Arch. Mech., 48, 1, pp. 67–88, Warszawa 1996

Nondestructive evaluation of cracks in conducting magnetoelastic materials

D. GAFKA (WARSZAWA)

THEORETICAL ANALYSIS of the alternating current flow around the crack appearing in the loaded material is given. The effective conductivity near the crack has been found to be nonhomogeneous, so the phenomenon of accumulation of the charge appears there. The charge density of accumulated charge near the crack is given. The potential drop has been calculated taking into account the effect of load (stress intensity factor) of the material for three basic modes of a cracked body: opening, sliding, and tearing. The nonhomogeneous conductivity tensors are reported for all modes. The analysis is based on general rotationally invariant nonlinear magnetoelastic equations (energy balance equation, Clausius–Duhem formula, and Gibbs function expansion). Bilinear constitutive relations for large quantities of the bias (loading) have been given and linear, but parametric constitutive formulas for small-field variables (superposed over the bias, for example current) have been derived in the reference or intermediate frame. The obtained effective material tensors of the biased magnetoelastic media, such as effective elastic, piezomagnetic, permeability tensors, as well as effective conductivity, allow to treat any other small-field phenomena in the intermediate frame in the same manner as the linear phenomena for the free (unbiased) media with the only change of material parameters.

1. Introduction

THERE ARE MANY METHODS of nondestructive testing (NDT) and evaluation (NDE) of materials. One of them is the ultrasonic technique, when acoustic waves are applied to the material under testing to get information about possible cracks or defects. The other can be magnetic technique, which is based on the distribution of magnetic field around the crack. Recently however, the potential drop technique has gained much interest of many researchers [15, 18, 27, 28]. Briefly, the technique involves application of a constant current to a cracked specimen or structure and the measurement of the potential drop across the crack. As the crack increases the effective electrical resistance of the material, the potential drop between two points on both sides of the crack rises. By monitoring the potential drop due to the crack and comparing it with a reference value of potential drop measured elsewhere far from the crack, the crack depth can be determined. The method can be used with either a direct current (d.c.) or an alternating current (a.c.). The older of these two techniques is the d.c. method, which is simple and relatively cheap. Some studies associated with development of the a.c. potential drop (ACPD) technique for detection and measurement of surface cracks in metals have been reported, for example, in [15, 16, 27]. A method for evaluation of the stress intensity factor for a 3-D surface crack by means of the a.c. potential drop technique was proposed in [25]. An advanced a.c. potential drop technique has been developed in [26] for nondestructive inspection of a crack.

All of these works are based on experimental techniques of evaluation of the crack size. The purpose of this paper is to give the theoretical derivation of the potential drop on the crack when the a.c. current is applied to the loaded, cracked magnetoelastic body in general. It is well known that the region very near to the crack or defect tip, regardless their origin, exhibits extremely large deformation and mechanical stress. Both fields are singular (tend to infinity, theoretically) at the end of the wedge, which can be the model of the defect. This explains why the linear magnetoelasticity for ferromagnetic conducting materials is not valid and can not be used near defects. Therefore, this work is based on nonlinear thermodynamics for magnetoelastic bodies [1 – 15] to derive constitutive relations for the material under large bias. This is the subject of the first few sections. Next, the proper relations derived are applied to the ACPD NDE problem for three basic modes of loading of the cracked body.

2. Basic definitions

What follows here, we start from general phenomenological, thermodynamic, nonlinear and rotationally invariant formulation. The theory is given for nonpolarizable, but conducting magnetoelastic solid, taking into account magnetostriction and piezomagnetic effects for wider use. The material is assumed to be anisotropic but homogeneous. Nonlinearity is reduced to quadratic terms only, as they play the most important role in nonlinear phenomena of coupling between predeformation and small-field vibration (bilinear model).

Two basic configurations are used. The reference frame is associated with material Cartesian coordinates, X_I, $I = 1,2,3$, which are denoted always by capital letters and indices as well as every quantity given in this frame. The second is the actual configuration associated with spatial Cartesian coordinates, x_i, $i = 1,2,3$, which are denoted by small letters and indices as well as every quantity given in this frame. Full advantage of well-known relations and tensor variables will be taken in the paper [2, 14, 20, 21, 22]:

- mapping of the material point

$$x_i = x_i(X_K, t); \tag{2.1}$$

- motion gradient

$$F_{iK} = x_{i,K} = \frac{\partial x_i}{\partial X_K}, \qquad J = \det F \neq 0; \tag{2.2}$$

- displacement gradient tensor

$$H = \nabla U = F - I, \tag{2.3}$$

where I is the identity tensor, and $U_K = \delta_{Ki} x_i - X_K$;

• Cauchy strain tensor

$$C = F^T F; \tag{2.4}$$

• Lagrange strain tensor

$$S = \frac{1}{2}(C - I) = \frac{1}{2}(H + H^T + H^T H); \tag{2.5}$$

• velocity gradient tensor

$$L = (\nabla V)^T; \tag{2.6}$$

• rate of strain tensor

$$D = \frac{1}{2}(L + L^T). \tag{2.7}$$

Also, the electric strength in the actual frame will be denoted by $e_i = -\phi_{,i}$ (ϕ is electric scalar potential), conduction current by j_i, magnetic induction vector by $b_i = \epsilon_{ijk} a_{k,j}$ (a_k is the magnetic vector potential), and magnetization vector by m_i. Then, the Maxwell stress tensor in the actual frame can be written as [10, 20, 21, 22, 31]

$$\begin{aligned} t_{ij}^M &= t_{ij}^{Me} + t_{ij}^{Mm}, \\ t_{ij}^{Me} &= \varepsilon_0 e_i e_j - 0.5\varepsilon_0 e_k e_k \delta_{ij}, \\ t_{ij}^{Mm} &= \frac{1}{\mu_0} b_i b_j - m_i b_j - \frac{1}{2}\left(\frac{1}{\mu_0} b_k b_k \delta_{ij} - 2 m_k b_k \delta_{ij}\right), \end{aligned} \tag{2.8}$$

which is split above into two parts. Electric permittivity is denoted by ε_0 and magnetic permeability is μ_0. The electric part t_{ij}^{Me} is symmetric, because of absence of polarization, but the magnetic part t_{ij}^{Mm} is not symmetric as well as the total Maxwell stress tensor t_{ij}^M. The term $-m_i b_j$, which causes the asymmetry, can be added to the mechanical Cauchy stress tensor, t_{ij}^m (nonsymmetric), and then they both will become symmetric as well as the total stress tensor

$$t_{ij} = t_{ij}^m + t_{ij}^M = t_{ij}^{ms} + t_{ij}^{Ms}, \tag{2.9}$$

where

$$\begin{aligned} t_{ij}^{Ms} &= \varepsilon_0 e_i e_j + \frac{1}{\mu_0} b_i b_j - \frac{1}{2}\left(\varepsilon_0 e_k e_k + \frac{1}{\mu_0} b_k b_k - 2 m_k b_k\right)\delta_{ij}, \\ t_{ij}^{ms} &= t_{ij}^m - m_i b_j. \end{aligned} \tag{2.10}$$

The constitutive equations will be derived not in the actual frame, but in the reference frame like in [1, 5, 6, 7, 11, 13, 20, 21, 24, 29]. Next, the straightforward substitution of the sum of the bias and small-field quantities will be done to get a small-field variables, parametric constitutive equations with coefficients depending on large quantities of the bias. At the end, we will get new effective material constants for magnetoelastic conducting body upon initial external stress or magnetic field. It can be said that these effective tensors describe the nonlinear coupling between the solid itself and predeformation or initial magnetic field.

3. Nonlinear thermodynamic formulation

We will start our considerations from the first and second law of thermodynamics, which are reproduced here for completeness. The global balance of energy states that the change in time of the kinetic energy $\mathcal{K}$ and the internal energy $\mathcal{E}$ must be equal to the work $\mathcal{W}$ done upon the body by external forces and heat $\mathcal{Q}$ delivered to the body [8, 9, 10, 14, 20, 21, 22]:

$$\dot{\mathcal{K}} + \dot{\mathcal{E}} = \mathcal{W} + \mathcal{Q}. \tag{3.1}$$

In the isothermal conditions, the local balance equation, equivalent to the above, for magnetoelastic conducting body can be written as [20, 21]

$$\varrho \dot{\epsilon} = \operatorname{tr}\left(t^m L^T\right) + j \cdot e - m \cdot \dot{b} \tag{3.2}$$

(notation remark: vector z corresponds to its Cartesian coordinates z_i, where $z = j, e, m, b...$).

For purposes of this paper it is more convenient to use instead of the internal energy density ϵ, the scalar state function known as a Gibbs function, which can be obtained from ϵ by the Legendre transformation

$$\psi = \epsilon + \frac{1}{\varrho} m \cdot b, \tag{3.3}$$

so we can use local energy balance in the form

$$\varrho \dot{\psi} = \operatorname{tr}\left(t^m L^T\right) + j \cdot e + b \cdot \dot{m}. \tag{3.4}$$

The axiom of entropy (second law of thermodynamics) states that the time rate of the total entropy of the density η is never less than the entropy supply in the body (for the case of no influx through the boundary). In the isothermal conditions (or neglecting the change of temperature) for the simple and admissible process of magnetoelastic conducting body, the local law can be expressed in the form of Clausius - Duhem inequality written below for the Helmholtz free energy function $\Psi = \psi - \eta\theta$, where θ is the temperature [8, 12, 20]

$$-\varrho \dot{\Psi} + \operatorname{tr}\left(t^m L^T\right) + j \cdot e + b \cdot \dot{m} \geq 0. \tag{3.5}$$

The internal energy ψ will be rotationally invariant if it is expressed in terms of material measures of strain, magnetization, and electric field. We choose Lagrange strain tensor S, Eq. (2.5), and axial magnetization vector in the reference configuration [20]

$$M_I = J F^{-1}_{jI} m_j, \tag{3.6}$$

as well as electric field in the reference configuration [20]

$$E_I = e_j F_{jI} \tag{3.7}$$

as independent variables. Then, the magnetic induction and conduction current in the reference frame can be expressed for completeness as follows

$$B_I = b_j F_{jI}\,, \qquad \mathcal{J}_I = J F^{-1}_{jI} j_j\,. \tag{3.8}$$

To get the Gibbs function in a form $\psi = \psi(S, M, E)$, Eq. (3.4) must be rearranged to involve new variables. Differentiation of S and M in time gives

$$\dot{S} = F^T D F, \qquad \dot{m} = \dot{M} F^{-1} - mL + m\nabla \cdot L. \tag{3.9}$$

Then

$$\begin{aligned} \varrho \dot{\psi} &= \operatorname{tr}\left(t^m L^T\right) + J\mathcal{J} \cdot E + b \cdot \dot{M} F^{-1} - b \cdot mL + b \cdot m\nabla \cdot L \\ &= \operatorname{tr}\left((t^m - bm + b \cdot mI)L^T\right) + J\mathcal{J} \cdot E + \operatorname{tr}\left(b\dot{M}^T F^{-1^T}\right) \\ &\operatorname{tr}\left((t^{ms} + b \cdot mI)L^T\right) + J\mathcal{J} \cdot E + \operatorname{tr}\left(b\dot{M}^T F^{-1^T}\right). \end{aligned} \tag{3.10}$$

L^T can be now decomposed into $L^T = D - \Omega$, where $\Omega = 0.5(L - L^T)$ is the rate of rotation tensor always skewsymmetric, so the product of $(t^{ms} + b \cdot mI)$ and Ω is always zero. We get then

$$\varrho \dot{\psi} = \operatorname{tr}\left((t^{ms} + b \cdot mI)D\right) + J\mathcal{J} \cdot E + \operatorname{tr}\left(b\dot{M}^T F^{-1^T}\right), \tag{3.11}$$

and with help of the inverse of the first of Eqs. (3.9)

$$\varrho \dot{\psi} = \operatorname{tr}\left((t^m - mb + b \cdot mI)F^{-1}F^{-1^T}\dot{S}\right) + J\mathcal{J} \cdot E + \operatorname{tr}\left(b\dot{M}^T F^{-1^T}\right). \tag{3.12}$$

The last equation rewritten in components is

$$\begin{aligned} \varrho \dot{\psi} &= t^m_{ij} F^{-1}_{iK} F^{-1}_{jL} \dot{S}_{KL} - m_i b_j F^{-1}_{iK} F^{-1}_{jL} \dot{S}_{KL} + m_l b_l \delta_{ij} F^{-1}_{iK} F^{-1}_{jL} \dot{S}_{KL} \\ &\quad + J\mathcal{J}_I E_I + b_i \dot{M}_K F^{-1}_{iK}\,. \end{aligned} \tag{3.13}$$

4. Reference frame constitutive equations

To find constitutive relations we decompose the total derivative $\dot{\psi}$ into terms associated with independent variables:

$$\dot{\psi} = \frac{\partial \psi}{\partial S_{KL}} \frac{dS_{KL}}{dt} + \frac{\partial \psi}{\partial M_K} \frac{dM_K}{dt} + \frac{\partial \psi}{\partial E_K} \frac{dE_K}{dt}\,. \tag{4.1}$$

Multiplying Eq. (4.1) by $-\varrho$ and adding to Eq. (3.13) side by side one can obtain

$$\left(t^m_{ij}F^{-1}_{iK}F^{-1}_{jL} - m_ib_jF^{-1}_{iK}F^{-1}_{jL} + m_lb_l\delta_{ij}F^{-1}_{iK}F^{-1}_{jL} - \varrho\frac{\partial\psi}{\partial S_{KL}}\right)\dot{S}_{KL} + \left(b_iF^{-1}_{iK} - \varrho\frac{\partial\psi}{\partial M_K}\right)\dot{M}_K - \varrho\frac{\partial\psi}{\partial E_K}\dot{E}_K + J\mathcal{J}_IE_I = 0. \tag{4.2}$$

Additionally, the Clausius-Duhem inequality (3.5) gives

$$\left(t^m_{ij}F^{-1}_{iK}F^{-1}_{jL} - m_ib_jF^{-1}_{iK}F^{-1}_{jL} + m_lb_l\delta_{ij}F^{-1}_{iK}F^{-1}_{jL} - \varrho\frac{\partial\psi}{\partial S_{KL}}\right)\dot{S}_{KL} + \left(b_iF^{-1}_{iK} - \varrho\frac{\partial\psi}{\partial M_K}\right)\dot{M}_K - \varrho\frac{\partial\psi}{\partial E_K}\dot{E}_K + J\mathcal{J}_IE_I \geq 0. \tag{4.3}$$

Formulas (4.2) and (4.3) must hold for arbitrary nonzero independent time variations $\dot{S}_{KL}$, $\dot{M}_K$, and $\dot{E}_K$, so both can be written separately as

$$\frac{\partial\psi}{\partial E_K} = 0, \qquad \mathcal{J}_IE_I \geq 0, \tag{4.4}$$

and

$$\begin{aligned} t^m_{ij} &= \varrho F_{iK}F_{jL}\frac{\partial\psi}{\partial S_{KL}} + m_ib_j - m_lb_l\delta_{ij}, \\ b_i &= \varrho F_{iK}\frac{\partial\psi}{\partial M_K}, \end{aligned} \tag{4.5}$$

or

$$\begin{aligned} t^m_{ij} &= \varrho F_{iK}F_{jL}\frac{\partial\psi}{\partial S_{KL}} + \varrho m_iF_{jK}\frac{\partial\psi}{\partial M_K} - \varrho\delta_{ij}m_lF_{lK}\frac{\partial\psi}{\partial M_K}, \\ b_i &= \varrho F_{iK}\frac{\partial\psi}{\partial M_K}. \end{aligned} \tag{4.6}$$

The remaining constitutive relation, on the basis of (4.4), reads

$$\mathcal{J} = \mathcal{J}(S, M, E), \tag{4.7}$$

as the Thomson effect was neglected (homogeneous temperature distribution was assumed). Because the form $\mathcal{J}E$ must be positive definite (see inequality (4.4)), the continuity condition at the thermodynamic equilibrium must be fulfilled

$$\mathcal{J}(S, M, E) = 0 \quad \text{for} \quad E_I = 0, \tag{4.8}$$

what gives the linear representation in terms of E_I

$$\mathcal{J}_I = \Sigma_{IJ}(S, M)E_J, \tag{4.9}$$

where Σ must be the second order symmetric material tensor. Now, neglecting the Hall effect in the above relation and involving the isotropic representation of Σ due to the only independent variable S, one can find

$$\Sigma = \sigma I + \sigma_1 S + \sigma_2 S^2, \tag{4.10}$$

where three fundamental invariants of tensor S have been used. The first term is a natural (unbiased) media conduction term, whereas two next terms are due to the deformation of the solid. Coefficients σ, σ_1, σ_2 have to be found experimentally.

There are still three quantities t_{ij}^m, b_i, m_i in constitutive equations (4.5), (4.6) given in the actual frame (spatial coordinates). It is obvious that if we want to apply constitutive relations to practical problems, where we study the dynamical behaviour of a nonlinear elastic body of finite extent, it is much simpler to have these quantities convected to the reference frame (material coordinates). Then it will be easier to write the boundary conditions concerning either the mechanical displacement or the stress, and the magnetic induction on the fixed surface in the reference frame instead of the deforming surface in the actual configuration.

So, it is worth to introduce briefly the reference frame quantities [20, 21]:

- mechanical Piola - Kirchhoff stress tensor

$$T_{Ij}^m = J F_{kI}^{-1} t_{kj}^m; \tag{4.11}$$

- Maxwell Piola - Kirchhoff stress tensor

$$T_{Ij}^M = T_{Ij}^{Mm} + T_{Ij}^{Me} = J F_{kI}^{-1} t_{kj}^M; \tag{4.12}$$

- total Piola - Kirchhoff stress tensor

$$T_{Ij} = T_{Ij}^m + T_{Ij}^M = J F_{kI}^{-1} t_{kj}. \tag{4.13}$$

After substituting the second equation of (4.5) into the first of Eqs. (3.8) and using the conservation of mass equation in the form

$$\varrho^F = J\varrho \tag{4.14}$$

(where ϱ^F is the mass density in a natural state, when the body is acted upon neither by force nor by magnetic field), one can obtain

$$B_I = \varrho^F \frac{\partial \psi}{\partial M_I}. \tag{4.15}$$

With the help of the first equation of (4.5) and Eq. (4.11), one can express the mechanical Piola-Kirchhoff stress tensor as

$$T_{Ij}^m = \delta_{jJ} T_{IJ}^m = \varrho^F F_{jL} \frac{\partial \psi}{\partial S_{IL}} + \varrho^F F_{jK} M_I \frac{\partial \psi}{\partial M_K} - \varrho^F F_{jI} M_K \frac{\partial \psi}{\partial M_K}. \tag{4.16}$$

Equations (4.9) with (4.10), and (4.15) and (4.16) stand for the constitutive relations in which every quantity is related to the reference frame.

5. Bilinear expansion of Gibbs function

To obtain explicit forms (not with partial differentiation) of constitutive relations, we should expand the Gibbs thermodynamic function in terms of its independent variables in the reference frame. The expansion is cut after the third order terms to get constitutive equations in a bilinear form. Linear and quadratic terms of constitutive relations play the most important role in describing nonlinear phenomena in magnetoelastic solids [20, 21, 23]. The other, higher order terms can be neglected.

Let us introduce the commonly accepted expansion [20]

$$
\begin{aligned}
(5.1)\qquad \varrho^F\psi = {} & \frac{1}{2}c_{IJKL}S_{IJ}S_{KL} + \frac{1}{6}c_{IJKLMN}S_{IJ}S_{KL}S_{MN} \\
& + \frac{1}{2}f_{MIJKL}M_M S_{IJ}S_{KL} + f_{MIJ}M_M S_{IJ} + B_{IJKL}S_{IJ}M_K M_L \\
& + \frac{1}{2}\mu_{MN}M_M M_N + \frac{1}{6}\mu_{MNK}M_M M_N M_K ,
\end{aligned}
$$

where c_{IJKL} is the elastic tensor of the second order, c_{IJKLMN} is the elastic tensor of the third order, f_{MIJ} is the piezomagnetic tensor of the second order, f_{MIJKL} is the magnetoelastic tensor of the third order, μ_{MN} is the permeability tensor of the second order, μ_{MNP} is the magnetic anisotropy tensor of the third order, and B_{MNIJ} is the magnetostriction tensor of the third order (the contribution of exchange energy is neglected as it is an effect by one order higher than that taken into account).

Of course, in general the following symmetry conditions must be fulfilled [23]

$$
\begin{aligned}
& c_{IJKL} = c_{JIKL} = c_{IJLK} = c_{KLIJ}, \\
& c_{IJKLMN} = c_{JIKLMN} = c_{IJLKMN} = c_{IJKLNM} = c_{KLIJMN} = c_{MNKLIJ}, \\
& f_{MIJ} = f_{MJI}, \\
& f_{MIJKL} = f_{MJIKL} = f_{MIJLK} = f_{MKLIJ}, \\
& \mu_{MN} = \mu_{NM}, \\
& \mu_{MNP} = \mu_{NMP} = \mu_{PNM}, \\
& B_{MNIJ} = B_{NMIJ} = B_{MNJI},
\end{aligned}
\qquad (5.2)
$$

so we have only 21 independent coefficients of c_{IJKL}, 56 of $c_{IJKJLMN}$, 18 of f_{MIJ}, 63 of f_{MIJKL}, 6 of μ_{MN}, 10 of μ_{MNP}, and 36 of B_{MNIJ}, what together gives 210 independent material constants of magnetoelastic media in the bilinear theory (45 in a linear one). But if the solid has some symmetry points or axis or planes, what always happens, the number of material constants to be given is reduced further and usually is much less than 210.

Moreover, we want to get constitutive expressions in the form like the form below:

$$
T = T(H, M, E), \qquad B = B(H, M), \qquad (5.3)
$$

where M, H, E are material quantities in the reference frame, so the following is now substituted to (4.15) and (4.16)

$$S = \frac{1}{2}\left(H + H^T + H^T H\right), \qquad F = H + I, \tag{5.4}$$

and [20, 21]

$$\begin{aligned} F^{-1} &= I - H + H^T H + \mathcal{O}(H^3), \\ J &= 1 + \operatorname{tr} H + \frac{1}{2}(\operatorname{tr} H)^2 - \frac{1}{2}\operatorname{tr}(H^T H) + \mathcal{O}(H^3), \end{aligned} \tag{5.5}$$

where $\mathcal{O}(H^n)$ denotes the remaining terms of order higher or equal than n, which are to be neglected next. After substitution of Eqs. (5.4) and (5.1) into Eq. (4.15) one can obtain

$$\begin{aligned} B_I = {} & f_{IKL}S_{KL} + \frac{1}{2}f_{IKLMN}S_{KL}S_{MN} + \mu_{IJ}M_J + \frac{1}{2}\mu_{IJK}M_J M_K \\ & + 2B_{KLIJ}M_J S_{KL} = f_{IKL}H_{KL} + \frac{1}{2}f_{IKL}H_{ML}H_{MK} + \frac{1}{2}f_{IKLMN}H_{KL}H_{MN} \\ & + \mu_{IJ}M_J + \frac{1}{2}\mu_{IJK}M_J M_K + 2B_{KLIJ}M_J H_{KL} + \mathcal{O}(H^3) + \mathcal{O}(MH^2). \end{aligned} \tag{5.6}$$

After neglecting higher order terms and changing indices we have:

$$\begin{aligned} B_I = {} & f_{IKL}H_{KL} + \mu_{IJ}M_J + \frac{1}{2}\left(f_{IKLMN} + f_{IKN}\delta_{ML}\right)H_{KL}H_{MN} \\ & + \frac{1}{2}\mu_{IJK}M_J M_K + 2B_{KLIJ}M_J H_{KL}. \end{aligned} \tag{5.7}$$

After substitution of Eqs. (5.4) and (5.1) into Eq. (4.16) one can obtain

$$\begin{aligned} T^m_{IJ} = {} & (\delta_{JL} + H_{JL})(c_{ILMN}S_{MN} + \frac{1}{2}c_{ILMNPR}S_{MN}S_{PR} \\ & + f_{MIL}M_M + f_{MILPR}M_M S_{PR} + B_{ILMN}M_M M_N) \\ & + [(\delta_{JK} + H_{JK})M_I - (\delta_{IJ} + H_{IJ})M_K]\left(2B_{KLMN}S_{MN}M_L + f_{KMN}S_{MN}\right. \\ & \left. + \frac{1}{2}f_{KMNPR}S_{MN}S_{PR} + \mu_{KL}M_L + \frac{1}{2}\mu_{KLM}M_L M_M\right). \end{aligned} \tag{5.8}$$

Using the first of equations (5.4), the above formula can be simplified as follows

$$\begin{aligned} T^m_{IJ} = {} & c_{IJMN}H_{MN} + \frac{1}{2}c_{IJMN}H_{PM}H_{PN} + \frac{1}{2}c_{IJMNPR}H_{MN}H_{PR} \\ & + B_{IJMN}M_M M_N + f_{MIJ}M_M + f_{MIJPR}M_M H_{PR} + c_{ILMN}H_{MN}H_{JL} \\ & + f_{MIL}M_M H_{JL} + \mathcal{O}(H^3) + \mathcal{O}(HM^2) + f_{JMN}M_I H_{MN} + \mu_{JL}M_L M_I \\ & - \delta_{IJ}(f_{KMN}M_K H_{MN} - \mu_{KL}M_L M_K) + \mathcal{O}(MH^2) + \mathcal{O}(HM^2). \end{aligned} \tag{5.9}$$

Next, on the basis of Eqs. (2.8), (4.15), (5.4) and (5.6), the magnetic part of Maxwell Piola – Kirchhoff stress tensor can be found as:

$$
\begin{aligned}
(5.10)\qquad T_{IJ}^{Mm} &= -[(\delta_{JK}+H_{JK})M_I-(\delta_{IJ}+H_{IJ})M_K]\Big(2B_{KLMN}S_{MN}M_L \\
&\quad +f_{KMN}S_{MN}+\frac{1}{2}f_{KMNPR}S_{MN}S_{PR}+\mu_{KL}M_L+\frac{1}{2}\mu_{KLM}M_LM_M\Big) \\
&= -f_{JMN}M_IH_{MN}-\mu_{JL}M_LM_I+\delta_{IJ}(f_{KMN}M_KH_{MN}+\mu_{KL}M_LM_K) \\
&\quad +\mathcal{O}(MH^2)+\mathcal{O}(HM^2),
\end{aligned}
$$

and on the basis of Eqs. (2.8), (3.7), (4.12), and (5.6), the electric part of Maxwell Piola – Kirchhoff stress tensor can be written as:

$$
\begin{aligned}
(5.11)\qquad T_{IJ}^{Me} &= \delta_{jJ}T_{Ij}^{Me}=J\delta_{jJ}F_{kI}^{-1}\Big(\varepsilon_0F_{kJ}^{-1}E_JF_{jK}^{-1}E_K-\frac{1}{2}\varepsilon_0F_{iL}^{-1}E_LF_{iL}^{-1}E_L\delta_{kj}\Big) \\
&= J\delta_{jJ}\varepsilon_0E_JE_K\Big(F_{kI}^{-1}F_{kJ}^{-1}F_{jK}^{-1}-\frac{1}{2}F_{jI}^{-1}F_{kJ}^{-1}F_{kK}^{-1}\Big) \\
&= \delta_{jJ}\varepsilon_0E_LE_K\Big(\delta_{kI}\delta_{kL}\delta_{jK}-\frac{1}{2}\delta_{jI}\delta_{kL}\delta_{kK}\Big)+\mathcal{O}(H^2E) \\
&= \varepsilon_0E_LE_K\Big(\delta_{IL}\delta_{JK}-\frac{1}{2}\delta_{LI}\delta_{JK}\Big)=\frac{1}{2}\epsilon_0E_LE_K(\delta_{IL}\delta_{JK}+\delta_{IK}\delta_{JL}-\delta_{IJ}\delta_{KL}).
\end{aligned}
$$

So, the total Piola – Kirchhoff stress tensor $T_{IJ}=T_{IJ}^M+T_{IJ}^m$ after disregarding higher order terms and changing indices, is now

$$
\begin{aligned}
(5.12)\qquad T_{IJ} &= c_{IJKL}H_{KL}+\Big(c_{ILMN}\delta_{JK}+\frac{1}{2}c_{IJNL}\delta_{MK}+c_{IJKLMN}\Big)H_{KL}H_{MN} \\
&\quad +(f_{MIL}\delta_{KJ}+f_{MIJKL})M_MH_{KL}+f_{MIJ}M_M+B_{IJMN}M_MM_N \\
&\quad +\frac{1}{2}\epsilon_0(\delta_{IL}\delta_{JK}+\delta_{IK}\delta_{JL}-\delta_{IJ}\delta_{KL})E_LE_K.
\end{aligned}
$$

Additionally, substitution of Eq. (5.4) into Eqs. (4.9) with (4.10) gives the coincident formula for the conduction current,

$$
(5.13)\qquad \mathcal{J}_I=(\sigma\delta_{IJ}+\sigma^hH_{IJ})E_J,\qquad \sigma^h=\sigma_1.
$$

Two constitutive relations (5.7) and (5.12) can be simplified more by the introduction of new tensors

$$
\begin{aligned}
\tilde{c}_{IJKLMN} &= c_{ILMN}\delta_{JK}+\frac{1}{2}c_{IJNL}\delta_{MK}+\frac{1}{2}c_{IJKLMN}, \\
(5.14)\qquad \tilde{f}_{MIJKL} &= f_{MIL}\delta_{JK}+f_{MIJKL}, \\
\tilde{l}_{IJKL} &= 0.5\varepsilon_0(\delta_{IL}\delta_{JK}+\delta_{IK}\delta_{JL}-\delta_{KL}\delta_{IJ}).
\end{aligned}
$$

Then

$$
\begin{aligned}
T_{IJ} &= c_{IJKL}H_{KL} + \tilde{c}_{IJKLMN}H_{KL}H_{MN} + \tilde{f}_{MIJKL}M_M H_{KL} \\
&\quad + f_{MIJ}M_M + B_{IJMN}M_M M_N + \tilde{l}_{IJKL}E_K E_L, \\
B_I &= \mu_{IJ}M_J + \frac{1}{2}\mu_{IJK}M_J M_K + 2B_{KLIJ}M_J H_{KL} \\
&\quad + \frac{1}{2}\tilde{f}_{IKLMN}H_{KL}H_{MN} + f_{IKL}H_{KL}.
\end{aligned} \tag{5.15}
$$

There are less symmetries for the new third order tensors, because they contain the geometrical contribution to elasticity, piezomagnetism and electrostriction, respectively ($\tilde{l}$ is only the geometrical electrostriction, since the material one was neglected for magnetoelastic solids in (5.1)). It is as follows:

$$
\begin{aligned}
\tilde{c}_{IJKLMN} &= \tilde{c}_{JIKLMN} = \tilde{c}_{IJMNKL}, \\
\tilde{f}_{MIJKL} &= \tilde{f}_{MJIKL} = \tilde{f}_{MKLIJ}, \\
\tilde{l}_{IJKL} &= \tilde{l}_{JIKL} = \tilde{l}_{IJLK}.
\end{aligned} \tag{5.16}
$$

The relations (5.13) and (5.15) stand for bilinear constitutive equations for magnetoelastic solids even if the deformation or magnetization in the solid is large.

6. Small-field constitutive equations

Constitutive formulas obtained in the previous section are written in the reference frame. Let us now suppose that the initial nonzero strain field, as a result of external force or magnetic field, exists in the body at first. A small field is considered to be superposed on this state next. It means that we have now the third, intermediate configuration of the body, in addition to the undisturbed, reference configuration and the actual one. The material point identified by the material coordinate X_I first moves to intermediate coordinate $X_k^i(X_I)$ by a large initial deformation, and next to $x_k(X_I)$ by a small alternating vibration, for example. So, every field quantity in the reference frame can be decomposed into two parts. The first associated with the bias and the second, small one – with the small deformation in the body. Let the sign tilde denote the quantities associated with the initial state of large deformation. We have then, in the reference frame,

$$
Z^t = \tilde{Z} + Z, \qquad \text{where} \qquad Z = T, H, M, B. \tag{6.1}
$$

Additionally, small-field electric strength and conduction current is supposed to exist in the biased body, so

$$
E^t = E, \qquad \mathcal{J}^t = \mathcal{J}. \tag{6.2}
$$

These total quantities must fulfill the constitutive equations (5.13) and (5.15), so

$$
\begin{aligned}
(6.3)\qquad \tilde{T}_{IJ} + T_{IJ} = {} & \Big\{ c_{IJKL}\tilde{H}_{KL} + \tilde{c}_{IJKLMN}\tilde{H}_{KL}\tilde{H}_{MN} + \tilde{f}_{MIJKL}\widetilde{M}_M\tilde{H}_{KL} \\
& + f_{MIJ}\widetilde{M}_M + B_{-IJMN}\widetilde{M}_M\widetilde{M}_N \Big\} + c_{IJKL}H_{KL} + \tilde{c}_{IJKLMN}H_{KL}\tilde{H}_{MN} \\
& + \tilde{c}_{IJMNKL}H_{KL}\tilde{H}_{MN} + f_{MIJ}M_M + \tilde{f}_{MIJKL}M_M\tilde{H}_{KL} + \tilde{f}_{MIJKL}H_{KL}\widetilde{M}_M \\
& + B_{IJMN}M_M\widetilde{M}_N + B_{IJMN}M_N\widetilde{M}_M + [\tilde{c}_{IJKLMN}H_{KL}H_{MN} \\
& + \tilde{f}_{MIJKL}M_MH_{KL} + B_{IJMN}M_MM_N + \tilde{l}_{IJKL}E_KE_L] ,
\end{aligned}
$$

and

$$
\begin{aligned}
(6.4)\qquad \tilde{B}_I + B_I = {} & \Big\{ \mu_{IJ}\widetilde{M}_J + \frac{1}{2}\mu_{IJK}\widetilde{M}_J\widetilde{M}_K + 2B_{KLIJ}\widetilde{M}_J\tilde{H}_{KL} \\
& + \frac{1}{2}\tilde{f}_{IKLMN}\tilde{H}_{KL}\tilde{H}_{MN} + f_{IKL}\tilde{H}_{KL} \Big\} + \mu_{IJ}M_J + \frac{1}{2}\mu_{IJK}M_J\widetilde{M}_K \\
& + \frac{1}{2}\mu_{IJK}M_J\widetilde{M}_K + 2B_{KLIJ}H_{KL}\widetilde{M}_J + 2B_{KLIJ}M_J\tilde{H}_{KL} \\
& + \frac{1}{2}\tilde{f}_{IKLMN}H_{KL}\tilde{H}_{MN} + \frac{1}{2}\tilde{f}_{IMNKL}H_{KL}\tilde{H}_{MN} + f_{IKL}H_{KL} \\
& + \left[\frac{1}{2}\mu_{IJK}M_JM_K + 2B_{KLIJ}M_JH_{KL} + \frac{1}{2}\tilde{f}_{IKLMN}H_{KL}H_{MN}\right] ,
\end{aligned}
$$

and

$$
(6.5)\qquad \mathcal{J}_I = \sigma\delta_{IJ}E_J + \sigma^h\tilde{H}_{IJ}E_J + \left[\sigma^hH_{IJ}E_J\right] .
$$

The terms in the last three formulas have been regrouped so it can be said that expressions in the first brackets are equal to the biasing quantities $\tilde{T}$ and $\tilde{B}$, respectively, while terms in the square brackets can be neglected because they are second order terms of small-field quantities. After separation one can obtain nonlinear constitutive equations for large initial state in the form

$$
\begin{aligned}
\tilde{T}_{IJ} = {} & c_{IJKL}\tilde{H}_{KL} + \tilde{c}_{IJKLMN}\tilde{H}_{KL}\tilde{H}_{MN} + \tilde{f}_{MIJKL}\widetilde{M}_M\tilde{H}_{KL} \\
& + f_{MIJ}\widetilde{M}_M + B_{IJMN}\widetilde{M}_M\widetilde{M}_N ,
\end{aligned}
$$

$$
(6.6)\qquad
\begin{aligned}
\tilde{B}_I = {} & \mu_{IJ}\widetilde{M}_J + \frac{1}{2}\mu_{IJK}\widetilde{M}_J\widetilde{M}_K + 2B_{KLIJ}\widetilde{M}_J\tilde{H}_{KL} \\
& + \frac{1}{2}\tilde{f}_{IKLMN}\tilde{H}_{KL}\tilde{H}_{MN} + f_{IKL}\tilde{H}_{KL} ,
\end{aligned}
$$

and the linear ones for a small-field superposed on this biasing state

$$
(6.7)\qquad
\begin{aligned}
T_{IJ} &= c^{\text{eff}}_{IJKL}H_{KL} + f^{\text{eff}}_{MIJ}M_M , \\
B_I &= \mu^{\text{eff}}_{IJ}M_J + f^{\text{eff}}_{IKL}H_{KL} , \\
\mathcal{J}_I &= \sigma^{\text{eff}}_{IJ}E_J ,
\end{aligned}
$$

where effective tensors are of course dependent on the biasing magnetization and displacement gradient (parameters)

$$
\begin{aligned}
c^{\mathrm{eff}}_{IJKL} &= c_{IJKL} + 2\tilde{c}_{IJKLMN}\tilde{H}_{MN} + \tilde{f}_{MIJKL}\widetilde{M}_{M}\,,\\
f^{\mathrm{eff}}_{IKL} &= f_{IKL} + \tilde{f}_{IKLMN}\tilde{H}_{MN} + 2B_{KLIJ}\widetilde{M}_{J}\,,\\
\mu^{\mathrm{eff}}_{IJ} &= \mu_{IJ} + \mu_{IJK}\widetilde{M}_{K} + 2B_{KLIJ}\tilde{H}_{KL}\,,\\
\sigma^{\mathrm{eff}}_{IJ} &= \sigma\delta_{IJ} + \sigma^{h}\tilde{H}_{IJ}\,.
\end{aligned}
\tag{6.8}
$$

7. Loading of the cracked specimen

It is well known, that aside from ideally brittle materials, any loading of cracked body is accompanied by inelastic deformation in the neighbourhood of the crack tip due to stress concentration there. Usually three basic modes of loading are distinguished [19]: Mode I, called opening mode, Mode II, named sliding mode, and Mode III, described as tearing mode. All of them with the Cartesian coordinate system are depicted in Fig. 1. The stress concentrations around the crack

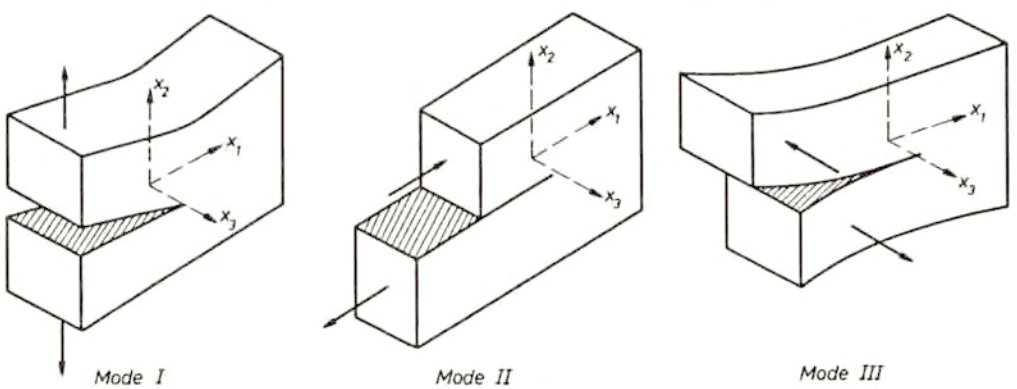

FIG. 1. Basic loading modes for a cracked body.

for different modes are given in [19, pages 138–147] in the formulas, following below, written in a polar coordinate system with the origin at the crack tip ($X_1 = r\cos(\theta)$, $X_2 = r\sin(\theta)$). For Mode I the nonzero stress components are

$$
\begin{Bmatrix} \tilde{T}_{11} \\ \tilde{T}_{12} \\ \tilde{T}_{22} \end{Bmatrix} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}\cos(\theta/2)\begin{Bmatrix} 1-\sin(\theta/2)\sin(3\theta/2) \\ \sin(\theta/2)\cos(3\theta/2) \\ 1+\sin(\theta/2)\sin(3\theta/2) \end{Bmatrix}.
\tag{7.1}
$$

For Mode II and Mode III they are, respectively

$$\begin{Bmatrix} \tilde{T}_{11} \\ \tilde{T}_{12} \\ \tilde{T}_{22} \end{Bmatrix} = \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}} \begin{Bmatrix} -\sin(\theta/2)[2+\cos(\theta/2)\cos(2\theta/2)] \\ \cos(\theta/2)[1-\sin(\theta/2)\sin(3\theta/2)] \\ \sin(\theta/2)\cos(\theta/2)\cos(3\theta/2) \end{Bmatrix}, \tag{7.2}$$

$$\begin{Bmatrix} \tilde{T}_{31} \\ \tilde{T}_{32} \end{Bmatrix} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \begin{Bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{Bmatrix}. \tag{7.3}$$

It can be seen that for $r = 0$ the stress in all cases is singular, and approaches infinity, but the stress intensity factors K_{I}, K_{II}, K_{III} are defined in such a way that they are finite

$$K_{\mathrm{I}} = \lim_{r\to 0} \left\{ \sqrt{2\pi r}\tilde{T}_{22}\Big|_{\theta=0} \right\}, \tag{7.4}$$

$$K_{\mathrm{II}} = \lim_{r\to 0} \left\{ \sqrt{2\pi r}\tilde{T}_{12}\Big|_{\theta=0} \right\}, \tag{7.5}$$

$$K_{\mathrm{III}} = \lim_{r\to 0} \left\{ \sqrt{2\pi r}\tilde{T}_{32}\Big|_{\theta=0} \right\}. \tag{7.6}$$

For the simple isotropic, nonmagnetostrictive material ($B_{KLIJ} = 0$, $f_{IKL} = 0$) possessing the elastic tensor in the form [14]

$$c_{IJKL} = 2\mu\delta_{KI}\delta_{LJ} + \lambda\delta_{KL}\delta_{IJ}, \tag{7.7}$$

where μ and λ are Lamé constants, the corresponding nonzero displacements near the crack tip can be written for Modes I, II, and III as [19]

$$\begin{Bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{Bmatrix} = \frac{K_{\mathrm{I}}}{2\mu}\sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos(\theta/2)[\kappa - 1 + 2\sin^2(\theta/2)] \\ \sin(\theta/2)[\kappa + 1 - 2\cos^2(\theta/2)] \end{Bmatrix}, \tag{7.8}$$

$$\begin{Bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{Bmatrix} = \frac{K_{\mathrm{II}}}{2\mu}\sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \sin(\theta/2)[\kappa + 1 + 2\cos^2(\theta/2)] \\ -\cos(\theta/2)[\kappa - 1 - 2\sin^2(\theta/2)] \end{Bmatrix}, \tag{7.9}$$

$$\{\tilde{U}_3\} = \frac{2K_{\mathrm{III}}}{\mu}\sqrt{\frac{r}{2\pi}}\{\sin(\theta/2)\}, \tag{7.10}$$

where for Mode I and II either $\kappa = 3 - 4\nu$ for plane strain or $\kappa = (3-\nu)/(1+\nu)$ for plane stress case, and ν is Poisson's constant. Taking into account the gradient description in polar coordinates (two-dimensional) of the form

$$\frac{\partial}{\partial X_1} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \qquad \frac{\partial}{\partial X_2} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}, \qquad \frac{\partial}{\partial X_3} = 0, \tag{7.11}$$

the displacement gradients can be expressed for Mode I, II, III, respectively, as:

$$\tilde{H} = \frac{K_{\mathrm{I}}}{4\mu}\frac{1}{\sqrt{2\pi r}}\begin{bmatrix} \cos(\theta/2)(\kappa - \cos\theta - 2\sin^2\theta) & & \\ -\sin(\theta/2)(\kappa - \cos\theta + 2\sin^2\theta) & & \\ 0 & & \\ & \sin(\theta/2)(\kappa + \cos\theta + 2\cos^2\theta) & 0 \\ & \cos(\theta/2)(\kappa + \cos\theta - 2\cos^2\theta) & 0 \\ & 0 & 0 \end{bmatrix}, \tag{7.12}$$

$$\tilde{H} = \frac{K_{\mathrm{II}}}{4\mu}\frac{1}{\sqrt{2\pi r}}\begin{bmatrix} -\sin(\theta/2)(\kappa + 2 + \cos\theta - 2\sin^2\theta) & & \\ \cos(\theta/2)(\kappa - 2 + \cos\theta - 2\sin^2\theta) & & \\ 0 & & \\ & \cos(\theta/2)(\kappa + 2 - \cos\theta + 2\cos^2\theta) & 0 \\ & -\sin(\theta/2)(\kappa - 2 - \cos\theta - 2\cos^2\theta) & 0 \\ & 0 & 0 \end{bmatrix}, \tag{7.13}$$

$$\tilde{H} = \frac{K_{\mathrm{III}}}{\mu}\frac{1}{\sqrt{2\pi r}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\sin(\theta/2) & \cos(\theta/2) & 0 \end{bmatrix}. \tag{7.14}$$

8. Current flow around the crack

Nondestructive testing (NDT) of materials is a group of methods used to distinguish if the material given has some defects (cracks, dislocations) or not. Nondestructive evaluation (NDE) is more advanced since it should give more information about defects (size, type, etc.). There are many experimental techniques which can be used to get proper information: ultrasonic or magnetic methods, and the potential drop technique (d.c. or a.c.), which is now gaining much interests of researchers [15, 16, 25, 26, 28]. In these kinds of methods the specimen tested is subjected to the current flow, which is assumed to form around the crack. It is especially the property of the a.c. current which occurs mainly very near to the surface. For the values of frequency f and conductivity large enough, the penetration depth is small,

$$\delta = \frac{1}{\sqrt{\pi\mu\sigma f}}. \tag{8.1}$$

The path of the a.c. current is shown in Fig. 2.

If the specimen with crack is loaded, the large gradient of displacement (Sec. 7, Eqs. (7.12), (7.13), (7.14)) which exists near the crack can be regarded as the

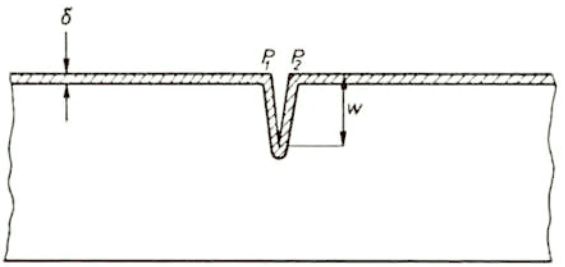

FIG. 2. A.c. current flow path around the crack.

mechanical bias for the current flow around the crack. Therefore, the conductivity near the crack is different from the natural (for unbiased material) one, and should be calculated with respect to predeformation as it is given in Sec. 6, Eq. (6.8). Substituting the gradient of displacement for different modes to the definition of effective conductivity, it is easy to obtain σ^{eff} for Mode I, II, III, respectively

$$(8.2)\qquad \sigma_{IJ}^{\text{eff}} = \begin{bmatrix} \sigma + \gamma_{\text{I}}\cos(\theta/2)(\kappa - \cos\theta - 2\sin^2\theta) & \gamma_{\text{I}}\sin(\theta/2)(\kappa + \cos\theta + 2\cos^2\theta) & 0 \\ -\gamma_{\text{I}}\sin(\theta/2)(\kappa - \cos\theta + 2\sin^2\theta) & \sigma + \gamma_{\text{I}}\cos(\theta/2)(\kappa + \cos\theta - 2\cos^2\theta) & 0 \\ 0 & 0 & \sigma \end{bmatrix},$$

$$(8.3)\qquad \sigma_{IJ}^{\text{eff}} = \begin{bmatrix} \sigma - \gamma_{\text{II}}\sin(\theta/2)(\kappa + 2 + \cos\theta - 2\sin^2\theta) & \gamma_{\text{II}}\cos(\theta/2)(\kappa + 2 - \cos\theta + 2\cos^2\theta) & 0 \\ \gamma_{\text{II}}\cos(\theta/2)(\kappa - 2 + \cos\theta - 2\sin^2\theta) & \sigma - \gamma_{\text{II}}\sin(\theta/2)(\kappa - 2 - \cos\theta - 2\cos^2\theta) & 0 \\ 0 & 0 & \sigma \end{bmatrix},$$

$$(8.4)\qquad \sigma_{IJ}^{\text{eff}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ -\gamma_{\text{III}}\sin(\theta/2) & \gamma_{\text{III}}\cos(\theta/2) & \sigma \end{bmatrix},$$

where

$$(8.5)\qquad \gamma_{\text{I}} = \frac{K_{\text{I}}\sigma^h}{4\mu\sqrt{2\pi r}}, \qquad \gamma_{\text{II}} = \frac{K_{\text{II}}\sigma^h}{4\mu\sqrt{2\pi r}}, \qquad \gamma_{\text{III}} = \frac{K_{\text{III}}\sigma^h}{\mu\sqrt{2\pi r}}.$$

It is clear that the conductivity is a function of θ and r, so the material near the crack tip can be modeled as nonhomogeneous. In such a case the flow of the current is no longer described by the formula $\mathcal{J}_{I,I} = 0$. To get the proper equation, it is necessary to consider

$$\mathcal{J}_{I,I} = \sigma^{\text{eff}}_{IJ} E_{J,I} + \sigma^{\text{eff}}_{IJ,I} E_J , \tag{8.6}$$

$$\mathcal{D}_{I,I} = \delta_{IJ}(\varepsilon E_{J,I} + \varepsilon_{,I} E_J), \tag{8.7}$$

and the Maxwell formula

$$\nabla \times \mathcal{H} = \mathcal{J} + j\omega\mathcal{D}, \tag{8.8}$$

where $\mathcal{H}$ is the magnetic intensity field, $\mathcal{D}$ is electric induction field, and $\omega = 2\pi f$ is angular frequency of the a.c. current applied. Calculating the divergence of both sides of the last formula, one can get the following equation

$$\mathcal{J}_{I,I} = -j\omega\varrho, \tag{8.9}$$

where ϱ is the charge density, and combining it with equations (8.6) and (8.7), the following formula is obtained

$$j\omega\varepsilon_0\delta_{IJ} + \sigma^{\text{eff}}_{IJ}\varrho = \sigma^{\text{eff}}_{IJ}\varepsilon_0 \mathcal{J}_K \sigma^{\text{eff}^{-1}}_{KL,L} . \tag{8.10}$$

The density of the charge, which is accumulated during the flow process near the crack tip, can be expressed for large conductivity or not too high frequency approximately as

$$\varrho \approx \varepsilon_0 \mathcal{J}_I \sigma^{\text{eff}^{-1}}_{IJ,J} . \tag{8.11}$$

9. Alternating current potential drop

When the a.c. current is flowing on the surface of the specimen (see Fig. 2), the potential drop far from the crack, which can be observed by the probe on the surface, can be simply calculated as

$$\Delta U_0 = \frac{Il}{\sigma S} , \tag{9.1}$$

where I is total current applied to the specimen, l is the distance between the probe pins along X_2, $S = \delta d$ is the cross-section of a.c. current path (d is the width of the specimen along X_3), and σ is the conductivity (scalar for the natural, not predeformed state of the material).

When the probe pins enclose the crack underneath, then the potential drop is composed of two parts

$$\Delta U = \Delta U_0 + \Delta U_c . \tag{9.2}$$

The first component is the same as (9.1) because it comes from the separation of pins in the probe, and the second one is the potential drop along the crack. It is assumed here that the crack is modeled by the wedge in the X_1 direction with the gap tending to zero ($P_1 \to P_2$). This means that the current is flowing approximately along X_1 with the constant density $\mathcal{J}_1 = I/S$, so the electric field is $E_1 = \sigma_{11}^{\text{eff}^{-1}} \mathcal{J}_1$, and it differs from point to point since the effective conductivity is nonhomogeneous. The total potential drop around the crack can be calculated as

$$\Delta U_c = \int_{P_1}^{P_2} E_1 \, dX_1 = \frac{I}{S} \int_{P_1}^{P_2} \sigma_{11}^{\text{eff}^{-1}} \, dX_1 . \tag{9.3}$$

The integration variable can be changed to polar, remembering that first it goes forward and second time back along X_1

$$\Delta U_c = \frac{I}{S} \left[\int_0^w \sigma_{11}^{\text{eff}^{-1}} \, dr \Big|_{\theta=-\pi} + \int_0^w \sigma_{11}^{\text{eff}^{-1}} \, dr \Big|_{\theta=\pi} \right], \tag{9.4}$$

where w is the depth of the crack under the surface. To perform the integration it is necessary to find $\sigma_{11}^{\text{eff}^{-1}}$ for $\theta = \pm\pi$.

On the left-hand side of the crack, $\theta = -\pi$, the effective conductivity is simplified as compared with Eqs. (8.2), (8.3), (8.4) for Mode I, II, III, respectively

$$\sigma_{IJ}^{\text{eff}}(r, -\pi) = \begin{bmatrix} \sigma & -\gamma_{\text{I}}(\kappa+1) & 0 \\ \gamma_{\text{I}}(\kappa-1) & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}, \tag{9.5}$$

$$\sigma_{IJ}^{\text{eff}}(r, -\pi) = \begin{bmatrix} \sigma + \gamma_{\text{II}}(\kappa+1) & 0 & 0 \\ 0 & \sigma + \gamma_{\text{I}}(\kappa-3) & 0 \\ 0 & 0 & \sigma \end{bmatrix}, \tag{9.6}$$

$$\sigma_{IJ}^{\text{eff}}(r, -\pi) = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ \gamma_{\text{III}} & 0 & \sigma \end{bmatrix}. \tag{9.7}$$

On the right-hand side of the crack, $\theta = \pi$, the effective conductivity is for Mode I, II, III, respectively

$$\sigma_{IJ}^{\text{eff}}(r, \pi) = \begin{bmatrix} \sigma & \gamma_I(\kappa+1) & 0 \\ -\gamma_{\text{I}}(\kappa-1) & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}, \tag{9.8}$$

$$\sigma_{IJ}^{\text{eff}}(r, \pi) = \begin{bmatrix} \sigma - \gamma_{\text{II}}(\kappa+1) & 0 & 0 \\ 0 & \sigma - \gamma_{\text{I}}(\kappa-3) & 0 \\ 0 & 0 & \sigma \end{bmatrix}, \tag{9.9}$$

$$\sigma_{IJ}^{\text{eff}}(r,\pi) = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ -\gamma_{\text{III}} & 0 & \sigma \end{bmatrix}. \tag{9.10}$$

Now, $\sigma_{11}^{\text{eff}^{-1}}$ for $\theta = \pm\pi$ can be obtained for three different modes as follows:

$$\sigma_{11}^{\text{eff}^{-1}} = \frac{1}{\sigma} \tag{9.11}$$

for Mode I and Mode III as well, whereas for Mode II it is different

$$\sigma_{11}^{\text{eff}^{-1}} = \frac{1}{\sigma \mp \gamma_{\text{II}}(\kappa+1)}. \tag{9.12}$$

For Mode I and III, $\sigma_{11}^{\text{eff}^{-1}}$ is constant, independent of r, thus the integration in Eq. (9.4) is very simple and gives

$$\Delta U_c = \frac{2Iw}{\sigma S}, \tag{9.13}$$

just like the potential drop on the straight path of the length $2w$. For the Mode II, however, $\sigma_{11}^{\text{eff}^{-1}}$ is dependent on r through γ_{II}, see formula (8.5), what can be written explicitly for $\theta = \pm\pi$ as

$$\sigma_{11}^{\text{eff}^{-1}} = \sigma \mp \frac{\gamma}{\sqrt{r}}, \tag{9.14}$$

where

$$\gamma = \frac{K_{\text{II}}\sigma^h(\kappa+1)}{4\mu\sqrt{2\pi}}. \tag{9.15}$$

Then, the integral in Eq. (9.4) takes the form

$$\Delta U_c = \frac{I}{S}\left[\int_0^w \frac{\sqrt{r}}{\sigma\sqrt{r}+\gamma}dr + \int_0^w \frac{\sqrt{r}}{\sigma\sqrt{r}-\gamma}dr\right], \tag{9.16}$$

what after integration gives

$$\Delta U_c = \frac{2I}{\sigma S}\left[w + \frac{\gamma^2}{\sigma^2}\ln\left|\frac{\sigma^2}{\gamma^2}w - 1\right|\right]. \tag{9.17}$$

It can be seen from the above that the potential drop consists of the term resulting from the flow around the crack of the path length $2w$, and the second one is the result of the stress concentration near the crack tip, and is dependent on the ratio of squared conductivity and its deformation sensitivity.

10. Conclusions

Since for Mode I, and II no influence of predeformation was found, so it can be said that Eq. (9.17) stands for the potential drop on the crack for any kind of load in general, which always can be decomposed into three basic modes. The second term of the formula (9.17) is the correction of the potential drop due to the load through stress intensity factor K_{II} for different material described by stress sensitivity to conductivity σ^h, conductivity itself σ, and elastic parameters μ, κ. The derivation of the potential drop was done for the isotropic elastic material, its conductivity being sensitive to deformation. The coupling of the magnetic field was neglected as well as the change of the penetration depth due to the change of conductivity.

Following the form of the relations (6.8), the influence of the bias on the magnetoelastic solids can have the important implications:

- The properties of the biased material can be changed so far as the natural crystallographic symmetries are not kept the same for the biased state of material.
- The medium can become strongly piezomagnetic even if its piezomagnetic natural state tensor is zero (or very small, what is common); the reason can be the third order magnetoelastic term for very high strain, but mainly the magnetostriction in the magnetically biased solid (this is a stronger effect) – this is called biased piezomagnetism.
- On the other hand the material having small permeability can become ferromagnetic, when large initial strain is assumed due to the same magnetostriction effect.
- The conductivity can become even a nonsymmetric tensor (in general) instead of its natural state scalar value.

Equations (6.6), (6.7) and (6.8) seem to be very useful to make a numerical application. As we compare to the natural state case, we can see that the sole change of material tensors from second order tensors given in tables to effective tensors (6.8), which are dependent on the second and the third order tensors as well as on the biasing quantities, is sufficient to consider the biased solid as an unbiased one. We have to keep in mind to write the equations of motion and the boundary conditions also in the reference frame. It is easy to find effective tensors and use well-known methods or even numerical algorithms (developed earlier for linear problems), and to solve also the nonlinear problems. These effective parameters describe the nonlinear coupling between a crystal and predeformation of a mechanical or magnetic origin.

Acknowledgement

This work was supported in part by KBN grant No. 3 P404 056 06.

References

1. A.C. PIPKIN, R.S. RIVILIN, *Electrical conduction in deformed isotropic materials*, J. Math. Phys., **1**, 127–130, 1960.
2. W. PRAGER, *Introduction to mechanics of continua*, McGraw-Hill, 1961.
3. A.C. ERINGEN, *Nonlinear theory of continuous media*, McGraw-Hill, 1962.
4. J.W. DUNKIN and A.C. ERINGEN, *On the propagation of waves in electromagnetic elastic solid*, Int. J. Engng. Sci., **1**, 461–495, 1963.
5. N.F. JORGAN and A.C. ERINGEN, *On the static nonlinear theory of electromagnetic thermoelastic solids – I*, Int. J. Engng. Sci., **2**, 59–95, 1964.
6. N.F. JORDAN and A.C. ERINGEN, *On the static nonlinear theory of electromagnetic thermoelastic solids – II*, Int. J. Engng. Sci., **2**, 97–114, 1964.
7. H.F. TIERSTEN, *Coupled magnetomechanical equations for magnetically saturated insulators*, J. Math. Phys., **5**, 1298–1318, 1964.
8. A.C. ERINGEN, *A unified theory of thermomechanical materials*, Int. J. Engng. Sci., **4**, 179–202, 1966.
9. M. SOKOLOWSKI, *Theory of couple-stresses in bodies with constrained rotations*, Springer-Verlag, 1970.
10. H. PARKUS, *Magneto-thermoelasticity*, Springer-Verlag, 1972.
11. Y. PAO and C. YEH, *A linear theory for soft ferromagnetic elastic solids*, Int. J. Engng. Sci., **11**, 415–436, 1973.
12. D.G.B. EDELEN, *Primitive thermodynamics: a new look at the Clausius–Duhem inequality*, Int. J. Engng. Sci., **12**, 121–141, 1974.
13. E. KIRAL and G.F. SMITH, *On the constitutive relations for anisotropic materials – triclinic, monoclinic, rhombic, tetragonal and hexagonal crystal systems*, Int. J. Engng. Sci., **12**, 471–490, 1974.
14. W. NOWACKI, *Dynamic problems of thermoelasticity*, Polish Scientific Publishers, 1975.
15. R.P. WEI and R.L. BRAZILL, *An assessment of a-c and d-c potential systems for monitoring fatigue crack growth*, [in:] Fatigue Crack Growth Measurement and Data Analysis, S.J. HUDAK, Jr. and R.J. BUCCI [Eds.], ASTM STP, **738**, 103, 1981.
16. D.H. MICHAEL, R.T. WAECHER and R. COLLINS, *The measurement of surface cracks in metals by using a.c. electric fields*, Proc. Roy. Soc. Lond. A, **381**, 139–148, 1982.
17. S.W. MEEKS, *Piezomagnetic and elastic properties of metallic glass alloys* $Fe_{67}Co_{18}B_{14}Si_1$ *and* $Fe_{81}B_{13,5}Si_{3,5}C_2$, J. Appl. Phys., **54**, 6584–6593, 1983.
18. W. LORD, *Electromagnetic methods of nondestructive testing*, Gordon and Breach, 1985.
19. M. F. KANNINEN and C. H. POPELAR, *Advanced fracture mechanics*, Oxford University Press, 1985.
20. G.A. MAUGIN, *Continuum mechanics of electromagnetic solids*, Elsevier, 1988.
21. A.C. ERINGEN and G.A. MAUGIN, *Electrodynamics of continua I*, Springer-Verlag, 1989.
22. H.F. TIERSTEN, *A development of the equations of electromagnetism in material continua*, Springer-Verlag, 1990.
23. E. KIRAL and A.C. ERINGEN, *Constitutive equations of nonlinear, electromagneto-elastic crystals*, Springer-Verlag, 1990.
24. G.A. MAUGIN and M. SABIR, *Mechanical and magnetic hardening of ferromagnetic bodies: influence of residual stress and application to nondestructive testing*, Int. J. Plasticity, **6**, 573–589, 1990.
25. M. SAKA, M. NAKAYAMA, T. KANEKO and H. ABÉ, *Measurement of stress-intensity factor by means of a-c potential drop technique*, Experimental Mech., **31**, 209, 1991.
26. M. SAKA, T. KANEKO and H. ABÉ, *Evaluation of stress intensity factor for a 3-D surface crack by means of a.c. potential drop technique*, Trans. JSME(A), **57**, 2222, 1991.
27. W.D. DOVER, R. COLLINS and D.H. MICHEL, *Review of developments in ACPD and ACFM*, British J. of NDT, **3**, 121, 1991.

28. M. SAKA, M. NAKAYAMA, T. KANEKO and H. ABÉ, *Measurement of stress-intensity factor by means of ac potential drop technique*, Experimental Mech., **31**, 209–212, 1991.

29. D. GAFKA and J. TANI, *Parametric constitutive equations for electroelastic crystals upon electrical or mechanical bias*, J. Appl. Phys., **70**, 6679–6686, 1991.

30. D. GAFKA and J. TANI, *Sensitivity of surface acoustic wave velocity in lithium niobate to electric field or biasing stress*, J. Appl. Phys., **73**, 7145–7151, 1993.

31. J.S. LEE, G.A. MAUGIN and Y. SHINDO, *Mechanics of electromagnetic materials and structures*, AMD-Vol. 161, MD-Vol. 42, 1993.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received February 2, 1995.