Arch. Mech., 48, 1, pp. 3–23, Warszawa 1996

Differential manifolds of the optimal solutions of a system of three ordinary equations

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OPTIMAL SYNTHESIS of an ordinary differential equation with control consists in ascribing the motion laws determined by control values to the domains of a decomposition of the state space, that is in establishment of a space of optimal solutions, or in rendering a definite structure to this space. In the paper are presented the considerations relating to the properties of the optimal structure of a three-dimensional space. Necessary conditions of optimality have been given, in geometrical form, of a three-dimensional differential equation.

1. Introduction

THE CONSIDERATIONS concern the differential equation of the following form

$$\dot{x} = Ax + Bu, \qquad x \in R^3 \backslash K, \qquad \mu K = 0, \qquad u \in U \subset R, \tag{1.1}$$

where U is a compact set. The aim of these considerations is the optimal synthesis of Eq. (1.1), that is determination of the function: $u^* : R^3 \backslash K \to U$ such, that every solution $x(x^0, t)$, $x^0 \in R^3$, $t \in \langle t_0, t_\mathcal{O} \rangle$, of Eq. (1.1) attains the origin $\mathcal{O}$ with $u = u^*$ at the shortest time $t_\mathcal{O} - t_0$.

The solution of the problem of synthesis consists in a decomposition of the state space into some geometrical manifolds of various dimensions with the assigned constant value controls, correspoding to apexes of rectangle controls U. Such decomposition of the space, together with assignation of the controls is connected with a problem of the optimal structure of the differential equation, which has been defined in Sec. 4, and which is considered as a correlate of the notion of entirety, built in some way from elements being manifolds of optimal solutions of Eq. (1.1).

The notion of a structure is generally valid for differential equations of the form

$$\dot{x} = f(x), \qquad x \in D \backslash K \subset R^n, \qquad \mu K = 0 \tag{1.2}$$

and it is connected with a decomposition $\mathcal{D}$ of set D into some n-dimensional domains D_i $(i = 1, 2, ...)$, to which some forms f_i $(i = 1, 2, ...)$ of the function f are referred, where f_i: $D_i \to R^n$ are continuous and Lipschitz functions. Definition of structure and optimal structure, as referred to Eq. (1.1), will be given below.

For the equation (1.1) the following assumptions have been made: A is a constant matrix 3×3 with negative real eigenvalues, B is a constant matrix 3×2, $u = (u_1, u_2)$, $U = \{u : c_l^1 \leq u_l \leq c_l^2\}$, $c_l^1 < 0$, $c_l^2 > 0$, $l = 1, 2$.

The optimal structure of Eq. (1.1) for the assumptions given above will be based upon a fact well known from the control theory [1]:

STATEMENT 1. The number of switchings on the optimal solution of Eq. (1.1) attains its minimum value.

Let $x(x^0, u, t)$ be the solution of Eq. (1.1), corresponding to control u with the initial condition $x(x^0, u, O) = x^0$, when it should be emphasized, that control on the solution is u, and let $x(x^0, t)$ be the solution of the equation (1.1) if such emphasizing is not necessary; let $T_u(x^0; (t_1, t_2))$ be a segment of a trajectory of the equation (1.1), corresponding to control u and interval (t_1, t_2) of parameter t; let $T_u(x^0, I^-)$ be a negative semi-trajectory of the equation (1.1): $\{x : x \in T(x^0; \langle 0, -\infty))\}$, and let, finally, $T_u(x^0)$ be a segment of trajectory corresponding to the interval $\langle 0, t_0 \rangle$.

It is known, [2], that Eq. (1.1) has the following properties: (1) – the domain of controllability coincides with R^3, (2) – the optimal control u^* is a piece-wise constant function with values $\mathrm{col}(c_1^j, c_2^k)$, $j, k \in \{1, 2\}$, such that each of its coordinates has in the interval $\langle t_0, t \rangle$ not more than 2 switchings.

We are interested in the optimal solutions only and for this reason it will be convenient, because of property (2), to use the form of Eq. (1.1) adequate to this property:

$$\dot{x} = Ax + Bu \quad (= f(x, u)), \qquad x \in R^n \backslash K, \qquad \mu K = 0, \tag{1.3}$$

$u \in \mathfrak{A}$, where $\mathfrak{A} = (a, b, c, d)$ and where $a, \ldots, d$ are control values u, corresponding to apexes of rectangle of controls U, that is two-element columns $\mathrm{col}(c_1^j, c_2^k)$, $j, k \in \{1, 2\}$. We will speak about the motion laws $(a), \ldots, (d)$, having in mind Eq. (1.3) with $u = \mathrm{col}(c_1^j, c_2^k)$, where (j, k) are the respective sequences. $\mathfrak{A}$ will mean everywhere a sequence of controls corresponding to a sequence of sucessive apexes of rectangle U in some ordering.

Concept of the optimal structure of a differential equation contains the notion of a set of semi-slides of solutions of this equation, [3].

2. Sets of semi-slides of solutions

Let V be a domain of R^3 and let almost everywhere on V vector field $f : V \backslash S \to R^3$, $\mu S = 0$, continuous and satisfying the Lipschitz conditions in domains W_i, $i = 1, \ldots, m$, $\bigcup_i \overline{W}_i = \overline{V}$ be determined. Let Γ ($\dim \Gamma \geq 1$) be a smooth boundary set in V such that $\Gamma \subset S$ (Fig. 1). Γ will be then a set belonging to the boundaries of m ($m \geq 2$) domains $W_i \subset V$ ($i = 1, \ldots, m$).

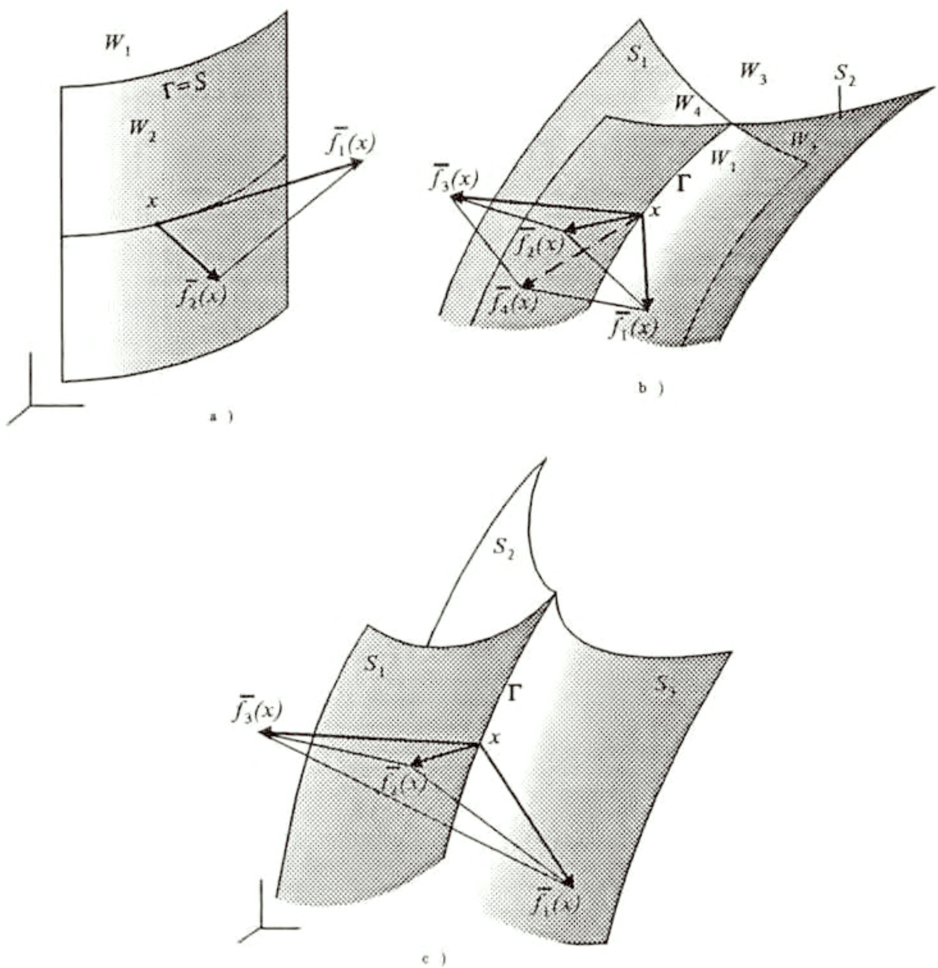

FIG. 1. a) dim $\Gamma = 2$, b) dim $\Gamma = 1$ ($S = S_1 \cap S_2$), c) dim $\Gamma = 1$ ($S = \overline{S}_1 \cap \overline{S}_2 \cap \overline{S}_3$).

If $x^0 \in \Gamma$, then there are m boundaries $\overline{f}_i(x^0)$ of the function f at point x^0:

$$\overline{f}_i(x) = \lim_{\substack{y \to x \\ y \in W_i}} f(y).$$

Let $\kappa(x)$ denote an arbitrary vector tangent to Γ at point x^0.

DEFINITION 1. *If for $x \in \Gamma$*

$$\exists\ i \in (1, ..., m)\ \ \exists\ k \in R\ \ (k \neq 0), \qquad k\overline{f}_i(x^0) = \kappa(x), \tag{2.1}$$

wherein $\overline{f}_i(x^0)$ is the edge of the smallest convex pyramid built on vectors $\overline{f}_i(x^0)$, $i = 1, ..., m$, then we say that x^0 is a point of semi-slide of vector field f on set Γ.

If the statement (2.1) *is true for any point* $x \in \Gamma \cap V$, *that is if*

$$(2.2) \qquad \exists\, i \in (1, \dots, m) \quad \forall\, x \in \Gamma \cap V \quad \exists\, k \in R \ \ (k \neq 0), \qquad k\overline{f}_i(x^0) = \kappa(x),$$

then we say that Γ *is a set of semi-slides of vector field* f *in set* V.

Let $W = V \backslash S$. In conformity with the assumption, $W = \bigcup_i W_i$. Consider the differential equation

$$(2.3) \qquad \dot{x} = f(x), \qquad x \in W,$$

where f is a function defined, continuous and satisfying the Lipschitz condition in each of domains W_i ($i = 1, \dots, m$). Let $x^0 \in \Gamma \subset S$. It is known, [4], that there exists a solution $x\colon O(t_0) \to R^3$ of the equation (2.3) at point x^0; $O(t_0)$ is the neighbourhood of t_0.

DEFINITION 2. *If for* x^0 *the statement* (2.1) *is true, then we say that the solution* x *is semi-sliding on* Γ *at point* x^0, *or that* x^0 *is a point of semi-slide of solutions of Eq.* (2.3) *on* Γ. *If the statement* (2.2) *is true, then solutions of Eq.* (2.3) *are semi-sliding on* Γ, *or* Γ *is a set of semi-slides of solutions of Eq.* (2.3) *in set* V.

The point $x^0 \in \Gamma$ such that

$$\exists\, y \in R^3 \quad \forall\, i \in (1, \dots, m) \quad \exists\, k \in R \ \ (k \neq 0), \qquad (f_i(x^0) = y, \quad ky = \kappa(x))$$

is, according to the definition 1, the point of a semi-slide of vector field f.

Point x^0, which is not a semi-slide point of vector field f, is either a semi-slide point, [4], or a point of a strict passage of the solution x of the equation (2.3) through Γ.

3. Basic decomposition, basic frame

Consider Eq. (1.3). Let (i) ($i \in \mathfrak{A}$) be a motion law, and T_i be a negative half-trajectory of Eq. (1.3), $T_i = T_i\langle \mathcal{O}, I^- \rangle$, corresponding to that motion law. T_i is the unique sub-set R^3 having the property:

$$\forall\, x^0 \in T_i \ \ \exists\, t_0 \geq 0, \qquad x(x^0, i, t_0) = 0.$$

STATEMENT 2. If $x(x^0, t)$, $x^0 \in R^3 \backslash \mathcal{O}$ is the optimal solution of Eq. (1.3), then

$$\exists\, i \in \mathcal{A} \quad \exists\, \bar{t} \geq 0 \quad \forall\, t \in \langle \bar{t}, t_0 \rangle, \qquad x(x^0, t) \in T_i\,.$$

From the Statement 1 there follows

STATEMENT 3. Every solution $x(x^0, i, t)$, $x^0 \in T_i$, is optimal.

Let (i), (j) be two motion laws, $i, j \in \mathfrak{A}$, $i \neq j$, with i, j being the neighbouring elements of the sequence $\mathfrak{A}$ (cf. Sec. 1). Let $\overline{S}_{(i,j)}$ be a two-dimensional set:

$$\overline{S}_{(i,j)} = T_j(T_i, I^-) = \{x : x \in \{T_j(x^0, T^-), \quad x^0 \in T_i\}\}$$

having a bundle structure and generated by semi-trajectory T_i. It may be noted that each half-trajectory T_i, $i \in \mathfrak{A}$, generates two sets $\overline{S}_{(i,j)}$, j being the element of the sequence $\mathfrak{A}$ neighbouring i (Fig. 2). Each of the sets $S_{(i,j)} = \overline{S}_{(i,j)} \backslash T_j \backslash T_i$ is by definition a differential manifold, on which the motion law (j) is valid. We will use the notation S_{ij}, if we don't indicate which of the half-trajectories T_i, T_j generates the set S_{ij}, that is the set $S_{(ij)}$ or $S_{(ji)}$. We will use the notation $S_{ij}^{(i)}$, if the motion law (i) will be valid on set S_{ij}, that is

$$S_{ij}^{(i)} = S_{(j,i)}, \qquad S_{ij}^{(j)} = S_{(i,j)}.$$

S_{ij} are manifolds on half-trajectories T_i, T_j, $i, j \in \mathfrak{A}$, $i \neq j$.

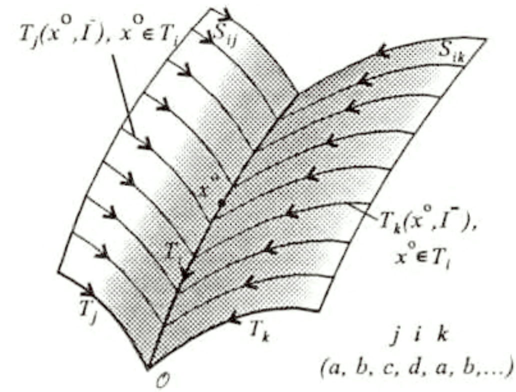

FIG. 2.

It may be easily noticed that no manifold S_{ij} contains singular points of the equation (1.3).

From definition of the sets S_{ij} it follows that, if $x(x^0, j, t)$ is the solution of Eq. (1.3), and if $x^0 \in S_{ij}$, then

$$(3.1) \qquad \exists\, t_1 > 0, \quad (\forall\, t \in \langle t_0, t_1 \rangle), \qquad x(x_0, j, t) \in S_{ij}, \qquad x(x^0, j, t_i) \in T_i).$$

Let $x(x^0, t)$ be the optimal solution of the equation (1.3). In this case the following statements hold.

STATEMENT 4, [5]. The sequence of controls on the optimal solution is a sequence with elements of $\mathfrak{A}$, neighbouring in $\mathfrak{A}$.

From definition of sets S_{ij} and from Statement 4 we will obtain the following

STATEMENT 5. If $x(x^0, t)$ is the optimal solution reaching the set T_i ($i \in \mathfrak{A}$) for $t = t_1 > 0$, then

$$\exists\ j \in \mathfrak{A} \quad (j \neq i, \quad i, j\text{– neighbouring in } \mathfrak{A})$$

$$\exists\ \varepsilon > 0 \quad \forall\ t \in \langle t_1 - \varepsilon, t_1), \qquad x(x^0, t) \in S_{(i,j)}\,.$$

LEMMA 1. All manifolds S_{ij}, $i, j \in \mathfrak{A}$, $i \neq j$, where i, j are neighbouring elements of the sequence $\mathfrak{A}$, are the sets of optimal solutions of Eq. (1.3).

P r o o f. Let $x^0 \in S_{(i,j)} \in \mathfrak{A}$, $i \neq j$, i, j being neighbours in $\mathfrak{A}$ and let $x(x^0, t)$ be the optimal solution of Eq. (1.3). Then

$$\text{(3.2)} \qquad \exists\ t_1 > 0, \quad (\forall\ t \in \langle t_0, t_1), \qquad x(x^0, t) \in S_{(i,j)}$$
$$\text{and} \qquad \forall\ t \in \langle t_1, t_{\mathcal{O}}), \qquad x(x^0, t) \in T_i),$$

that is there exists only one switching for $t = t_1$ and the sequence of controls on the solution is (j, i). Indeed, let us assume, to the contrary, that the sequence of controls on the optimal solution is different from (j, i), e.g. $(k, ...)$, $k \neq j$. The solution $x(x^0, k, t)$, $x^0 \in S_{(i,j)}$, in order to reach the origin $\mathcal{O}$, must previously reach some of the sets T_p, $p \in \mathfrak{A}$ (see Statement 2). Let us assume that it has reached this set for $t = t_2 > 0$. In order to reach T_p it must earlier reach some manifold $S_{(p,r)}$, $r \in \mathfrak{A}$, $r \neq p$, $r \neq j$, where p, r are neighbouring in the sequence $\mathfrak{A}$; x^0 belonging to manifold $S_{(i,j)}$ cannot belong to manifold $S_{(o,r)}$ (case $r \neq j$). Let us assume that $x(x^0, t)$ has reached $S_{(p,r)}$ for $t = t_1$, $0 < t_1 < t_2$. In such a case, on solution $x(x^0, t)$ there are two switchings: t_1 and t_2, and the sequence of controls will be (k, r, p). Hence a contradiction with the assumption on the optimality of the solutions follows: at the sequence of controls (j, i) on the solution $x(x^0, t)$ there is one switching only, cf. Statement 1.

Similarly it can be shown that the solution $x(x^0, t)$ with sequence of controls $(j, r, ...)$, $r \in \mathfrak{A}$, $r \neq i$ is not optimal with regard to the number of switchings.

The sequence of controls (j, i) on the solution $x(x^0, t)$ with the switching time different from t_1, causes that the point $\mathcal{O}$ is not being reached.

Satisfaction of (3.2) proves the Lemma.

■

Let K^* be a geometrical figure of zero measure in R^3:

$$\text{(3.3)} \qquad K^* = \mathcal{O} \cup \bigcup_{i \in \mathfrak{A}} T_i \cup \bigcup_{i,j \in \mathfrak{A}} S_{i,j}\,, \quad i, j - \text{neighbouring in } \mathfrak{A}.$$

From the definition it follows that K^* has the properties given below:

i. K^* divides the space R^3 into six discontinuous three-dimensional domains Ω_i ($i = 1, ...6$). The elements of the figure K^*, together with the domains Ω_i, are elements of decomposition $\mathcal{D}^*$ of the space R^3. Such decomposition will be called the basic one, its elements being respectively: edges (T_i), walls ($S_{(i,j)}$) and cells (Ω_i), Fig. 3. Figure K^* is called the basic frame of decomposition $\mathcal{D}^*$. $\mathcal{D}^*$ is determined uniquely by Eq. (1.3).

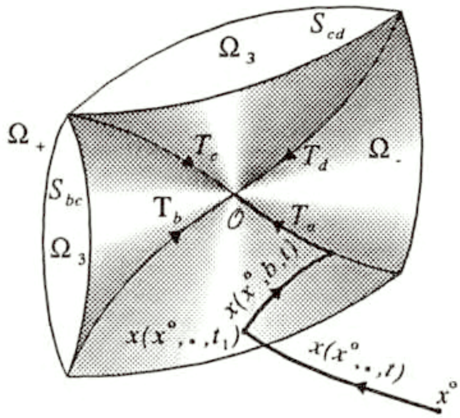

FIG. 3.

ii. To each element of the basic frame K^*, except the point $\mathcal{O}$, one of the motion laws: (a), (b), (c), (d) is uniquely ascribed, that is the following surjection is determined:

(3.4) $$\gamma : K^* \to \mathfrak{A}.$$

iii. All elements K^*, except the point $\mathcal{O}$, are sets of optimal solutions of Eq. (1.3), cf. Statement 3 and Lemma 1.

LEMMA 2. If $x(x^0, t)$, $x^0 \neq \mathcal{O}$ is the optimal solution of the equation (1.3), then

(3.5) $$\exists\, \bar{t} \geq 0 \quad \forall\, t \in \langle t, t_0 \rangle, \qquad x(x^0, t) \in K^*.$$

P r o o f. The sequence of controls on the optimal solution of Eq. (1.3) is composed of elements $\mathfrak{A}$.

If $x^0 \in K^*$, then in conformity with the property iii, (3.5) is true for $\bar{t} = 0$.

If $x^0 \notin K^*$, then in order for the solution $x(x^0, t)$ to reach the point $\mathcal{O}$ for $t = t_{\mathcal{O}}$ it is necessary:

$$\exists\, i \in \mathfrak{A} \quad \exists\, \bar{\bar{t}} > 0 \quad \forall\, t \in \langle \bar{\bar{t}}, t_{\mathcal{O}} \rangle, \qquad x(x^0, t) \in T_i.$$

Having assumed $\bar{\bar{t}} = \bar{t}$ we will have (3.5).

■

On the basis of Lemma 2 and Statement 5, the following statement is true.

STATEMENT 6. If $x(x^0, t)$, $x^0 \notin K^*$ is the optimal solution of the equation (1.3), then

$$\exists\,(i,j)\ \ (i,j \in \mathfrak{A},\quad i,j\ -\ \text{neighbouring in } \mathfrak{A})$$
$$\exists\,\bar{t} > 0\quad \exists\,\bar{\bar{t}} > \bar{t}\quad \forall\, t \in \langle \bar{t}, \bar{\bar{t}}),\qquad x(x^0, t) \in S_{ij}$$

that is the optimal solution beginning in a cell of decomposition $\mathcal{D}^*$ reaches the wall of that decomposition.

COROLLARY 1. The optimal solution of the equation (1.3), beginning in the cell of the decomposition $\mathcal{D}^*$, cannot reach its edge without having previously reached its wall.

K^* is an orientable figure. The cells of the decomposition $\mathcal{D}^*$ (see Fig. 3) are of two kinds: two-wall ones, bounded by walls $S_{(i,j)}$, $S_{(j,i)}$ ($i, j \in \mathfrak{A}$, $i \neq j$, i, j neighbouring in $\mathfrak{A}$), designated as Ω_{ij}, and four-wall ones, designated as Ω_-, Ω_+ bounded by four walls $S^{(.)}_{ab}$, $S^{(.)}_{bc}$, $S^{(.)}_{cd}$ and $S^{(.)}_{ad}$ of the basic frame. Let us recall that each element of decomposition $\mathcal{D}^*$, and hence also each cell, contains only its own elements: the elements of $\mathcal{D}^*$ are disjoint sets.

4. Structure of equation (1.3)

Let us assume that in Eq. (1.3) in which the set K has the following form:

$$K = K^* \cup L. \tag{4.1}$$

K is a frame of a new decomposition $\mathcal{D}$ of the space R^3 which has been built on the basic frame K^* by joining to it some new zero measure elements in such a way that these new elements do not intersect the basic frame:

$$L \cap K^* = \emptyset\,.$$

The assumption of a non-empty set L means a division of the cells of the basic decomposition $\mathcal{D}^*$ into the domains of definition (and continuity) of function f.

Let $\{\Omega\}$ denote a set of all cells of decomposition, and let δ be an injection

$$\delta:\ \{\Omega\} \to \mathfrak{A}\,. \tag{4.2}$$

DEFINITION 3. *Couple (K, δ) is called the structure of Eq.* (3.1)*, if all elements of the basic frame K^* are sets of semi-slides of solutions of Eq.* (1.3)*. The structure (K, δ) is called optimal, if all solutions $x(x^0, t)$, $x^0 \in R^3$ of this structure are optimal ones.*

DEFINITION 4. *We can say that two structures (K_1, δ_1), (K_2, δ_2) are equal, that is $(K_1, d_1) = (K_2, \delta_2)$, if:*

i) *their frames are isomorphic, that is if there exists isomorphism* $g: R^3 \to R^3$ *such that*

$$K_2 = \wp(K_1), \tag{4.3}$$

ii) $\delta_1 = \delta_2$.

If $\{K\}$ denotes a set of all frames of Eq. (1.3), then (4.3) is a relation of equivalence in set $\{K\}$. For a given number l of elements of frame K, the relation (4.3) divides the set $\{K\}$ into a finite number of equivalence classes $\{K\}_j$ ($j = 1, ..., J$, $J < \infty$). We will indentify in the next part of the paper the frame K, that is the representation of some equivalence class $\{K\}_j$ with that class. Indeed

$$\forall\ K \in \{K\} \quad \exists\ i \in (1, ..., J), \qquad K \in \{K\}_i,$$

that means that each frame is a representation of some class.

It is obvious that

$$\forall\ (i,j)\ \ (i,j \in (1, ...J),\ \ i \neq j)\ \ \forall\ K' \in \{K\}_i \quad \forall\ K'' \in \{K\}_j \qquad \sim \exists\ \wp,\ \ K'' = \wp(K'). \tag{4.4}$$

DEFINITION 5. *Decomposition* $\mathcal{D}$ *of space* R^3 *is called possible, if it generates structure* (K, δ) *of Eq.* (1.3).

Let $\{\mathcal{D}\}$ be a set of all possible decompositions in space R^3, defined by (4.1). From Statement 1 we obtain:

STATEMENT 7. In the optimal structure (K, δ) of Eq. (1.3) the numbers of cells and elements of the set L reach their minima on set $\{\mathcal{D}\}$.

LEMMA 3. Couple (K^*, δ), where K^* is a basic frame of Eq. (1.3), is not the optimal structure of this equation, irrespective of the map δ.

P r o o f. Let there exist such mapping δ that (K^*, δ) is the optimal structure of Eq. (1.3). To fix the attention, let the control a be referred to the four-walled cell Ω_-: $\Omega_-^{(a)}$. Three different cases can be considered, concrening systems of elements of boundaries of the cell $\Omega_-^{(a)}$, on which the motion law (a) is valid and beyond which it is not valid (cf. Fig. 3): (1) – edge T_a, (2) – edge T_a and one of the walls $S_{ab}^{(a)}$ or $S_{ad}^{(a)}$, (3) – edge T_a and two walls S_{ab} and S_{ad}. These cases correspond to three different basic frames (as equivalence classes) K_1^*, K_2^* and K_3^*, respectively. Since, by the contrary assumption, every solution $x(x^0, t)$, $x^0 \in \Omega_-^{(a)}$, is the optimal one of Eq. (1.3), then, in conformity with the Lemma 2 and Corollary 1, it will reach the respective basic frame K_i^* ($i = 1, 2, 3$) for $t = \bar{t} < \infty$ in the subset (see Fig. 3):

$$S_{ab}^{(b)} \cup S_{bc}^{(\cdot)} \cup S_{cd}^{(\cdot)} \cup S_{ad}^{(d)} \quad \text{for } i = 1 \qquad \text{case 1;}$$

$$(S_{ab}^{(b)} \cup S_{ad}^{(d)}) \cup S_{bc}^{(\cdot)} \cup S_{cd}^{(\cdot)} \quad \text{for } i = 2 \qquad \text{case 2;}$$

$$S_{bc}^{(\cdot)} \cup S_{cd}^{(\cdot)} \qquad \text{for } i = 3 \qquad \text{case 3.}$$

Since in both these sums a term $S^{(\cdot)}_{bc} \cup S^{(\cdot)}_{cd}$ appears that is the sum of two walls adjacent to edge T_c, then from the continuous dependence of initial conditions we have

$$\forall\, i \quad \exists\, x^0 \in \Omega^{(a)}_- \quad \exists\, \bar{t} > 0 \qquad \begin{cases} \forall\, t \in \langle 0, \bar{t}), \quad x(x^0, t) \in \Omega^{(a)}_- \\ \text{and} \\ x(x^0, \bar{t}) \in T_c\,, \end{cases}$$

which contradicts: (1) with Corollary 1, and (2) Statement 4 – switching $a \to c$. Hence, the thesis of the lemma is true.

■

COROLLARY 2. The basic frame K^* is a subset of the frame K of the optimal structure of Eq. (1.3), that is $L \neq \emptyset$.

CONCLUSION 1. The optimal structure of the equation (1.3) does not contain four-wall cells: Ω_-, Ω_+.

5. Structure of four-wall cells

Let us make the simplest, in the sense of the number of elements, division of the cells Ω_-, Ω_+. Such a division of a four-wall cell consists in construction of two three-wall cells of it by introduction of one "diagonal" wall stretched on the edges corresponding to controls of diagonals of a rectangle of controls U, and which does not change the number of frame edges. Let us assume that the frame K has the following form:

$$K = K^* \cup S_- \cup S_+\,, \tag{5.1}$$

where S_- and S_+ are the walls of division of the cells Ω_-, Ω_+ accordingly.

Let us consider, to fix the attention, the cell Ω_-. On the grounds of the assumption that the numbers of edges of frames K and K^* are the same, we may conclude that a closure of the wall S_- contains the origin $\mathcal{O}$ and the pairs of edges T_a and T_c, or T_b and T_d, see Fig. 3. Let, again for fixing attention, this be the pair T_a, T_c. Hence, decomposition $\mathcal{D}$ contains two three-walled cells, which will be denoted by Ω_b and Ω_d from the edges T_b and T_d, respectively, which belong to their closures and do not belong to $\overline{S}_-$. Let us assume that in these cells the motion laws (j), (i), $i, j \in \mathfrak{A}$, $i \neq j$ are valid, respectively. We will show later that such decomposition is possible (see Definition 5).

Let us assume that the wall S_- is a set of points of a strict transition of the solutions of Eq. (3.1). Let us assume further, for fixing attention, that the solutions pass through S_- from the cell Ω_d to Ω_b: control on solutions of Eq. (1.3) changes its value from i to j.

LEMMA 4. If the couple $(\Omega_d, (i))$ is a cut-off $(K, \delta)\Big|_{\Omega_d}$ of the optimal structure (K, δ) to set Ω_d, then

(i) $i = d$,

(ii) $\overline{\Omega}_d \supset S_{ad}^{(d)}$, $\overline{\Omega}_d \supset S_{cd}^{(d)}$.

P r o o f. It may easily be noted that only in the case determined by the conditions of the Lemma, the following relation will hold

$$\forall\ x^0 \in \Omega_d^{(d)} \quad \exists\ \bar{t} > 0 \qquad \begin{cases} \forall\ t \in \langle 0, \bar{t}), \quad x(x^0, t) \in \Omega_d^{(d)} \\ \text{and} \\ x(x^0, \bar{t}) \in S_- \,. \end{cases}$$

In all the remaining cases of combinations $(i, (S_{ad}^{(\cdot)}, S_{cd}^{(\cdot)}))$ we will have for the cell $\Omega_d^{(d)}$:

$$\exists\ k \in \{a, c, d\} \ \ (k \neq i) \quad \exists\ x^0 \in \Omega_d^{(d)} \quad \exists\ \bar{t} > 0 \quad \begin{cases} \forall\ t \in \langle 0, \bar{t}), \quad x(x^0, t) \in \Omega_d^{(i)} \\ \text{and} \\ x(x^0, \bar{t}) \in T_k \,, \end{cases}$$

which is excluded, on the grounds of the assumption that $x(x^0, t)$ is the optimal solution of Eq. (3.1), see Conclusion 1.

■

Figure 4 shows an example of the couple (K, δ) such that

$$(K, \delta)\Big|_{\Omega_d} = (\Omega_d, (a)), \qquad \overline{\Omega}_d \supset S_{ad}^{(a)}, \qquad \overline{\Omega}_d \supset S_{cd}^{(c)}.$$

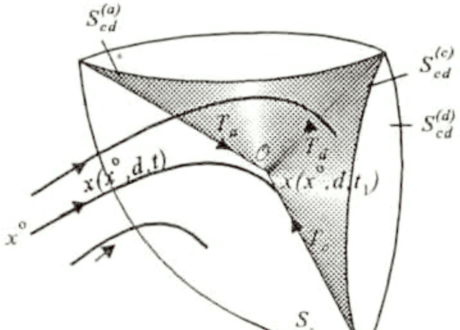

FIG. 4. Cell Ω_d.

LEMMA 5. If the couple $(\Omega_b, (i))$ is a cut-off $(K, \delta)\Big|_{\Omega_b}$ of the optimal structure (K, δ) to the set Ω_b, then one of the following two conditions is satisfied:

$$1)\quad i = a, \qquad \overline{\Omega}_b \supset S^{(a)}_{ab}, \qquad \overline{\Omega}_b \supset S^{(b)}_{bc},$$

$$2)\quad i = c, \qquad \overline{\Omega}_b \supset S^{(b)}_{ab}, \qquad \overline{\Omega}_b \supset S^{(c)}_{bc}.$$

P r o o f. (i) is the motion law in cell Ω_b, wherein $i = a$, or $i = c$. Indeed, $i \neq d$, because (i) is, on the grounds of the assumption, different from the motion law valid for the cell $\Omega^{(d)}_d$ (see Lemma 4), and $i \neq b$, because switching $d \to b$ (Statement 4) at a passage of the solution $x(x^0, t)$ of the structure (K, δ) through wall S_- is excluded.

To fix the attention, assume $i = a$. In such a case the following combinations of the walls of cell Ω_b are excluded:

$$(S^{(b)}_{ab},\ S^{(b)}_{bc}), \quad (S^{(b)}_{ab},\ S^{(c)}_{bc}).$$

Obviously, would it be true, then in the cell $\Omega^{(a)}_b$

$$\exists\ x^0 \in \Omega_b \cup S_- \quad \exists\ \bar{t} > 0 \quad \begin{cases} \forall\ t \in (0, \bar{t}), \quad x(x^0, t) \subset \Omega_b \\ \text{and} \\ x(x^0, \bar{t}) \in T_b\,. \end{cases}$$

In view of Statement 4, combination of the walls $(S^{(a)}_{ab},\ S^{(c)}_{bc})$ is also excluded because, if this were true, switching of controls $a \to c$ on the wall $S^{(c)}_{bc}$ would take place.

Hence a single possible wall combination is $(S^{(a)}_{ab},\ S^{(c)}_{bc})$, for which (see Fig. 5):

$$\exists\ x^0 \in \Omega_b \cup S_- \quad \exists\ \bar{t} > 0 \quad \begin{cases} \forall\ t \in (0, \bar{t}), \quad x(x^0, t) \subset \Omega_b \\ \text{and} \\ x(x^0, \bar{t}) \in S^{(b)}_{bc}\,. \end{cases}$$

For $i = c$ the proof is identical as before: only one combination of walls of the cell Ω_{bc} is possible: $(S^{(b)}_{ab}, S^{(c)}_{bc})$, for which (see Fig. 6)

$$\exists\ x^0 \in \Omega_b \cup S_- \quad \exists\ \bar{t} > 0 \quad \begin{cases} \forall\ t \in (0, \bar{t}), \quad x(x^0, t) \subset \Omega_b \\ \text{and} \\ x(x^0, \bar{t}) \in S^{(b)}_{ab}\,. \end{cases}$$

■

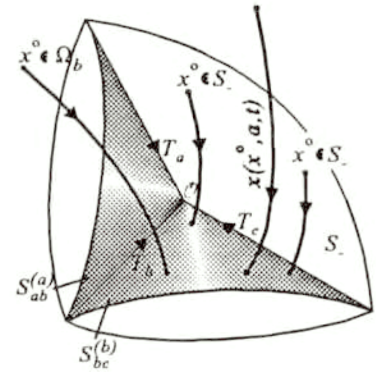

FIG. 5. Cell Ω_b.

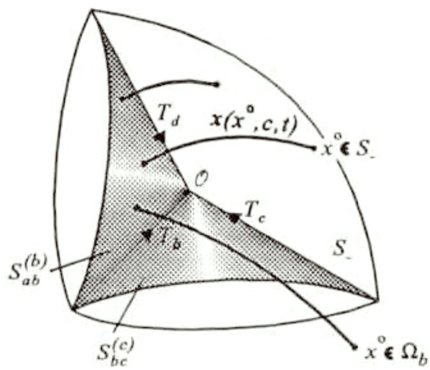

FIG. 6. Cell Ω_b.

Lemmas 4 and 5 can be generalized by formulation of the following

COROLLARY 3. If Ω_i ($i \in \mathfrak{A}$) is a three-wall cell defined above, whose one wall is S_k ($k \in \{-,+\}$), then the necessary condition for ensuring that the couple $(\Omega_i, (j))$ ($j \in \mathfrak{A}$) is a cut-off of the optimal structure (K, δ) to set Ω_i, is the satisfaction of one of the following two conditions:

1) $j = i$, when two walls adjacent to T_i have the motion law (i),

2) $j = l$, $l \neq i$, when there are two walls $S^{(l)}_{il}$ and $S^{(m)}_{im}$ ($l, m \in \mathfrak{A}$) adjacent to edge T_i, and the pairs (i, l) and (i, m) are sides of rectangle of controls U.

In the case 1) all solutions of the optimal structure (K, δ), which begin in cell Ω_i, will reach in a finite time the wall S_k; in the case 2) the cell Ω_i is a set of segments of trajectories of that structure with starting points belonging to S_k and end points belonging to $S^{(m)}_{im}$.

Let us recall that K is a frame built on the basic frame K^* by joining it with the walls S_-, S_+. From Lemmas 4 and 5 there follows

COROLLARY 4. K allows for two optimal structures (Fig. 7): (K_1, δ_1), (K_2, δ_2) with the frames:

$$K_1: \ \mathrm{Fr}\,\Omega_- \supset S^{(d)}_{ad} \cup S^{(d)}_{cd} \cup S^{(a)}_{ab} \cup S^{(b)}_{bc},$$
$$K_2: \ \mathrm{Fr}\,\Omega_- \supset S^{(d)}_{ad} \cup S^{(d)}_{cd} \cup S^{(b)}_{ab} \cup S^{(c)}_{bc},$$

and functions δ_1, δ_2, respectively:

$$(5.2) \qquad \begin{aligned} (K_1,\delta_1)\Big|_{\Omega_d} &= (\Omega_d,(d)), \qquad & (K_1,\delta_1)\Big|_{\Omega_b} &= (\Omega_b,(a)),\\ (K_2,\delta_2)\Big|_{\Omega_d} &= (\Omega_d,(d)), \qquad & (K_2,\delta_2)\Big|_{\Omega_b} &= (\Omega_b,(c)). \end{aligned}$$

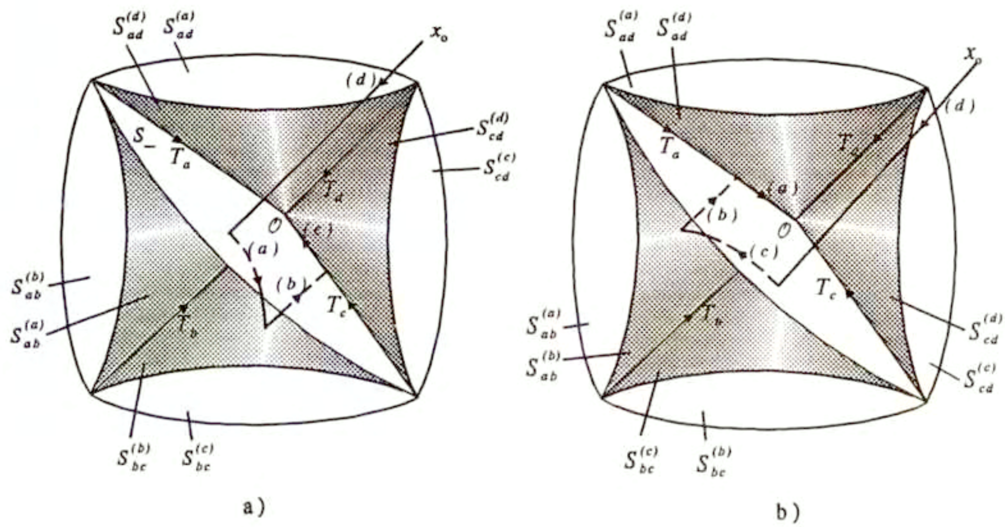

FIG. 7. a) Structure (K_1, δ_1), b) Structure (K_2, δ_2).

COROLLARY 5. Determining the boundary of the set Ω_-, the optimal structures (K_1, δ_1), (K_2, δ_2) lead to determination of the boundary of set Ω_+ (see Fig. 7):

$$(5.3) \qquad \begin{aligned} &\mathrm{Fr}\,\Omega_+ \supset S^{(d)}_{ad} \cup S^{(d)}_{cd} \cup S^{(b)}_{cb} \cup S^{(c)}_{bc} \qquad \text{for struct. } (K_1,\delta_1),\\ &\mathrm{Fr}\,\Omega_+ \supset S^{(d)}_{ad} \cup S^{(d)}_{cd} \cup S^{(a)}_{ab} \cup S^{(c)}_{bc} \qquad \text{for struct. } (K_2,\delta_2). \end{aligned}$$

We may note that for both structures, the system of the walls of set Ω_+ is analogous to the system of the walls of Ω_-: there are two walls with the same motion law: $S^{(c)}_{cd}$ and $S^{(c)}_{bc}$ for the structure (K_1, δ_1) and $S^{(a)}_{ad}$ and $S^{(a)}_{ab}$ for (K_2, δ_2). This, in turn, in view of Lemma 4 (after renumeration of the edges) leads to a method of division of the set Ω_- by wall S_+: similarly to the case of division of set Ω_-, the wall S_+ passes through the origin $\mathcal{O}$ and, in the case of both structures, it passes also through the edges T_b, T_d – Fig. 8.

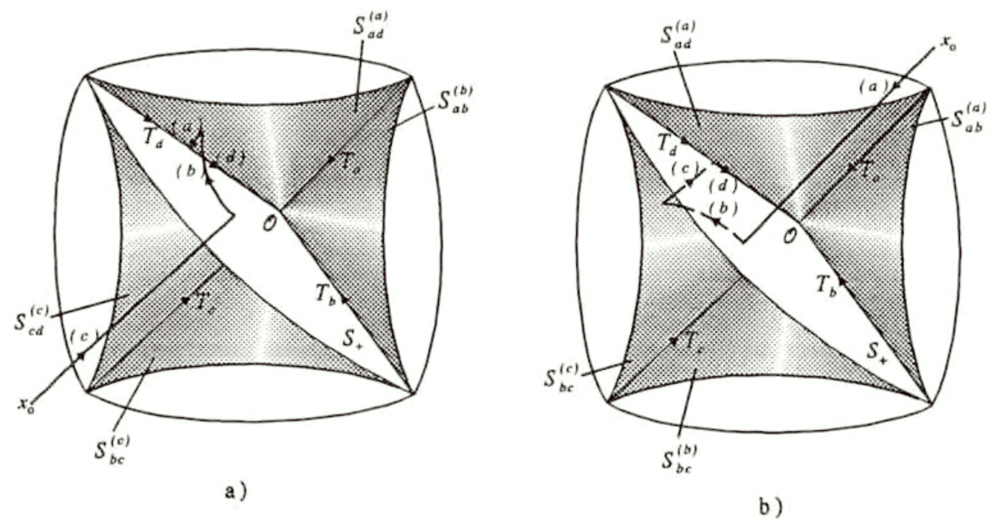

FIG. 8. a) Structure (K_1, δ_1), b) Structure (K_2, δ_2).

Hence, after the same analysis as that applied to the set Ω_-, we can draw the following conclusion.

COROLLARY 5. If (K_1, δ_1), (K_2, δ_2) are the optimal structures of the equation (1.3) then (see Fig. 8):

(1) conditions (5.3) must be satisfied, and

(2)

$$(5.4)\qquad \begin{aligned} &(K_1, \delta_1)\Big|_{\Omega_c} = (\Omega_c, (c)), \qquad &&(K_1, \delta_1)\Big|_{\Omega_a} = (\Omega_c, (b)),\\ &(K_2, \delta_2)\Big|_{\Omega_c} = (\Omega, (b)), \qquad &&(K_2, \delta_2)\Big|_{\Omega_a} = (\Omega_c, (a)). \end{aligned}$$

Here a question may be posed whether there exist two walls S_-, S_+, which would ensure the switchings $d \to a$, $d \to c$, $c \to b$ and $a \to b$, respectively. A positive answer to this question can be given, if the following hypothesis is assumed.

HYPOTHESIS. If (K, δ) is the optimal structure of Eq. (1.3), then all switchings are effected on differential manifolds of that equation in the form $T_{\cdot}$, or $S^{(\cdot)}_{\cdot\cdot}$.

This leads to the following conclusion.

COROLLARY 6. Walls S_-, S_+ are differential manifolds of Eq. (1.3), that is of the form $S^{(\cdot)}_{ac}$, $S^{(\cdot)}_{bd}$, respectively, where (see Figs. 7 and 8):

$$S_- = \begin{cases} S^{(c)}_{ac} & \text{for structure } (K_1, \delta_1),\\ S^{(a)}_{ac} & \text{for structure } (K_2, \delta_2), \end{cases}$$

$$S_- = S^{(d)}_{bd} \quad \text{for the both structures } (K_1, \delta_1),\ (K_2, \delta_2).$$

These walls are sets of a strict passage of solutions of the optimal structure (K, δ), cf. Definition 3.

Three-wall cells Ω_b, Ω_d are thus formed by a division of four-wall cell Ω_- by wall S_- of the form $S_{ac}^{(i)}$ $(i \in \{a, c\})$; the cells Ω_a, Ω_c are formed by a division of the cell Ω_+ by wall S_+ of the form $S_{bd}^{(d)}$.

6. Variants of the optimal structure

Let us have the basic decomposition $\mathcal{D}^*$ of space R^3 defined by equation (1.3) on the basic frame K^*. Let us have a definite ordering (one of two) of apexes of rectangle of controls U without ascribing the definite apexes to controls a, b, c and d of U.

Let us assume that:

1. Among the walls of each four-wall cell there are exactly two walls with a common edge and with a common motion law for these walls.

2. Motion laws for the common edge of those walls correspond to controls being apexes of controls rectangle U, the apexes being the ends of one of its sides.

Let us distinguish (arbitrarily) one of the four-wall cells, and let us denote the edge of two walls having the same motion law by T_d, and the cell itself by Ω_- (the second four-wall cell will thus be denoted by Ω_+). The ordering controls have been determined and hence the sequence (a, b, c, d) is determined by control values u corresponding to the apexes of U. Let us note that, depending upon the ordering in the cell Ω_+, either T_c or T_a can then be the common edge of the wall pair with a common motion law.

Hence, for the assumptions given above and taking into account the considerations presented in Sec. 5, the following theorem is true.

STATEMENT 8. A necessary condition of optimality of the structure (K, δ), where K is a frame defined in Sec. 5, is that the walls of the four-wall cell Ω_- of basic frame K^* must form one of two systems given below:

$$(S_{ab}^{(a)}, S_{bc}^{(b)}, S_{cd}^{(d)}, S_{ad}^{(d)})_{\mathrm{I}} \quad \text{or} \quad (S_{ab}^{(b)}, S_{bc}^{(c)}, S_{cd}^{(d)}, S_{ad}^{(d)})_{\mathrm{II}}. \tag{6.1}$$

The resulting wall systems for the cell Ω_+ are, respectively

$$(S_{ab}^{(b)}, S_{bc}^{(c)}, S_{cd}^{(c)}, S_{ad}^{(a)})_{\mathrm{I}} \quad \text{or} \quad (S_{ab}^{(a)}, S_{bc}^{(b)}, S_{cd}^{(c)}, S_{ad}^{(a)})_{\mathrm{II}}. \tag{6.2}$$

Let us assume, further, that there exists a division of the cells Ω_-, Ω_+ by the walls S_- and S_+ of the following form:

$$S_- = \begin{cases} S_{ac}^{(c)} & \text{for system I, or} \\ S_{ac}^{(a)} & \text{for system II,} \end{cases}$$

$$S_+ = S_{bd}^{(d)} \quad \text{for both systems}$$

into three-wall cells: S_d, S_b, S_c, S_a.

Let K_1 and K_2 denote respectively the decomposition frames $\mathcal{D}_1$ and $\mathcal{D}_2$ formed from $\mathcal{D}^*$ (Sec. 5) and corresponding to the systems I and II of the walls S_- and S_+ (see (5.1)).

It may easily be noticed that by changing the order of notation of apexes of the rectangle of controls U, the frame K_2 can be transformed into frame K_1. It is sufficient to assume in Ω_+ as T_c the edge of the wall pair with the common motion law.

Let us assume that the frame K_1 has been determined by the differential equation (1.3). Then the results of analysis given in Secs. 5 and 6 can be presented in the form of the following lemma.

LEMMA 6. If (K, δ) is the optimal structure of the equation (1.3), the following conditions are satisfied:

1 $K = K_1$,

2 δ is any function of δ_1 defined by its free cuts to three-wall cells Ω_d, Ω_b, Ω_c, Ω_a according to (5.2) and (5.4).

7. Two-wall cells

Let Ω_{ij} ($i, j \in \mathfrak{A}$, $i \neq j$, i, j neighbouring in the sequence $\mathfrak{A}$), be a cell of the decomposition $\mathcal{D}$.

LEMMA 7. If (K, δ) is the optimal structure of Eq. (1.3), then

$$(K, \delta)\Big|_{\Omega_{ij}} = (\Omega_{ij}, (k)),$$

where

$$k = i \vee j.$$

P r o o f. Let (K, δ) be the optimal structure of Eq. (1.3). The walls of the cell Ω_{ij} (Sec. 3) have the form $S_{ij}^{(i)}$ and $S_{ij}^{(j)}$. Let (k) ($k \in \mathfrak{A}$) be the motion law valid in Ω_{ij}. In conformity with the Statement 6, the control k transfers an arbitrary solution $x(x^0, k, t)$, $x^0 \in \Omega_{ij}$, on one of the walls of that cell, that is

$$\forall\, x^0 \in \Omega_{ij} \quad \exists\, l \in \{i, j\} \quad \exists\, \bar{t} > 0 \quad \begin{cases} \forall\, t \in \langle 0, \bar{t}) \subset \Omega_{ij}\,, \\ x(x^0, k, \bar{t}) \in S_{ij}^{(l)}\,. \end{cases}$$

Let us assume that the thesis of the lemma is false, that is that

$$k \neq i \wedge j.$$

On the grounds of the assumption on the optimality of the structure (K, δ), both walls $S_{ij}^{(i)}$ and $S_{ij}^{(j)}$ are sets of semi-slides of solutions of Eq. (1.3), with the motion laws (i) and (j) $(i, j \neq k)$, accordingly. Thus both the walls are the sets of points such that

$$\forall\, y \in S_{ij}^{(i)} \cup S_{ij}^{(j)} \quad \exists\, x^0 \in \Omega_{ij} \quad \exists\, \bar{t} > 0 \qquad \begin{cases} \forall\, t \in \langle 0, \bar{t}) \in \Omega_{ij}\,, \\ x(x^0, k, \bar{t}) = y, \end{cases}$$

and hence

$$\forall\, l \in \{i, j\} \quad \exists\, x^0 \in \Omega_{ij} \quad \exists\, \bar{t} > 0 \qquad \begin{cases} \forall\, t \in \langle 0, \bar{t}) \in \Omega_{ij}\,, \\ x(x^0, k, \bar{t}) \in T_l\,, \end{cases}$$

which contradicts the Statement 6 and Corollary 1 and proves the lemma.

■

Let $\Omega_{ij}^{(k)}$ $(i, j, k \in \mathfrak{A},\ i \neq j,\ k \in \{i, j\},\ i, j$ – neighbouring in $\mathfrak{A})$ be a cell of the structure (K, δ).

LEMMA 8. If (K, δ) is the optimal structure of the equation (1.3) then at least one of the walls $S_{ij}^{(i)}$ $(l \in \{j, j\})$ of the cell $\Omega_{ij}^{(k)}$ belongs to the boundary of a three-wall cell $\Omega^{(k)}.(l \neq k)$.

P r o o f. If Ω_{ij} is a two-wall cell of decomposition $\mathcal{D}$, then, in conformity with Sec. 5, its walls are also the walls of the respective three-wall cells Ω (Fig. 9).

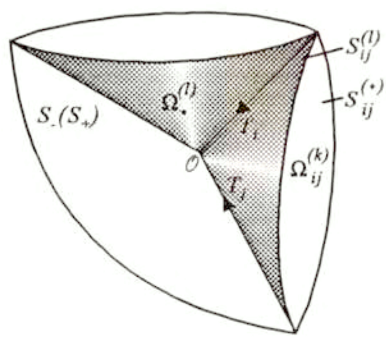

FIG. 9.

Let us assume that the thesis of the lemma is not true, that is let the motion laws in the three-wall cells adjacent to $\Omega_{ij}^{(k)}$ $(k = i \vee j$ – see Lemma 7) be different from the motion laws valid for the walls $S_{ij}^{(i)}$, $S_{ij}^{(j)}$: $\Omega_{\cdot}^{(l)}$, $\Omega_{\cdot}^{(m)}$, $l, m \in \mathfrak{A}$, $l \neq i$, $m \neq j$. To fix the attention, assume $k = i$, that is $\Omega_{ij}^{(i)}$. Then the wall $S_{ij}^{(j)}$ will not be a set of semi-slides of solutions of Eq. (1.3). Indeed, in the cells with a common wall $S_{ij}^{(j)}$, the motion laws are valid: $(i) \neq (j)$ in the two-wall cell $\Omega_{ij}^{(i)}$ and $(m)(\neq (j))$ in the three-wall cell $\Omega_{\cdot}^{(m)}$ the motion law will be valid $(m) \neq (j)$, whereas the motion law on the wall $S_{ij}^{(j)}$ is (j) – Fig. 10. Hence the wall $S_{ij}^{(j)}$ is not a set of semi-slides of solutions of equation (1.3), what contradicts the assumption of optimality of the structure (K, δ).

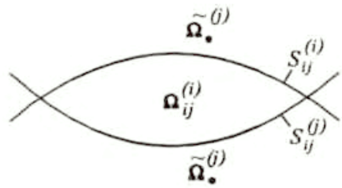

FIG. 10.

■

LEMMA 9. If Ω_{ij} and $\Omega_{\cdot}$ ($i, j, l \in \mathfrak{A}$, $i \neq j$, i, j – neighbouring in $\mathfrak{A}$) are, respectively, the two-walled and three-wall cells of the decomposition $\mathcal{D}$ adjacent to the wall $S_{ij}^{(i)}$, if $l \neq i$ and if (K, δ) is the optimal structure of Eq. (1.3), then

$$(K, \delta)\Big|_{\Omega_{ij}} = (\Omega_{ij}, (i)).$$

Proof. If $l \neq i$ and $S_{ij}^{(i)}$, on the grounds of the assumption on the optimality of the structure (K, δ), is a set of semi-slides of solutions of Eq. (1.3), then the motion law in the cell Ω_{ij} is (i), see Lemma 7. Indeed, assumption of the motion law (j) in Ω_{ij} according to Lemma 7 would be contradictory to the assumption of optimality of (K, δ) – see Definition 3.

■

LEMMA 10. Let Ω_{ij} be a two-wall cell of the decomposition $\mathcal{D}$, and $\Omega_{\cdot}^{(l)}(i)$, $\Omega_{\cdot}^{(m)}(j)$ ($l, m \in \mathfrak{A}$) be three-wall cells, adjacent to the respective walls $S_{ij}^{(i)}$, $S_{ij}^{(j)}$ of the cell Ω_{ij}. If (K, δ) is the optimal structure of Eq. (1.3) and

$$(l = i) \wedge (m = j),$$

then (Fig. 11)

$$(K, \delta)\Big|_{\Omega_{ij}} = (\Omega_{ij}, (i)) \vee (\Omega_{ij}, (j)).$$

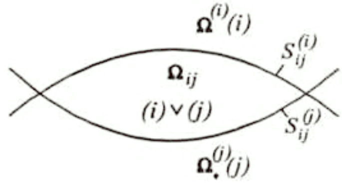

FIG. 11.

8. Necessary condition of optimality

Let there be a differential equation (1.3). Let K^* be the basic frame of this equation and K the frame of the form (5.1), where

$$S_- = S_{ac}^{(c)}, \qquad S_+ = S_{bd}^{(d)}$$

and where T_d, T_c have been defined as the edges of the respective wall pairs with a common motion law of three-wall cells of the decomposition $\mathcal{D}^*$. From the analysis given in Sec. 5 it follows that the frame K $(= K_1)$ exists – see Lemma 6. Decomposition $\mathcal{D}$ contains 8 cells: 4 three-wall and 4 two-wall ones.

On the grounds of the Statement 8 and Lemmas 7–10, the following necessary conditions of optimality of the structure (K, δ) may be given.

THEOREM 8.1. *The necessary conditions for* (K, δ) *to be the optimal structure of Eq.* (1.3) *are as follows:*

$$\text{(8.1)} \qquad \begin{aligned} &\left.\begin{aligned} (K,\delta)\big|_{\Omega_d} &= (\Omega_d, (d)) \\ (K,\delta)\big|_{\Omega_b} &= (\Omega_b, (a)) \end{aligned}\right\} \text{ in four-wall cell,} \\ &\left.\begin{aligned} (K,\delta)\big|_{\Omega_c} &= (\Omega_c, (c)) \\ (K,\delta)\big|_{\Omega_a} &= (\Omega_a, (b)) \end{aligned}\right\} \text{ in four-wall cell,} \\ &(K,\delta)\big|_{\Omega_{ab}} = (\Omega_{ab}, (a) \vee (b)), \\ &(K,\delta)\big|_{\Omega_{bc}} = (\Omega_{bc}, (b)), \\ &(K,\delta)\big|_{\Omega_{cd}} = (\Omega_{cd}, (c) \vee (d)), \\ &(K,\delta)\big|_{\Omega_{cd}} = (\Omega_{cd}, (a)). \end{aligned}$$

It may be noted that in the case of two two-wall cells Ω_{ab}, Ω_{cd} the motion law valid for these cells has been formulated alternatively. Assuming this alternative we would obtain from (8.1) the necessary and sufficient conditions of existence of optimality of the structure (K, δ).

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Received May 6, 1994.