beta}^h + 2\widetilde{A}_c^{1212}\left[(\varepsilon_{12}^h)^2 + (\varepsilon_{21}^h)^2\right],$$

$$2W_h^c = \sum_{\alpha,\beta} \widetilde{A}_c^{\alpha\alpha\beta\beta}\varepsilon_{\alpha\alpha}^h\varepsilon_{\beta\beta}^h + 2\widetilde{A}_c^{1212}\left[(\varepsilon_{12}^h)^2 + (\varepsilon_{21}^h)^2\right].$$

The potential W_h^2 for the case of open cracks can be expressed as follows

(4.48)
$$W_h^c = W_h^0 - \frac{1}{2}A_v^{1111}F_{11}(\varrho)(E_h)^2.$$

By virtue of the above expression one easily verifies that the potential W_h is of class C^1, $E_h = 0$ being its line of non-smoothness of its first order derivatives. The complementary effective potential $W_h^*(N_h^{\alpha\beta})$ can be calculated by using the Fenchel transformation of W_h, cf. Remarks 3.1 and 4.1. The potential W_h^*, defined on the space $\mathbb{E}_s^2$ remains smooth, the equation $N_h = 0$ (cf. Eq. (3.43)) determines the line of the non-smoothness of its first order derivatives, cf. Fig. 4b. We recall that $\mathbb{E}_s^2$ is the space of symmetric 2×2 matrices, here identified with its dual.

5. Final remarks

Accuracy of the formulae for effective moduli of the cracked laminates found in this work is examined in the second part of the paper [11]. There we refer to other known analytical models concerning aligned, regularly distributed cracks as well as to available experimental data. We show that for the case of aligned cracks the predictions of the model (h_0, l) lie closely to results of McCartney [12, 13]. In their principles, however, these models are completely different, see Introductions to Refs. [9, 10].

Possible generalization of the formulae found in this paper to the case of other damage modes and, in general, to the case of angle-ply laminates would be of vital interest. For instance one can choose a different way: use the homogenization scheme of Caillerie – Kohn – Vogelius (cf. Ref. [7]) and then apply the finite element method to solve the local problems. The recent paper of Adolfsson and Gudmundson [1] goes in this direction, yet in the manner that circumvents the homogenization formalism of the passage from the original problem to the effective macroscopic problem and the underlying local analysis.

Appendix

The following non-dimensional parameters depending on the quantities defined in Ref. [10] are used in the present paper:

$$\alpha_{\lambda\mu} = A_1^{\lambda\lambda\mu\mu}/A_v^{1111}, \qquad \beta_{\lambda\mu} = A_2^{\lambda\lambda\mu\mu}A_v^{1111},$$

$$(\alpha, \beta, \gamma, \delta) = (A_{vu}^{1111}, h^2 A_{vw}^{11}, A_{u}^{1111}, h^2 H^{11})/A_v^{1111},$$

$$(\lambda_1, \mu) = (h^2 A_{uw}^{11}, h^4 A_w)/A_v^{1111}, \qquad \lambda = \delta + \lambda_1,$$

(A.1)

$$(\beta_1, \beta_2, \beta_3, \beta_4, \gamma_3) = (A_v^{1122}, h^2 A_{vw}^{22}, h^2 A_{uw}^{22}, A_{vu}^{1122}, A_u^{1122})/A_v^{1111},$$

$$(\widehat{\alpha}, \widehat{\gamma}, \widehat{\delta}) = (A_{vu}^{2121}, A_u^{2121}, h H^{22})/A_v^{2121}.$$

Since $D_{1212}^{NL} = -D_{1212}^{N}$ we have $A_{vu}^{1212} = A_u^{1212}$. Hence one can prove that

$$A_v^{1212} = 2hG_A$$

and

(A.2)

$$\widehat{\alpha} = \widehat{\gamma} = \frac{c}{h}, \qquad \widehat{\delta} = 3\left[\frac{dG_A}{hG_T} + \frac{c}{h}\right]^{-1},$$

$$\widehat{\lambda} = h(\widehat{\delta}/cd)^{1/2}.$$

The parameters appearing in Eqs. (4.10), (4.13), (4.19) and (4.20) are defined by

$$\mu_{11} = 1 - \alpha\beta/\lambda, \qquad \mu_{12} = \alpha - \beta\gamma/\lambda, \qquad \mu_{13} = \delta\beta/\lambda,$$

$$\mu_{21} = \alpha\mu_{13}, \qquad \mu_{22} = \beta^2 - \mu, \qquad \mu_{23} = \gamma\mu_{13},$$

$$\mu_{24} = \beta\lambda - \delta\mu_{13} - \mu\alpha, \qquad \mu_{44} = \mu_{13}/\mu_{11};$$

$$a_1 = \mu_{12}\mu_{21} - \mu_{11}\mu_{23}, \qquad a_3 = \mu_{13}\mu_{22},$$

(A.3)

$$a_2 = \mu_{22}\mu_{12} + \mu_{13}\mu_{21} - \mu_{24}\mu_{11};$$

$$f_{11} = \alpha - \mu_{12}/\mu_{11},$$

$$\gamma_{11} = \mu_{44}(\beta - \mu_{44}), \qquad \gamma_{12} = \frac{\mu_{12}}{\mu_{11}}f_{11},$$

$$\gamma_{22} = \frac{\delta}{\beta}(f_{11})^2, \qquad \gamma_{13} = \mu_{44}f_{11},$$

$$\gamma_{21} = \frac{\delta}{\beta}(\beta - \mu_{44})^2, \qquad \gamma_{23} = \frac{\delta}{\beta}f_{11}(\beta - \mu_{44}).$$

The parameters $\alpha_{\lambda\mu}$, $\beta_{\lambda\mu}$ defined by (A.1)$_{1,2}$ can be expressed in terms of other parameters as follows

$$\alpha_{11} = 1 - \beta^2/\mu, \qquad \alpha_{12} = \beta_1 - \beta\beta_2/\mu, \qquad \rule{3cm}{0.4cm}$$

(A.4) $\qquad \beta_{11} = \alpha - \beta\lambda_1/\mu, \qquad \beta_{12} = \beta_4 - \beta\beta_3/\mu,$

$$\beta_{21} = \beta_4 - \beta_2\lambda_1/\mu, \qquad \beta_{22} = \gamma_2 - \beta_2\beta_3/\mu.$$

Note that $\alpha_{12} = \alpha_{21}$ but $\beta_{12} \neq \beta_{21}$.

Acknowledgements

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