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Invariance of sliding motions with respect to control matrix

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THE OBJECT of this paper is the proof of the theorem that the sliding solutions of systems of differential equations with discontinuous functions, with a linear control input, which describe mechanical systems, are independent of parameters of the control input matrix.

1. Introduction

DISTURBANCES are an element which usually accompanies physical phenomena. They can be produced either by changes of environment, in which the given phenomenon is observed, or by changes of the object itself. Hence, determination of characteristic features of the objects insensitive to disturbances is of great importance, similarly to the problem of formulation of conditions making possible a synthesis of systems which would be invariant with respect to disturbances. Also the number of the developed systems insensitive to disturbances is still growing.

Systems with variable structure belong to the systems which are insensitive to disturbances. For such systems, an invariance principle has been formulated which gives the conditions of invariance of a variable structure system, which performs sliding motions, resistant to possible disturbances [3]. The aforementioned invariance principle specifies the conditions of insensibility to a variable structure system to disturbances without separation of a special class of such systems. The object of consideration in this article is the invariance of mechanical variable structure systems. It appears that such systems are independent of the disturbances of control input parameters. This peculiar feature of mechanical variable structure systems is the object of analysis in this article.

Mechanical systems are described by differential equations of the second order. These equations can have various forms, depending upon the form of description (Newton, Lagrange). Moreover, the form of the equations also depends on the fact whether we have to do with holonomic, or non-holonomic systems. But nevertheless, irrespective of the kind of the description, we obtain, as a result, a system of differential equations of the second order. Transformation of such equations into a system of differential equations of the first order can lead to a situation, where the obtained matrices of coefficients of those equations contain many zero elements. When the equations which describe the sliding motion of a system with discontinuous functions are derived, these matrices must be appropriately transformed. A considerable number of zero elements makes it possible to put forward a hypothesis that elements of some matrices can be eliminated during the process of construction of the equations which describe the sliding motion. Full elimination of matrices would mean independence of the sliding solutions of the elements of that matrix. This article will be devoted to determination of the conditions of such independence.

2. Systems with discontinuous functions and sliding solutions

Variable structure systems subject to sliding motion are most often described by differential equations with a linear control input. They have the following form:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ s &= Kx, \\ u &= \begin{cases} u^{+} & \text{for } s > 0, \\ u^{-} & \text{for } s < 0. \end{cases} \end{aligned} \tag{2.1}$$

The system so described has a variable structure, determined by the control function u, varying over the surface s (in our case s is a hyperplane). The author will consider the properties of a motion, which takes place on a switching surface, which is called a sliding motion. When the sliding conditions [2, 4] are satisfied, then a sliding motion described by the differential equation (2.2) takes place on the discontinuity surface:

$$\dot{x} = Ax - B(KB)^{-1}KAx\,. \tag{2.2}$$

Let us note that the sliding solutions, that is the solutions of Eq. (2.2), depend upon the matrix B. Invariance of those equations with respect to B means a reduction of matrix B in the quotient $B(KB)^{-1}$. This means, that there exist conditions, whose satisfaction enables the elimination of matrix B in Eq. (2.2). However, in a general case, such a reduction is impossible. Another important and interesting problem is the determination of special cases, when all elements of matrix B are reduced after all the operations are performed in the case of matrices $B(KB)^{-1}$, even if such a reduction was not possible *a priori*. In such a case the sliding motion would be independent of the elements of the matrix B, in spite of the fact, that the structure of the equation (2.2) depends upon the structure of matrix B. The same can also be said about the sliding solutions. In the paper the author has verified the hypothesis that mechanical variable structure systems perform sliding motions independently of the elements of the matrix B, in spite of the fact that they depend upon its structure (even the dimensions and positions of nonzero elements).

A natural system is a system of n differential equations with one-dimensional control of the following form [1]:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
&\vdots \\
\dot{x}_{n-1} &= x_n, \\
\dot{x}_n &= a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n + bu, \\
s &= k_1x_1 + k_2x_2 + \ldots + k_nx_n, \\
u &= \begin{cases} u^+ & \text{for } s > 0, \\ u^- & \text{for } s < 0, \end{cases}
\end{aligned} \tag{2.3}$$

and hence the system in a matrix form can be written as follows:

$$\begin{aligned}
\dot{x} &= Ax + Bu, \\
s &= k_1x_1 + k_2x_2 + \ldots + k_nx_n \\
u &= \begin{cases} u^+ & \text{for } s > 0, \\ u^- & \text{for } s < 0. \end{cases}
\end{aligned} \tag{2.4}$$

Matrices A and B will have the following form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ a_1 & a_2 & a_3 & \ldots & a_n \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix}.$$

Matrix $B(KB)^{-1}$ will have the following form:

$$B(KB)^{-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} \left([k_1, k_2, \ldots, k_n] \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} \right)^{-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} (k_nb)^{-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{1}{k_n} \end{bmatrix}. \tag{2.5}$$

As a result, we will obtain an expression independent of matrix B, and hence also the sliding solution will be independent of matrix B.

In this way the following lemma has been proved.

LEMMA 1. For the system (2.3), or for an equivalent system (2.4), both the sliding solution, as well as the equation which describes it, are independent of elements of matrix B.

The hypothesis stating that the sliding solution for the mechanical systems is independent of matrix B, will be verified on an example of the system with two degrees of freedom with two intersecting discontinuity surfaces, which have the following structure:

$$\begin{aligned}
\dot{x}_1 &= x_2\,,\\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + b_2 u,\\
\dot{x}_3 &= x_4\,,\\
\dot{x}_4 &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + b_4 u,\\
s_1 &= k_{11}x_1 + k_{12}x_2 + k_{13}x_3 + k_{14}x_4\,,\\
s_2 &= k_{21}x_1 + k_{22}x_2 + k_{23}x_3 + k_{24}x_4\,,\\
u &= \begin{cases} u^+ & s_i > 0,\\ u^- & s_i < 0.\end{cases}
\end{aligned} \tag{2.6}$$

Matrices A and B from (2.6) will have the following form:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0\\ a_{21} & a_{22} & a_{23} & a_{24}\\ 0 & 0 & 0 & 1\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0\\ b_2 & 0\\ 0 & 0\\ 0 & b_4\end{bmatrix}, \qquad K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14}\\ k_{21} & k_{22} & k_{23} & k_{24}\end{bmatrix}.$$

We will verify the result of the operation $B(KB)^{-1}$ being an element of description of the sliding motions for the matrices described above:

$$B(KB)^{-1} = \begin{bmatrix} 0 & 0\\ b_2 & 0\\ 0 & 0\\ 0 & b_4\end{bmatrix} \left(\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14}\\ k_{21} & k_{22} & k_{23} & k_{24}\end{bmatrix} \begin{bmatrix} 0 & 0\\ b_2 & 0\\ 0 & 0\\ 0 & b_4\end{bmatrix} \right)^{-1}$$

$$= \frac{1}{k_{12}k_{24} - k_{14}k_{22}} \begin{bmatrix} 0 & 0\\ k_{24} & -k_{42}\\ 0 & 0\\ -k_{22} & k_{12}\end{bmatrix}.$$

As a result of the operations performed, we have obtained for the example considered a matrix with elements independent of matrix B. However, the structure of the matrix thus obtained is dependent upon the structure of matrix B. In consequence, the sliding motion is dependent structurally on matrix B, but it is independent of the values of elements of matrix B. As a result, we have obtained a partial invariance of sliding solutions with respect to matrix B.

In a general case, matrix $B(KB)^{-1}$ does not satisfy the conditions of the hypothesis of independence of matrix B. This can be illustrated by the following example:

Let $B(KB)^{-1} = C$, where C is some unknown matrix. Multiplying the left-hand side of that expression by KB we obtain

$$B = CKB.$$

If our hypothesis is true (without any exceptions), then the following condition should be satisfied:

$$CK = I \qquad (I \text{ -- identity matrix, that is } \ B = IB).$$

On the contrary, for the matrix

$$\overline{K} = \begin{bmatrix} \dfrac{b_1 - b_2 a_{12}}{b_1} & a_{12} \\ a_{21} & \dfrac{b_2 - b_1 a_{21}}{b_2} \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

the relationship $B = \overline{K}B$ is satisfied, where matrix $\overline{K}$ is not an identity matrix.

3. Mechanical systems with discontinuous functions

The preceding section included an example of a mechanical system with discontinuous functions, the sliding solutions of which were independent of matrix B. Under these circumstances a question arises, what was the real cause of elimination of elements of matrix K. We may presume that the real cause of elimination of those elements was a specific nature of mechanical systems, described by the equations, their matrices having many zero elements and a specific structure (half of elements of matrix B are zero); this was due to transformation of the second order differential equations into a system of differential equations of the first order.

For mechanical system the following theorem can be formulated.

THEOREM 1. *Consider a mechanical system of the form*

$$\dot{x} = Ax + Bu$$

with discontinuity surfaces

$$s = Dx \tag{3.1}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots\ldots & 0 \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2,2k} \\ 0 & 1 & 0 & 1\ldots\ldots & 0 \\ a_{41} & a_{42} & a_{43} & \ldots\ldots & a_{4,2k} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & 1 \\ a_{2k,1} & a_{2k,2} & a_{2k,3} & \ldots\ldots & a_{2k,2k} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ b_2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & b_4 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & b_{2k} \end{bmatrix}, \qquad D = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1,2k} \\ d_{21} & d_{22} & \ldots & d_{2,2k} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ d_{k,1} & d_{k,2} & \ldots & d_{k,2k} \end{bmatrix},$$

and matrix DB is nonsingular,

$$u_i = \begin{cases} u_i^+ & s_i > 0, \\ u_i^- & s_i < 0. \end{cases}$$

Sliding solution obtained for a mechanical variable structure system of the form (3.1) is independent of elements of the control matrix B.

P r o o f.

$$(3.2) \qquad (DB)^{-1} = \left(\begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1,2k} \\ d_{21} & d_{22} & \ldots & d_{2,2k} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ d_{k,1} & d_{k,2} & \ldots & d_{k,2k} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ b_2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & b_4 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & b_{2k} \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} \dfrac{|D_{11}|}{b_2 b_4 \ldots b_{2k}\,|D|} & \dfrac{|D_{21}|}{b_2 b_4 \ldots b_{2k}\,|D|} & \ldots & \dfrac{|D_{k1}|}{b_2 b_4 \ldots b_{2k}\,|D|} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{|D_{1k}|}{b_2 b_4 \ldots b_{2k}\,|D|} & \dfrac{|D_{2k}|}{b_2 b_4 \ldots b_{2k}\,|D|} & \ldots & \dfrac{|D_{kk}|}{b_2 b_4 \ldots b_{2k}\,|D|} \end{bmatrix},$$

D_{ji} *is a matrix without the i-th line and j-th column,*

$$(DB)^{-1} = \left[\frac{(-1)^{j+i}|D_{ji}|}{b_2 b_4 \dots b_{2k}|D|}\right],$$

where

$$|D_{ji} = b_2 b_4 \dots b_{2(j-1)} b_{2(j+1)} \dots b_{2k}|D^*_{ji}| = \frac{\prod b_i}{b_{2j}}(-1)^{j+i}|D^*_{ji}|,$$

where D^*_{ji} *is*

$$D^*_{ji} = \begin{bmatrix} d_{12} & d_{14} & \dots & d_{1,2(j-1)} & d_{1,2(j+1)} & \dots & d_{1,2k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ d_{i-1,2} & d_{i-1,4} & \dots & d_{i-1,2(j-1)} & d_{i-1,2(j+1)} & \dots & d_{i-1,2k} \\ d_{i+1,2} & d_{i+1,4} & \dots & d_{i+1,2(j-1)} & d_{i+1,2(j+1)} & \dots & d_{i-1,2k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ d_{k,2} & d_{k,4} & \dots & d_{k,2(j-1)} & d_{k,2(j+1)} & \dots & d_{k,2k} \end{bmatrix},$$

$$(DB)^{-1} = \left[(-1)^{j+i}\frac{1}{b_{2i}}\frac{|D^*_{ji}|}{|D|}\right].$$

From the results presented above we can obtain the form of the expression $B(DB)^{-1}$ by multiplying of the matrix $(DB)^{-1}$ in the form (3.2) by the matrix B.

$$\text{(3.3)} \qquad B(DB)^{-1} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ b_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & b_4 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_{2k} \end{bmatrix}$$

$$\times \begin{bmatrix} \dfrac{|D^*_{11}|}{b_2\,|D|} & -\dfrac{|D^*_{21}|}{b_2\,|D|} & \dots & \dfrac{(-1)^{k+1}|D^*_{k1}|}{b_2\,|D|} \\ -\dfrac{|D^*_{12}|}{b_4\,|D|} & \dfrac{|D^*_{22}|}{b_4\,|D|} & \dots & \dfrac{(-1)^{k+2}|D^*_{k2}|}{b_4\,|D|} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dfrac{(-1)^{k+1}|D^*_{1k}|}{b_{2k}\,|D|} & \dfrac{(-1)^{k+2}|D^*_{2k}|}{b_{2k}\,|D|} & \dots & \dfrac{(-1)^{k+k}|D^*_{kk}|}{b_{2k}\,|D|} \end{bmatrix}$$

(3.3) [cont.]

$$= \begin{bmatrix} 0 & 0 & \dots & 0 \\ (-1)^2\frac{|D^*_{11}|}{|D|} & -\frac{|D^*_{21}|}{|D|} & \dots & (-1)^{k+1}\frac{|D^*_{k1}|}{|D|} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ (-1)^{k+1}\frac{|D^*_{1k}|}{|D|} & (-1)^{k+2}\frac{|D^*_{2k}|}{|D|} & \dots & (-1)^{k+k}\frac{|D^*_{kk}|}{|D|} \end{bmatrix}.$$

Hence, matrix $B(DB)^{-1}$ does not contain the elements of matrix B, which concludes the proof.

In this way we have proved the theorem that the sliding solutions of the system (3.1) are independent of the elements of control input matrices B, being dependent on its structure. The structure of matrix B exerts influence on the sliding motion of a mechanical system. The theorem being proved in that form is restricted to holonomic systems. The questions whether non-holonomic systems have the same property deserves a separate consideration.

Invariance of the sliding solutions with respect to internal disturbances was analysed simultaneously with the first investigations of the sliding motions [3]. This property was one of the main causes of taking applications of sliding in the construction of the existing technical objects. The present paper extended the scope of investigations of the problems of parametric invariance, it has shown that the mechanical systems, as well as the systems of a different physical nature, wich can be described by the same mathematical model, possess sliding solutions, which are invariant with respect to the parameters of discontinuous control input matrix.

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