Arch. Mech., 48, 1, pp. 211–215, Warszawa 1996

BRIEF NOTES

Some explicit formulae for heteroclinic solutions to scalar second order ordinary differential equations

B. KAŹMIERCZAK (WARSZAWA)

We derive some explicit formulae for heteroclinic pairs for scalar ordinary differential equations of the form $u'' + v\lambda(u)u' - su(u-a)(u-1)(1+\phi(u)) = 0$, where λ is an arbitrary polynomial in ϕ and is properly chosen. The results are used in a simple model of plasma sustained by a laser beam.

Problem

Let us consider the equation:

$$u'' + vu' - su(u-a)(u-1) = 0, \tag{1}$$

where $'$ denotes differentiation with respect to $\xi \in (-\infty, \infty)$. If $a \in (0,1)$, then it is known (see e.g. [1]) that there exists a unique heteroclinic pair $(v, u(\xi))$ satisfying the conditions $u(0) = 1/2$, $u(-\infty) = 0$ and $u(\infty) = 1$. It has a very simple form:

$$v = -\left(\frac{1}{2} - a\right)\sqrt{2s}\,, \qquad u(\xi) = \left(1 + \exp\left[-\sqrt{\frac{s}{2}}\,\xi\right]\right)^{-1}.$$

In many applications it seems that the coefficient multiplying u' should not be treated as a constant. On the other hand, one would like to have still at least one solution in an explicit form. It is rather difficult to satisfy these two demands. However, in this work we find a partial solution to these problems. First, we have the following lemma:

Lemma 1. The function

$$u(\xi) = \left(1 + \exp\left[-\sqrt{s}\sqrt{\frac{A}{B}}\,\xi\right]\right)^{-1},$$

where $A := ca + b$, $B := 2b + c$, satisfies the equation

$$u'' + v(b + cu)u' - su(u-a)(u-1) = 0$$

for $v = -\sqrt{s}(1-2a)\left(\sqrt{AB}\right)^{-1}$. □

P r o o f. The claim of the lemma follows by straightforward calculation. □

REMARK 1. It is easy to note that in the case of equation of the form $u'' + vu' + Luu' - su(u-a)(u-1) = 0$, a heteroclinic pair is the following one:

$$\left(v = -(2M)^{-1}[LM + s(1-2a)], \quad u(\xi) = \left(1 + \exp\left[-\frac{1}{4}\xi M\right]\right)^{-1}\right),$$

where $M = L + \sqrt{L^2 + 8s}$. □

REMARK 2. If a function $f(u)$ behaves qualitatively as $-u(u-a)(u-1)$, i.e. so that $f_{,u}(0) < 0$, $f_{,u}(1) < 0$ and $\lambda(u)$ is of one sign, the heteroclinic pair (v, u) for the equation $u'' + v\lambda(u)u' + f(u) = 0$ is uniquely determined. The proof follows, for example, from the usual phase plane analysis (see e.g. [2]). □

In general, determination of an explicit heteroclinic solution for the equation $u'' + v\lambda(u)u' - su(u-a)(u-1) = 0$ is rather difficult, even if λ is in a polynomial form. However, one can analyze an equation of the kind

$$u'' + v\lambda(u)u' - su(u-a)(u-1)(1+\phi(u)) = 0 \tag{2}$$

and try to find a term $\phi(u)$ so that such an equation still has a solution of the form $(1 + \exp[-\sigma\xi])^{-1}$ for a unique value of the constant v. Below, we will consider two simple and important cases.

CASE A. $\lambda(u)$ – *arbitrary polynomial*

Let

$$\lambda(u) := b + \sum_{i=1}^{I} c_i u^i.$$

Let $I \geq 1$ be natural,

$$A_I := b + \sum_{i=1}^{I} c_i a^i, \qquad B_I = 2b + \sum_{i=1}^{I} c_i a^{i-1}.$$

Let b and c_i be such that A_I and B_I are positive. Then, the following lemma is true:

LEMMA 2. Let

$$v = -\sqrt{s}(1-2a)\left(\sqrt{A_I B_I}\right)^{-1} \qquad \text{and} \qquad \phi(u) = (1-2a)B_I^{-1}\sum_{j=1}^{I-1}\chi_j u^j,$$

where

$$\chi_j = \left(\sum_{i=j+1}^{I} c_i a^{i-j-1}\right).$$

Then, for any finite I, the function

$$u(\xi) = \left(1 + \exp\left[-\xi\sqrt{s}\sqrt{A_I B_I^{-1}}\right]\right)^{-1}$$

satisfies Eq. (2).

P r o o f. The claim of the lemma follows by straightforward calculation. □

By means of the above lemma one can check the following

COROLLARY. Suppose that $b = 0$ and $c_i = 0$ for $i \neq \zeta$, $\zeta \geq 2$.
Then, for

$$v = -\sqrt{\frac{s}{a}}(1-2a)c_\zeta a^\zeta \qquad \text{and} \qquad \phi(u) = (1-2a)a^{-\zeta+1}uS(a,u,\zeta-2),$$

the function

$$u(\xi) = \left(1 + \exp\left[-\xi\sqrt{as}\right]\right)^{-1}$$

satisfies Eq. (2). Here $S(a,u,\zeta)$ denotes the symmetric polynomial of the variables a and u of the order $(\zeta-2)$ for $\zeta > 2$, i.e.

$$S(a,u,\zeta) = \sum_{i=0}^{\zeta-2} a^i u^{\zeta-i-2}.$$

Thus, this time, ϕ is independent of λ. □

CASE B. $\lambda(u) = b + \gamma(\chi + u)^{-1}$

Now, we assume that

$$\gamma + b(a+\chi) \neq 0 \qquad \text{and} \qquad (\chi + u) \neq 0 \quad \text{for} \quad u \in (0,1).$$

LEMMA 3. If

$$v = -\sqrt{\frac{s}{2}}(1-2a)\frac{a+\chi}{\gamma + b(a+\chi)} \qquad \text{and} \qquad \phi(u) = \frac{1}{2}\,\frac{\gamma(1-2a)}{\gamma + b(a+\chi)}(\chi+u)^{-1},$$

then the function

$$u(\xi) = \left(1 + \exp\left[-\sqrt{\frac{s}{2}}\xi\right]\right)^{-1}$$

satisfies Eq. (2). □

P r o o f. The claim of the lemma follows by straightforward calculation. □

By means of the above lemma one can check the following

COROLLARY. Suppose that $b = 0$. Then, for

$$v = -\sqrt{\frac{s}{2}}(1-2a)(a+\chi)\gamma^{-1} \qquad \text{and} \qquad \phi(u) = (1-2a)\frac{1}{2}(u+\chi)^{-1},$$

the function

$$u(\xi) = \left(1 + \exp\left[-\xi\sqrt{\frac{s}{2}}\right]\right)^{-1}$$

satisfies Eq. (2). If additionally $\chi = 0$, then ϕ is independent of λ. However, this time the source function

$$-su(u-a)(u-1)\left[1 + (1-2a)\frac{1}{2u}\right]$$

does not tend to 0 for $u \to 0$ (for $a \neq 2^{-1}$). Likewise, if $\chi = -1$, then $\phi(u) = \frac{1}{2}(1-2a)(u-1)^{-1}$ and the source function does not tend to 0 for $u \to 1$. □

A simple application

Let us consider the plasma sustained by a laser beam. It can be described by the equation (see [3, 4]):

$$\varrho c_p \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\mathbf{grad}\right) T = \operatorname{div}(k\,\mathbf{grad}\,T) + F, \tag{3}$$

where T the temperature of the gas, ϱ is the mass density, c_p the heat capacity per unit volume (under a constant pressure), k is the effective heat conductivity coefficient and $\mathbf{v}$ is a velocity of the flowing gas. F is a nonlinear source term and (after rescaling T) it can be qualitatively modeled by a cubic polynomial $-ST(T_0 - T)(1-T)$, $T_0 \in (0, 1)$. We are interested in temperature front solutions connecting the two asymptotic states of gas: 1. Cold gas corresponding to $T = 0$ (about 300 K before scaling), and 2. Hot gas (partially ionized plasma) corresponding to $T = 1$ (15000 – 20000 K before scaling). When the velocity of the cold incoming gas is parallel to the direction of the laser beam, the problem has a cylindrical symmetry with respect to this direction. Let us analyze Eq. (3) at the points lying on the axis of symmetry. It is convenient to introduce a curvilinear system of coordinates with base vectors parallel and perpendicular to the lines of constant temperature. When we assume that $(\partial T/\partial t) = 0$, then at the axis of symmetry Eq. (3) becomes

$$\varrho c_p \|\mathbf{v}\| T_{,\xi} = k_{,T}(T_{,\xi})^2 + k(T_{,\xi\xi} + \kappa(x)T_{,\xi}) + F, \tag{4}$$

where $\kappa(x)$ is the curvature of an isotherm passing through the point x. This curvature is nonzero due to the finite width of the laser beam. If we replace

$\varrho c_p\|\mathbf{v}\|$ and k (which, in general, depend on T) by suitable constants, then Eq. (4) can be rewritten as:

$$T_{,\xi\xi} - \left(q - \kappa(x)k^{-1}\right) T_{,\xi} + f = 0, \tag{5}$$

where $f(T) = F(T)k^{-1} = -sT(T_0 - T)(1 - T)$, $s = Sk^{-1}$, and q corresponds to the mass speed of the gas. According to the experimental, numerical and theoretical results [4, 5], the curvature can be qualitatively modeled as a function of T in the following way: $\kappa(T)k^{-1} = [L_0 + TL]$, with appropriate positive L and L_0. Now, using Remark 1 after Lemma 1 we can find $q(L_0, L)$ for which there exists a heteroclinic solution to Eq. (5) joining the states 0 and 1. Namely, we have

$$q = (2M)^{-1}[LM + s(1 - 2a)] + L_0\,,$$

where

$$M = L + \sqrt{L^2 + 8s} \qquad \text{and} \qquad u(\xi) = \left(1 + \exp\left[-\frac{1}{4}\xi M\right]\right)^{-1}.$$

Acknowledgements

The author wishes to express his gratitude to Prof. Z. PERADZYŃSKI for pointing out the problem.

References

1. O. DIEKMANN and M.N. TEMME, *Nonlinear diffusion problems*, Matematisch Centrum, Amsterdam 1976.
2. P. FIFE, *Mathematical aspects of reacting and diffusing systems*, Lecture Notes in Biomathematics, **28**, Springer, 1979.
3. Iu.P. RAIZER, *Fizika gazovogo razrjada*, Nauka, Moskva 1987.
4. Z. PERADZYŃSKI, *Continuous optical discharge, properties and modelling*, Invited Papers, ICPIG XXI, Bochum 1993.
5. J. KURZYNA, Z. PERADZYŃSKI and Z. SZYMAŃSKI, *Free burning laser sustained plasma in a forced flow*, J. Phys. D, Applied Phys., **27**, 2074–2079, 1994.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received June 1, 1995.