Arch. Mech., 48, 1, pp. 163–190, Warszawa 1996

Stiffness loss in laminates with intralaminar cracks
Part II. Periodic distribution of cracks and homogenization

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IN THE SECOND PART of the paper, under the assumption that periodically distributed cracks in the internal layer can open or close according to Signorini's conditions, the homogenization procedure is carried out. With the help of two types of scaling, three effective models of the laminate with smeared-out cracks are proposed: a model of moderately thick laminates with cracks of high density, a model for thin laminates with cracks of arbitrary density, and for thin laminates with densely distributed cracks. The models derived show nonlinear and hyperelastic features, the relevant hyperelastic potentials being convex or strictly convex. The homogenized constitutive relations assume Kachanov's form in which the damage moduli tensor is uniquely determined.

1. Introduction

BY APPLYING the new two-dimensional laminate model developed in Part I [8], two methods of homogenization will be applied in the second part of the paper. The first of them is based on the in-plane scaling of the longitudinal dimensions of periodicity cells, whilst the second method uses simultaneous scaling of all dimensions of cells.

The formulae interrelating the loss of effective characteristics with components of macrodeformations will be rigorously derived. It will be proved that the effective laminate has nonlinear hyperelastic properties, its hyperelastic potential being convex or strictly convex. The homogenization approach enables us to determine all components of the tensor of effective stiffnesses within the framework of one scheme. The reason for that is that the homogenization scheme satisfies Hill's lemma of equivalence of mutual works of stresses stored in the effective and periodic (here: periodically cracked) composites, being sufficiently flexible to produce all effective stiffnesses, see SUQUET [I.48]; cf. also TELEGA [I.49], where a generalization of this lemma to the case of discontinuous fields has been put forward. HILL'S [4] lemma can be symbolically written as

$$\langle \sigma' \varepsilon'' \rangle = \langle \sigma' \rangle \langle \varepsilon'' \rangle,$$

where $\langle \cdot \rangle$ means averaging (in the periodic case – over the basic cell of periodicity) and σ', ε'' are admissible stress and strain fields, respectively.

The papers which are based neither on the homogenization method nor on Hill's lemma, usually make use of a more restricted lemma of equivalency of energy (σ' and ε'' are then interrelated by a constitutive relation), capable of evaluating only diagonal components of the stiffness matrices, viz. the effective

Young and Kirchhoff moduli, cf. HASHIN [I.17], ABOUDI [I.1]. Then the Poisson ratios have to be assessed independently, cf. HASHIN [I.18].

Another feature of the homogenization method is that it rigorously distinguishes between micro- and macrofields, defining them precisely. Correctness of the homogenization results follows from the proof of convergence of the solution of the ε-problem to the solution to the homogenized problem as $\varepsilon \to 0$. In the problem studied here the following two reasons: presence of cracks and non-conventional space scaling make the convergence proof difficult. This proof is included in the paper by TELEGA and LEWIŃSKI [14].

The micromechanics model put forward in the present paper can include neither the effects due to delamination nor those due to fibre breakage. An extended micromechanics model has been recently proposed by YANG and BOEHLER [I.58]. Their model is capable of describing the onset of interlamination and its interaction with transverse cracks.

Supported by the own experimental results, the papers of ALLEN *at al.* [1], GROVES *et al.* [I.11], HARRIS *et al.* [3], LEE *et al.* [I.25], MOTOGI and FUKUDA [10], MOTOGI *et al.* [11] show that also a phenomenological continuum damage mechanics approach can be helpful in the description of stiffness degradation of composite laminates.

Throughout the second part of the paper Roman numeral I refers always to the first part of our contribution [8].

2. Regular crack system

In this section we assume that the internal layer incurs transverse cracks which form a fixed layout. No attempt will be made to interrelate the crack pattern geometry neither with the directions of principal stresses nor with the directions of anisotropy of the plies. For fiber-reinforced polymeric composites this would also be unrealistic since the onset of matrix cracking is caused mainly by the mismatch between the thermal expansion of the fiber and the resin, cf. YALVAÇ *et al.* [I.57].

Our aim is to find a relation between the layout of cracks and the stiffness loss of the laminate. To arrive at transparent and useful formulae we confine our attention to the case when the layout of cracks is periodic. Otherwise we would have to resort to stochastic methods or bounding techniques that require different mathematical tools and result in less viable final formulae, cf. TELEGA and LEWIŃSKI [I.53].

Consider the laminate of Sec. I.2 weakened by a family $\mathcal{F}$ of fissures F_j described in Sec. I.3. One can divide the domain Ω (except for a boundary zone) into homothetic rectangular cells Z_j, each of which is weakened by a crack F_j. Z_j may contain a finite number of cracks. We observe only that F_j should not intersect the boundary of the rectangle Z_j (cf. Fig. 1) as well as the boundary

Γ of the domain Ω. This assumption can be weakened so as to admit cracks intersecting the boundaries, cf. CHACHA and SANCHEZ–PALENCIA [I.8]. For the effective models which will be derived in the succeeding sections, the properties of the homogenized elastic potentials will also be discussed when cracks intersect the boundaries.

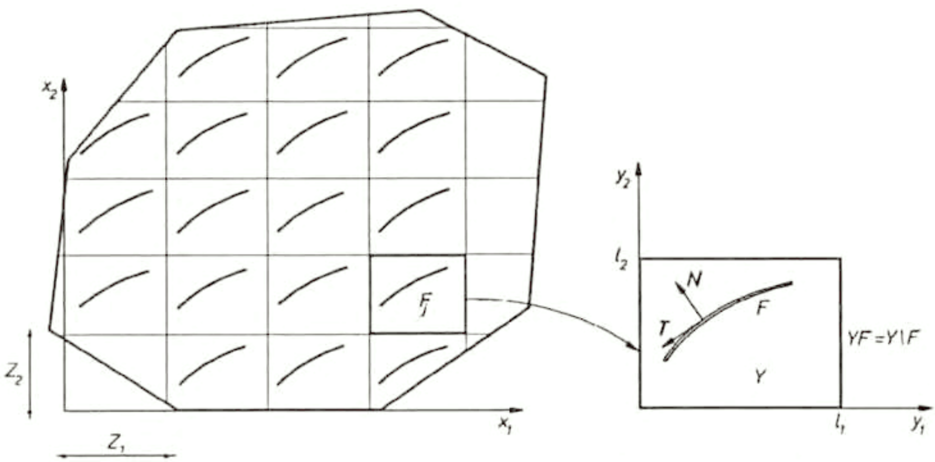


FIG. 1. Z-periodic layout of cracks. Geometry of the basic cell $YF = Y\setminus F$.

For future convenience we define a cell Z with a crack F such that all cells Z_j with cracks F_j are homothetic to it; $Z = (0, Z_1) \times (0, Z_2)$.

Mathematical description of Sec. I.3 applies here. A single crack F should be replaced by $\mathcal{F}$ which is a sum of a finite number of cracks. Both strong and variational formulations of Sec. I.3 remain formally unchanged.

If the number of cracks is very large, the solution to the relevant problem becomes unattainable even by employing numerical methods. On the other hand, a natural question arises about the overall properties of such laminates. It is the homogenization method that provides an answer to this question. As a method of averaging, this method provides a unique and perfect algorithm. As a method belonging to the family of small parameter methods, it depends upon the manner in which a small parameter ε is introduced. Two versions of the homogenization method corresponding to two methods of introducing ε into the original problem will be discussed in the sequel.

3. Moderately thick laminate weakened by transverse cracks of high density Model (h, l_0)

This section is aimed at deriving formulae for the static analysis and assessing effective stiffnesses of a moderately thick three-layer laminate densely cracked in the internal layer. The derivation will be based upon the conventional homogenization approach applied to the problem posed in the previous section. The model thus derived will be referred to as a (h, l_0) one, which means that it applies to the case when the distances between cracks are much smaller than h, and that the thickness $2h$ can be considered as moderately thick as compared with the longitudinal dimensions of Ω.

3.1. Family of ε-problems

As a method that belongs to the family of small parameter methods, the homogenization approach requires an introduction of a small parameter. The physical nature of the problem should provide us with the hints of how to introduce this parameter. In our problem, two small parameters are present: $\max(Z_1, Z_2)$ and h (or c and d). In this section we shall assume that only the former dimension is small and we replace Z_α by εl_α and F by εF. The rectangles of periodicity are homothetic to the basic cell $Y = (0, l_1) \times (0, l_2)$. The domain $\Omega \backslash F$ is now replaced by $\Omega^\varepsilon = \Omega \backslash F^\varepsilon$, where F^ε is the sum of all cracks εF_j. The set of the kinematically admissible fields assumes the form

$$\mathbb{K}_\varepsilon := \mathbb{K}(\Omega^\varepsilon) = H_{\Gamma_w}(\Omega)^2 \times K(\Omega^\varepsilon) \times H_{\Gamma_w}(\Omega), \tag{3.1}$$

where

$$K(\Omega^\varepsilon) = \left\{ \mathbf{u} \in H^1(\Omega^\varepsilon)^2 \mid \mathbf{u} = \mathbf{0} \text{ on } \Gamma_w \text{ and } [\![u_n]\!] \geq 0 \text{ on } F^\varepsilon \right\}. \tag{3.2}$$

For a fixed $\varepsilon > 0$ the equilibrium problem assumes the form

$$(P^1_{\Omega^\varepsilon}) \left| \begin{array}{l} \text{Find } (\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon) \in \mathbb{K}_\varepsilon \text{ such that} \\ a_{\Omega^\varepsilon}(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon; \mathbf{v}', \mathbf{u}' - \mathbf{u}^\varepsilon, w') \geq f(\mathbf{v}', \mathbf{u}' - \mathbf{u}^\varepsilon, w') \ \forall (\mathbf{v}', \mathbf{u}', w') \in \mathbb{K}_\varepsilon . \end{array} \right. \tag{3.3}$$

The bilinear form $a_{\Omega^\varepsilon}(\cdot, \cdot)$ is defined by Eq. (I.2.27), the integration over Ω being replaced here by integration over Ω^ε.

The scaling: $Z_\alpha \to \varepsilon l_\alpha$, $h \to h$ ($c \to c$, $d \to d$) used here will be referred to as an in-plane scaling; the sign ($\to$) means replacement.

REMARK 3. 1. The problem $(P^1_{\Omega^\varepsilon})$ is posed on a highly irregular domain Ω^ε, for which Korn's inequality does not apply in its standard form. However, in the paper by TELEGA and LEWIŃSKI [I.52], Korn's inequality has been derived in a form directly applicable to domains like Ω^ε presented in Fig. 1. Essential for such a domain is the assumption: $F \subset Y$, where F is closed as a set. In this case F does not intersect ∂Y. Consequently, the bilinear form a_{Ω^ε} is coercive on

$$V(\Omega^\varepsilon) = H_{\Gamma_w}(\Omega)^2 \times H_{\Gamma_w}(\Omega^\varepsilon)^2 \times H_{\Gamma_w}(\Omega),$$

and on $\mathbb{K}_\varepsilon$; the linear form f is continuous in this space. Applying now Th. 2.1 of KINDERLEHRER and STAMPACCHIA [I.21] we conclude that there exists a unique solution $(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon)$ to $P^1_{\Omega^\varepsilon}$.

The case when F intersects the boundary ∂Y of Y is obviously more complicated, cf. Fig. 2 and the paper by CHACHA and SANCHEZ-PALENCIA [I.8]. We still assume that $Y_\eta = Y \backslash F^\eta$ $(\eta > 0)$ is a connected set with a Lipschitzian boundary. Denoting by F_η^ε the sum of all holes εF_j^η $(\varepsilon > 0)$ we deduce that $\Omega_\eta^\varepsilon = \Omega \backslash F_\eta^\varepsilon$ is a domain of type I in the sense of OLEINIK *et al.* [12]. Now the boundary of the domain Ω^ε may be intersected by cracks from F^ε. By combining the results concerning extension theorems, presented in Chapter I of the book by OLEINIK *et al.* [12], with the approach used by TELEGA and LEWIŃSKI [I.52], we arrive at Korn's inequality for $V(\Omega^\varepsilon)$. Consequently, the problem $P^1_{\Omega^\varepsilon}$ still admits a unique solution for this particularly important cracking mode.

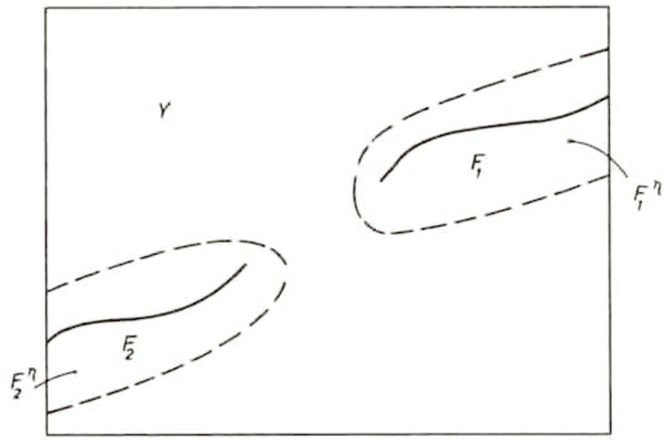


FIG. 2. Fissure $F = F_1 \cup F_2$ intersecting ∂Y.

3.2. Asymptotic solution

The asymptotic solution to the problem P^ε_Ω can be found by the method similar to that proposed by SANCHEZ-PALENCIA [I.47, Ch. 6, Sec. 6 & 7) and extended in LEWIŃSKI and TELEGA [I.29,I.30], TELEGA and LEWIŃSKI [I.52]. Therefore we shall expound here and in the succeeding sections only the ideas and results. The convergence theorem is provided in the paper by TELEGA and LEWIŃSKI [14].

Let us define the space

$$H^1_{\text{per}}(Y) = \Big\{ v \in H^1(Y) \mid v \text{ assumes equal values on opposite sides of } Y \Big\}.$$

For $YF = Y\backslash F$ the space $H^1_{\text{per}}(YF)$ is defined similarly. Let $\mathbf{N}$, $\mathbf{T}$ be the unit vectors: outward normal to F and tangent to F, respectively, cf. Fig. 1. Let the brackets $[\![\cdot]\!]$ denote jump on F. Let us define the sets of kinematical fields:

$$\tag{3.4} \begin{aligned} K_{YF} &= \Big\{ \mathbf{u} \in H^1_{\text{per}}(YF)^2 \mid [\![u_N]\!] \geq 0 \text{ on } F \Big\}, \\ \mathbb{K}_{YF} &= H^1_{\text{per}}(Y)^2 \times K_{YF} \times H^1_{\text{per}}(Y). \end{aligned}$$

The solution to the problem $(P^1_{\Omega^\varepsilon})$ is predicted in the following form

$$(3.5)\qquad \begin{aligned} v^\varepsilon_\alpha &= v^0_\alpha(x) + \varepsilon v^1_\alpha(x,y) + \varepsilon^2 v^2_\alpha(x,y) + \dots,\\ u^\varepsilon_\alpha &= u^0_\alpha(x) + \varepsilon u^1_\alpha(x,y) + \varepsilon^2 u^2_\alpha(x,y) + \dots,\\ w^\varepsilon &= w^0(x) + \varepsilon w^1(x,y) + \varepsilon^2 w^2(x,y) + \dots, \end{aligned}$$

where $y = x/\varepsilon$ and v^0_α, u^0_α, $w^0 \in H_{\Gamma_w}(\Omega)$, $v^1_\alpha(x,.)$, $w^1(x,.) \in H^1_{\text{per}}(Y)$, $v^k_\alpha(x,.)$, $w^k(x,.)$, $k \geq 2$ are Y-periodic and sufficiently regular; $\mathbf{u}^1(x,.) \in K_{YF}$, $v^k_\alpha(.,y)$, $w^k(.,y)$, $u^k_\alpha(.,y)$ are defined on Ω and are sufficiently regular; $y \in YF$.

The trial fields involved in the variational inequality (3.3) are expanded similarly

$$(3.6)\qquad \begin{aligned} v'_\alpha &= v'^0_\alpha(x) + \varepsilon v'^1_\alpha(x,y) + \varepsilon^2 v'^2_\alpha(x,y) + \dots,\\ u'_\alpha &= u'^0_\alpha(x) + \varepsilon u'^1_\alpha(x,y) + \varepsilon^2 u'^2_\alpha(x,y) + \dots,\\ w' &= w'^0(x) + \varepsilon w'^1(x,y) + \varepsilon^2 w'^2(x,y) + \dots, \end{aligned}$$

where

$$y = x/\varepsilon \qquad \text{and} \qquad v'^0_\alpha,\ u'^0_\alpha,\ w'^0 \in H_{\Gamma_w}(\Omega);$$
$$v'^1_\alpha(x,.),\ w'^1(x,.) \in H^1_{\text{per}}(Y), \qquad \mathbf{u}'^1(x,.) \in K_{YF}\,.$$

The stress resultants associated with the kinematic fields (3.5) assume the form

$$(3.7)\qquad \begin{aligned} N^{\alpha\beta}_\varepsilon &= N^{\alpha\beta}_0 + O(\varepsilon), \qquad R_\varepsilon = R_0 + O(\varepsilon),\\ L^{\alpha\beta}_\varepsilon &= L^{\alpha\beta}_0 + O(\varepsilon), \qquad Q^\alpha_\varepsilon = Q^\alpha_0 + O(\varepsilon), \end{aligned}$$

where

$$(3.8)\qquad \begin{aligned} N^{\lambda\mu}_0 &= A^{\lambda\mu\alpha\beta}_v \varepsilon^0_{\alpha\beta} + A^{\lambda\mu\alpha\beta}_{vu} \gamma^0_{\alpha\beta} + A^{\lambda\mu}_{vw} w^0,\\ L^{\lambda\mu}_0 &= A^{\lambda\mu\alpha\beta}_{vu} \varepsilon^0_{\alpha\beta} + A^{\lambda\mu\alpha\beta}_u \gamma^0_{\alpha\beta} + A^{\lambda\mu}_{uw} w^0,\\ R_0 &= A^{\alpha\beta}_{vw} \varepsilon^0_{\alpha\beta} + A^{\alpha\beta}_{uw} \gamma^0_{\alpha\beta} + A_w w^0; \end{aligned}$$

$$(3.9)\qquad Q^\alpha_0 = H^{\alpha\beta}\left(\kappa^h_\beta - \frac{\partial w^1}{\partial y_\beta}\right)\Big|_{y=x/\varepsilon}.$$

The deformations are defined by

$$\varepsilon^0_{\alpha\beta} = \varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)|_{y=x/\varepsilon}\,, \qquad \gamma^0_{\alpha\beta} = \gamma^h_{\alpha\beta} + \gamma^y_{\alpha\beta}(\mathbf{u}^1)|_{y=x/\varepsilon}\,,$$

where

$$(3.10)\qquad \varepsilon^h_{\alpha\beta} = \varepsilon_{\alpha\beta}(\mathbf{v}^0), \qquad \gamma^h_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{u}^0), \qquad \kappa^h_\beta = u^0_\beta - \frac{\partial w^0}{\partial x_\beta};$$

$$(3.11)\qquad \varepsilon^{y}_{\alpha\beta}(\mathbf{v}^1)=\frac{1}{2}\left(\frac{\partial v^1_\alpha}{\partial y_\beta}+\frac{\partial v^1_\beta}{\partial y_\alpha}\right),\qquad \gamma^{y}_{\alpha\beta}(\mathbf{u}^1)=\frac{1}{2}\left(\frac{\partial u^1_\alpha}{\partial y_\beta}+\frac{\partial u^1_\beta}{\partial y_\alpha}\right).$$

Quantities (3.10) play the role of homogenized (or averaged by the procedure of smearing-out the cracks) deformation measures. The following simple averages are the stress resultants of the model:

$$(3.12)\qquad N^{\alpha\beta}_h=\langle N^{\alpha\beta}_0\rangle,\qquad L^{\alpha\beta}_h=\langle L^{\alpha\beta}_0\rangle,\qquad R_h=\langle R_0\rangle,\qquad Q^{\alpha}_h=\langle Q^{\alpha}_0\rangle,$$

where

$$(3.13)\qquad \langle\cdot\rangle=\frac{1}{|Y|}\int\limits_{Y_F}(\cdot)\,dy,\qquad |Y|=l_1l_2\,.$$

To find the effective model of the laminate one should substitute expansions (3.5) and (3.6) into variational inequality (3.3) and let ε tend to zero. Following the lines of the derivation elaborated upon by SANCHEZ–PALENCIA [I.47] and LEWIŃSKI and TELEGA [I.29, I.30], one eventually arrives at the homogenized problem and at the local problems the solutions of which intervene in the homogenized stress-resultants (3.12) and the effective elastic potential, see Subsection 3.3.

The homogenized problem assumes the form:

$$(3.14)\qquad (P^1_h)\left|\begin{array}{l}\text{Find }\ (\mathbf{v}^0,\mathbf{u}^0,w^0)\in V\ \text{ such that}\\ a_h(\mathbf{v}^0,\mathbf{u}^0,w^0;\ \mathbf{v}',\mathbf{u}',w')=f(\mathbf{v}',\mathbf{u}',w'),\quad \forall(\mathbf{v}',\mathbf{u}',w')\in V,\end{array}\right.$$

where

$$(3.15)\qquad a_h(\mathbf{v}^0,\mathbf{u}^0,w^0;\ \mathbf{v}',\mathbf{u}',w')=\int\limits_{\Omega}\Big[N^{\alpha\beta}_h(\mathbf{v}^0,\mathbf{u}^0,w^0)\varepsilon_{\alpha\beta}(\mathbf{v}')$$
$$+L^{\alpha\beta}_h(\mathbf{v}^0,\mathbf{u}^0,w^0)\gamma_{\alpha\beta}(\mathbf{u}')+R_h(\mathbf{v}^0,\mathbf{u}^0,w^0)w'+Q^{\alpha}_h(\mathbf{u}^0,w^0)\kappa_\alpha(\mathbf{u}',w')\Big]\,dx.$$

The homogenized stress resultants can be expressed by the formulae

$$(3.16)\qquad \begin{aligned} N^{\lambda\mu}_h&=A^{\lambda\mu\alpha\beta}_{v}\varepsilon^h_{\alpha\beta}+A^{\lambda\mu\alpha\beta}_{vu}\left[\gamma^h_{\alpha\beta}+\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle\right]+A^{\lambda\mu}_{vw}w^h,\\ L^{\lambda\mu}_h&=A^{\lambda\mu\alpha\beta}_{vu}\varepsilon^h_{\alpha\beta}+A^{\lambda\mu\alpha\beta}_{u}\left[\gamma^h_{\alpha\beta}+\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle\right]+A^{\lambda\mu}_{uw}w^h,\\ R_h&=A^{\alpha\beta}_{vw}\varepsilon^h_{\alpha\beta}+A^{\alpha\beta}_{uw}\left[\gamma^h_{\alpha\beta}+\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle\right]+A_w w^h,\qquad Q^{\alpha}_h=H^{\alpha\beta}\kappa^h_\beta\,,\end{aligned}$$

since

$$(3.17)\qquad \langle\varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\rangle=0,\qquad \left\langle\frac{\partial w^1}{\partial y_\beta}\right\rangle=0.$$

We recall that v^1_α and w^1 are continuous on F. The notation $w^h\equiv w^0$ indicates that this field plays the role of a deformation field.

The fields $\mathbf{v}^1$, $\mathbf{u}^1$, w^1 depend upon the homogenized deformations (3.10). This interrelation is provided for by the local or basic cell problem:

$$(3.18)\qquad (P^1_{\rm loc})\quad \left|\begin{array}{l}\text{Find } (\mathbf{v}^1,\mathbf{u}^1,w^1)\in\mathbb{K}_{YF}\ \text{ such that}\\ \langle N_0^{\alpha\beta}\varepsilon^y_{\alpha\beta}(\mathbf{v}')\rangle = 0,\qquad \langle L_0^{\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}'-\mathbf{u}^1)\rangle \ge 0,\\ \left\langle Q_0^\alpha \frac{\partial w'}{\partial y_\alpha}\right\rangle = 0,\quad \forall(\mathbf{v}',\mathbf{u}',w')\in\mathbb{K}_{YF}\,.\end{array}\right.$$

Here $N_0^{\alpha\beta}$, $L_0^{\alpha\beta}$, Q_0^α depend on $\mathbf{v}^1$, $\mathbf{u}^1$, w^1 according to the relations (3.8) and (3.9), the x-dependent fields $\varepsilon^h_{\alpha\beta}$, $\gamma^h_{\alpha\beta}$, w^h being viewed as given and x being treated as a parameter.

The problem $P^1_{\rm loc}$ is *equivalent* to the following minimization problem:

$$(\widetilde{P}^1_{\rm loc})\quad \left|\begin{array}{l}\text{Find}\\ \inf\left\{\dfrac{1}{|Y|}\displaystyle\int_{Y\setminus F} j_1\Big(x,\,\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\mathbf{v}),\gamma^h+\gamma^y(\mathbf{u}),w^h\Big)dy\mid(\mathbf{v},\mathbf{u})\in H^1_{\rm per}(Y)^2\times K_{YF}\right\}\\ \quad+\inf\left\{\dfrac{1}{2|Y|}\displaystyle\int_Y H^{\alpha\beta}(x)\left(\kappa^h_\alpha-\frac{\partial w}{\partial y_\alpha}\right)\left(\kappa^h_\beta-\frac{\partial w}{\partial y_\beta}\right)dy\mid w\in H^1_{\rm per}(Y)\right\},\end{array}\right.$$

where j_1 is defined by (I.3.13). Consequently $(\mathbf{v}^1,\mathbf{u}^1,w^1)\in\mathbb{K}_{YF}$ solves the problem $(\widetilde{P}^1_{\rm loc})$ and *vice versa*. On account of the properties of j_1 and $\mathbf{H}$, the local fields $\mathbf{v}^1$ and $\mathbf{u}^1$ are determined uniquely up to constant vectors while w^1 is unique up to a constant.

3.3. Hyperelastic potential

The elastic potential of the homogenized laminate is given by

$$(3.19)\qquad U_h = \frac{1}{2}\Big\langle N_0^{\alpha\beta}\left[\varepsilon^h_{\alpha\beta}+\varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\right] + L_0^{\alpha\beta}\left[\gamma^h_{\alpha\beta}+\gamma^y_{\alpha\beta}(\mathbf{u}^1)\right] + R_0 w^h + Q_0^\alpha\left(\kappa^h_\alpha-\frac{\partial w^1}{\partial y_\alpha}\right)\Big\rangle = U_1+U_2\,,$$

where

$$(3.20)\qquad U_1(x,\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,w^h)$$

$$= \inf\left\{\frac{1}{|Y|}\int_{Y\setminus F} j_1\left(x,\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\mathbf{v}),\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\mathbf{u}),w^h\right)dy\mid(\mathbf{v},\mathbf{u})\in H^1_{\rm per}(Y)^2\times K_{YF}\right\}$$

$$= \frac{1}{|Y|}\int_{Y\setminus F} j_1\left(x,\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\mathbf{v}^1),\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\mathbf{u}^1),w^h\right)dy,$$

(3.21)
$$U_2(x, \boldsymbol{\kappa}^h) = \frac{1}{2} H^{\alpha\beta}(x) \kappa^h_\alpha \kappa^h_\beta \,,$$

since the simple relation

$$\int_Y \frac{\partial w}{\partial y_\alpha} dy = \int_{\partial Y} w n_\alpha \, ds = 0 \qquad \alpha = 1,\ 2)$$

reduces the second infimum in the problem $(\widetilde{P}^1_{\rm loc})$ to $U_2(x, \boldsymbol{\kappa}^h)$.

Below it will be proved that

(3.22)
$$N^{\alpha\beta}_h = \frac{\partial U_h}{\partial \varepsilon^h_{\alpha\beta}}, \qquad L^{\alpha\beta}_h = \frac{\partial U_h}{\partial \gamma^h_{\alpha\beta}}, \qquad R_h = \frac{\partial U_h}{\partial w^h}, \qquad Q^\alpha_h = \frac{\partial U_h}{\partial \kappa^h_\alpha}.$$

It is worth noting that the expression (3.19) can be reduced to

(3.23)
$$2U_h = N^{\alpha\beta}_h \varepsilon^h_{\alpha\beta} + L^{\alpha\beta}_h \gamma^h_{\alpha\beta} + R_h w^h + Q^\alpha_h \kappa^h_\alpha \,.$$

To prove (3.23) let us insert $\mathbf{v}' = \mathbf{v}^1$ and $w' = w^1$ into $(3.18)_{1,3}$ and then $\mathbf{u}' = 2\mathbf{u}^1$ and once more $\mathbf{u}' = \mathbf{0}$ into $(3.18)_2$. On combining the identities obtained with (3.19), one easily finds (3.23).

Let us pass now to the study of the properties of the effective elastic potential $U_h = U_1 + U_2$.

(i) $U_h(x,.,.,.,.)$ is a strictly convex function provided that F does not separate Y into two (say) disjoint subdomains.

Firstly we demonstrate that $U_h(x,.,.,.,.)$ is *always* convex. Due to $(\mathrm{I}.3.14)_2$, the partial elastic potential $U_2(x,.)$ is *always* strictly convex. Therefore we must investigate the function U_1. Let $\mathbf{E}^{(\alpha)} = (\boldsymbol{\varepsilon}^{(\alpha)}, \boldsymbol{\gamma}^{(\alpha)}, w^{(\alpha)}) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}$, $\alpha = 1,\ 2$ and assume that $(\mathbf{v}^{(\alpha)}, \mathbf{u}^{(\alpha)}) \in H^1_{\rm per}(Y)^2 \times K_{YF}$ solves the following minimization problem:

$$U_1(x, \boldsymbol{\varepsilon}^{(\alpha)}, \boldsymbol{\gamma}^{(\alpha)}, w^{(\alpha)}) = \inf\Bigg\{ \frac{1}{|Y|}$$
$$\int_{Y\setminus F} j_1\left(x, \boldsymbol{\varepsilon}^{(\alpha)} + \boldsymbol{\varepsilon}^y(\mathbf{v}), \boldsymbol{\gamma}^{(\alpha)} + \boldsymbol{\gamma}^y(\mathbf{u}), w^{(\alpha)}\right) dy \mid (\mathbf{v}, \mathbf{u}) \in H^1_{\rm per}(Y)^2 \times K_{YF} \Bigg\}.$$

To prove that $U_1(x,.,.,.)$ is a convex function we calculate

(3.24)
$$U_1\left[x, \frac{1}{2}(\mathbf{E}^{(1)} + \mathbf{E}^{(2)})\right]$$
$$= U_1\left[x, \frac{1}{2}(\boldsymbol{\varepsilon}^{(1)} + \boldsymbol{\varepsilon}^{(2)}), \frac{1}{2}(\boldsymbol{\gamma}^{(1)} + \boldsymbol{\gamma}^{(2)}), \frac{1}{2}(w^{(1)} + w^{(2)})\right]$$

(3.24) [cont.]
$$\leq \frac{1}{|Y|}\int\limits_{Y\setminus F}\Big\{\frac{1}{2}j_1\left(x, \boldsymbol{\varepsilon}^{(1)} + \boldsymbol{\varepsilon}^y(\mathbf{v}^{(1)}), \boldsymbol{\gamma}^{(1)} + \boldsymbol{\gamma}^y(\mathbf{u}^{(1)}), w^{(1)}\right)$$
$$+\frac{1}{2}j_1\left(x, \boldsymbol{\varepsilon}^{(2)} + \boldsymbol{\varepsilon}^y(\mathbf{v}^{(2)}), \boldsymbol{\gamma}^{(2)} + \boldsymbol{\gamma}^y(\mathbf{u}^{(2)}), w^{(2)}\right)\Big\}\,dy$$
$$= \frac{1}{2}U_1(x, \boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\gamma}^{(1)}, w^{(1)}) + \frac{1}{2}U_1(x, \boldsymbol{\varepsilon}^{(2)}, \boldsymbol{\gamma}^{(2)}, w^{(2)})$$
$$= \frac{1}{2}U_1(x, \mathbf{E}^{(1)}) + \frac{1}{2}U_1(x, \mathbf{E}^{(2)}).$$

If F does not intersect Y, like for instance in Fig. 1, then $U_1(x,.,.,.)$ is strictly convex even in the situations such as in Fig. 2.

(ii) $U_h(x,.,.,.,.)$ is of class C^1.

To corroborate this statement it is sufficient to consider the partial effective potential U_1 once again. As we know, U_1 is convex and finite, thus subdifferentiable, cf. ROCKAFELLAR [I.46, Corollary 10.1.1 and Th. 23.4]. The straightforward proof, however, is more instructive also for our further developments. Let the microscopic generalized stresses $\mathbf{N}_0$, $\mathbf{L}_0$ and R_0 be specified by Eq. (3.8) for a prescribed $(\boldsymbol{\varepsilon}^h, \boldsymbol{\gamma}^h, w^h) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}$. For each $(\overline{\boldsymbol{\varepsilon}}, \overline{\boldsymbol{\gamma}}, \overline{w}) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}$ we have

$$\text{(3.25)} \qquad U_1(x, \overline{\boldsymbol{\varepsilon}}, \overline{\boldsymbol{\gamma}}, \overline{w}) - U_1(x, \boldsymbol{\varepsilon}^h, \boldsymbol{\gamma}^h, w^h)$$
$$= \frac{1}{|Y|}\int\limits_{Y\setminus F} j_1\left(x, \overline{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}^y(\mathbf{v}), \overline{\boldsymbol{\gamma}} + \boldsymbol{\gamma}^y(\mathbf{u}), \overline{w}\right) dy$$
$$-\frac{1}{|Y|}\int\limits_{Y\setminus F} j_1\left(x, \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}^1), \boldsymbol{\gamma}^h + \boldsymbol{\gamma}^y(\mathbf{u}^1), w^h\right) dy$$
$$\geq \frac{1}{|Y|}\int\limits_{Y\setminus F}\Big\{N_0^{\alpha\beta}(y)\left[(\overline{\varepsilon}_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\overline{\mathbf{v}}(y))) - (\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1(y)))\right]$$
$$+L_0^{\alpha\beta}(y)\left[(\overline{\gamma}_{\alpha\beta} + \gamma^y_{\alpha\beta}(\overline{\mathbf{u}}(y))) - (\gamma^h_{\alpha\beta} + \gamma^y_{\alpha\beta}(\mathbf{u}^1(y)))\right] + R_0(y)\left(\overline{w} - w^h\right)\Big\}dy$$
$$\geq N_h^{\alpha\beta}\left(\overline{\varepsilon}_{\alpha\beta} - \varepsilon^h_{\alpha\beta}\right) + L_h^{\alpha\beta}\left(\overline{\gamma}_{\alpha\beta} - \gamma^h_{\alpha\beta}\right) + R_h\left(\overline{w} - w^h\right).$$

Here (3.12) and (3.18) as well as the subdifferentiability of the function $j_1(x,.,.,.)$ have been taken into account. The subgradient $(\mathbf{N}_h, \mathbf{L}_h, R_h) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}$ is unique, hence $U_1(x,.,.,.)$ and consequently $U_h(x,.,.,.,.)$ are of class C^1. Consequently, relations (3.22) follow.

(iii) There exists a constant $C_1 > 0$ such that

$$\text{(3.26)} \qquad U_h(x, \boldsymbol{\varepsilon}^h, \boldsymbol{\gamma}^h, \boldsymbol{\kappa}^h, w^h) \leq C_1(|\boldsymbol{\varepsilon}^h|^2 + |\boldsymbol{\gamma}^h|^2 + |\boldsymbol{\kappa}^h|^2 + |w^h|^2),$$

for a.e. $x \in \Omega$ and all $(\boldsymbol{\varepsilon}^h, \boldsymbol{\gamma}^h, \boldsymbol{\kappa}^h, w^h) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}^2 \times \mathbb{R}$.

The proof is straightforward. □

We notice that the property (3.26) holds irrespective of the form of F, the case presented in Fig. 3 being obviously included.

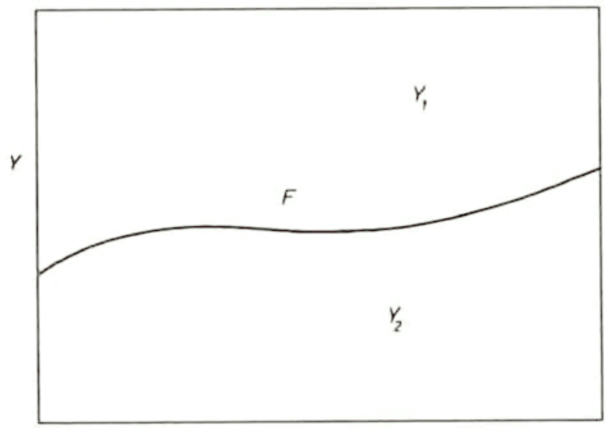


FIG. 3. Fissure dividing Y into two disjoint subdomains Y_1 and Y_2.

(iv) There exists a constant $C_0 > 0$ such that

$$U_h(x, \boldsymbol{\varepsilon}^h, \boldsymbol{\gamma}^h, \boldsymbol{\kappa}^h, w^h) \geq C_0(|\boldsymbol{\varepsilon}^h|^2 + |\boldsymbol{\gamma}^h|^2 + |\boldsymbol{\kappa}^h|^2 + |w^h|^2), \tag{3.27}$$

for a.e. $x \in \Omega$ and all $(\boldsymbol{\varepsilon}^h, \boldsymbol{\gamma}^h, \boldsymbol{\kappa}^h, w^h) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}^2 \times \mathbb{R}$.

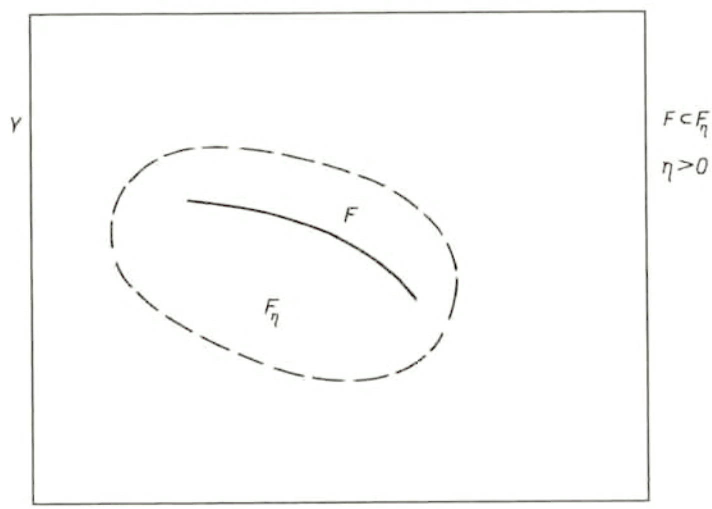


FIG. 4.

The coercivity condition (3.27) is valid only for $F \subset Y$ (Fig. 1, Fig. 4) and for situations like in Fig. 2. We assume that: a) $Y \backslash F_\eta$ has a Lipschitzian boundary, $F \subset F_\eta$, b) $|F_\eta| \to 0$ as $\eta \to 0$.

Now we have

$$
\begin{aligned}
(3.28)\qquad & U_h(x,\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,\boldsymbol{\kappa}^h,w^h) \\
& \geq \inf\left\{ \frac{1}{|Y|}\int\limits_{Y\setminus F} j_1\left(x,\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\mathbf{v}),\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\mathbf{u}),w^h\right)dy \right.\\
& \qquad \left. \mid (\mathbf{v},\mathbf{u})\in H^1_{\mathrm{per}}(Y)^2\times H^1_{\mathrm{per}}(Y F)^2 \right\} + C|\boldsymbol{\kappa}^h|^2 \\
& \geq \inf\left\{ \frac{1}{|Y|}\int\limits_{Y\setminus F_\eta} j_1\left(x,\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\mathbf{v}),\gamma^h+\gamma^y(\mathbf{u}),w^h\right)dy \right.\\
& \qquad \left. \mid (\mathbf{v},\mathbf{u})\in H^1_{\mathrm{per}}(Y)^2\times H^1_{\mathrm{per}}(Y)^2 \right\} + C|\boldsymbol{\kappa}^h|^2 \\
& \geq \frac{C_2}{|Y|}\int\limits_{Y\setminus F_\eta}\left(|\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\widetilde{\mathbf{v}})|^2+|\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\widetilde{\mathbf{u}})|^2\right)dy + C_3(|\boldsymbol{\kappa}^h|^2+|w^h|^2).
\end{aligned}
$$

Here $(\widetilde{\mathbf{v}},\widetilde{\mathbf{u}})\in H^1_{\mathrm{per}}(Y)^2\times H^1_{\mathrm{per}}(Y)^2$ is a minimizer.

We have

$$
\begin{aligned}
(3.29)\qquad \int\limits_{Y\setminus F_\eta}(|\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\widetilde{\mathbf{u}})|^2dy &\geq |Y\setminus F_\eta||\boldsymbol{\gamma}^h|^2+2\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{Y\setminus F_\eta}\gamma^y_{\alpha\beta}(\widetilde{\mathbf{u}})dy \\
&\geq |Y\setminus F_\eta||\boldsymbol{\gamma}^h|^2+2\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{Y}\gamma^y_{\alpha\beta}(\widetilde{\mathbf{u}})dy-2\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{F_\eta}\gamma^y_{\alpha\beta}(\widetilde{\mathbf{u}})dy \\
&= |Y\setminus F_\eta||\boldsymbol{\gamma}^h|^2-2\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{F_\eta}\gamma^y_{\alpha\beta}(\widetilde{\mathbf{u}})dy \\
&\geq |Y\setminus F_{\eta_0}||\boldsymbol{\gamma}^h|^2-2\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{F_\eta}\gamma^y_{\alpha\beta}(\widetilde{\mathbf{u}})dy,\qquad 0<\eta<\eta_0,\quad \eta_0\ -\ \text{fixed.}
\end{aligned}
$$

Let us examine the last term. One has

$$
(3.30)\qquad -\sum_{\alpha,\beta}\gamma^h_{\alpha\beta}\int\limits_{F_\eta}\gamma^y_{\alpha\beta}(\widetilde{\mathbf{u}})dy \geq -|F_\eta||\boldsymbol{\gamma}^h|\|\boldsymbol{\gamma}^y(\widetilde{\mathbf{u}})\|_{0,Y}\to 0 \qquad \text{as}\quad \eta\to 0.
$$

In the last inequality $\widetilde{\mathbf{u}}\in H^1_{\mathrm{per}}(Y)^2$ and $|F_\eta|\to 0$ as $\eta\to 0$. Thus we arrive at the condition (3.27).

When F intersects Y (cf. Fig. 3), the following estimate can only be obtained

$$
\begin{aligned}
(3.31)\qquad U_h(x,\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,\boldsymbol{\kappa}^h,w^h) &\geq C\left(|\boldsymbol{\varepsilon}^h|^2+|\boldsymbol{\kappa}^h|^2+|w^h|^2\right) \\
&\quad +\frac{C}{|Y|}\int\limits_{Y_1\cup Y_2}|\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\mathbf{u}^1)|^2\,dy,
\end{aligned}
$$

where C is a positive constant and $\mathbf{u}^1$ enters the solution of the local problem. The last term in the inequality (3.31) can be estimated from below by zero. Then the effective potential U_h is not coercive on the space $\mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}^2 \times \mathbb{R}$. However, the potential U_h may be coercive in a restricted sense, namely on $\mathbb{E}^2_s \times S \times \mathbb{R}^2 \times \mathbb{R}$, where S is a subset of $\mathbb{E}^2_s$. S depends on the solution of the local problem and particularly on F.

REMARK 3.2. Under the condition (3.27), the homogenized problem (P^1_h) admits a unique solution provided that the length of Γ_w is positive.

3.4. Strong formulation of the local problem

Prior to finding a strong form of (P^1_{loc}) we rearrange its variational formulation. As we already know, w^1 does not depend on y, viz. $w^1 = w^1(x)$. Let us decompose the stress resultants (3.8) and (3.9) as follows:

$$N_0^{\alpha\beta} = n^{\alpha\beta} + n_0^{\alpha\beta} \qquad L_0^{\alpha\beta} = l^{\alpha\beta} + l_0^{\alpha\beta}, \tag{3.32}$$
$$R_0 = r + r_0, \qquad Q_0^{\alpha} = q^{\alpha} + q_0^{\alpha},$$

where

$$n^{\lambda\mu} = A_v^{\lambda\mu\alpha\beta}\varepsilon^y_{\alpha\beta}(\mathbf{v}^1) + A_{vu}^{\lambda\mu\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}^1), \tag{3.33}$$
$$l^{\lambda\mu} = A_{vu}^{\lambda\mu\alpha\beta}\varepsilon^y_{\alpha\beta}(\mathbf{v}^1) + A_u^{\lambda\mu\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}^1),$$

$$q^{\lambda} = 0 \tag{3.34}$$

and

$$n_0^{\lambda\mu} = A_v^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta} + A_{vu}^{\lambda\mu\alpha\beta}\gamma^h_{\alpha\beta} + A_{vw}^{\lambda\mu}w^h,$$
$$l_0^{\lambda\mu} = A_{vu}^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta} + A_u^{\lambda\mu\alpha\beta}\gamma^h_{\alpha\beta} + A_{uw}^{\lambda\mu}w^h, \tag{3.35}$$
$$q^{\lambda} = H^{\lambda\mu}\kappa^h_{\mu}.$$

The local $(\overline{P}^1_{\text{loc}})$ problem reads:

Find $\mathbf{v}^1 \in H^1_{\text{per}}(Y)^2$ and $\mathbf{u}^1 \in K_{YF}$ such that

$$\left\langle n^{\alpha\beta}\frac{\partial v'_\alpha}{\partial y_\beta}\right\rangle = -\left\langle n_0^{\alpha\beta}\frac{\partial v'_\alpha}{\partial y_\beta}\right\rangle, \qquad \forall\, \mathbf{v}' \in H^1_{\text{per}}(Y)^2, \tag{3.36}$$

$$\left\langle l^{\alpha\beta}\frac{\partial(u'_\alpha - u^1_\alpha)}{\partial y_\beta}\right\rangle \geq -\left\langle l_0^{\alpha\beta}\frac{\partial(u'_\alpha - u^1_\alpha)}{\partial y_\beta}\right\rangle, \qquad \forall\, \mathbf{u}' \in K_{YF}. \tag{3.37}$$

Localization of $(\overline{P}^1_{\text{loc}})$ leads to the strong formulation: Find $\mathbf{v}^1$ and $\mathbf{u}^1$ such that:

(i) the following local equilibrium equations hold in YF:

$$A_v^{\alpha\beta\lambda\mu}\frac{\partial^2 v_\lambda^1}{\partial y_\beta \partial y_\mu} + A_{vu}^{\alpha\beta\lambda\mu}\frac{\partial^2 u_\lambda^1}{\partial y_\beta \partial y_\mu} = 0, \tag{3.38}$$

$$A_{vu}^{\alpha\beta\lambda\mu}\frac{\partial^2 v_\lambda^1}{\partial y_\beta \partial y_\mu} + A_u^{\alpha\beta\lambda\mu}\frac{\partial^2 u_\lambda^1}{\partial y_\beta \partial y_\mu} = 0. \tag{3.39}$$

(ii) $\mathbf{u}^1$ and $\mathbf{v}^1$ assume equal values at the opposite sides of YF.

(iii) $N_0^{\alpha\beta} n_\beta$, $L_0^{\alpha\beta} n_\beta$ assume opposite values at the opposite sides of Y; here $\mathbf{n} = (n_\alpha)$ represents a unit vector, outward normal to ∂Y.

(3.40) (iv) $\overset{1}{L}{}_N^0 = \overset{2}{L}{}_N^0 = L_N^0 \le 0, \qquad \overset{\sigma}{L}{}_N^0 = L_{0|\sigma}^{\alpha\beta} N_\alpha N_\beta\,,$

$L_N^0 [\![u_N^1]\!] = 0, \qquad [\![u_N^1]\!] \ge 0.$

(3.41) (v) $\overset{1}{L}{}_T^0 = \overset{2}{L}{}_T^0 = 0, \qquad \overset{\sigma}{L}{}_T^0 = L_{0|\sigma}^{\alpha\beta} N_\alpha T_\beta\,.$

(vi) $\mathbf{v}^1$ is continuous on Y.

(3.42) (vii) $N_{0|1}^{\alpha\beta} N_\alpha N_\beta = N_{0|2}^{\alpha\beta} N_\alpha N_\beta\,, \qquad N_{0|1}^{\alpha\beta} N_\alpha T_\beta = N_{0|2}^{\alpha\beta} N_\alpha T_\beta\,.$

3.5. Insensitivity to the l_α / h ratios

One can show that the solution to the problem $(P_{\rm loc}^1)$ is not sensitive to ratios $\varrho_\alpha = l_\alpha / 2h$ (crack spacing/laminate thickness), hence is not sensitive to the transverse shape of the periodicity cell. This property follows from the fact that all equations and inequalities of $(P_{\rm loc}^1)$ involve derivatives of the same order, hence this system is free from length scales. Consequently, the effective potential U_h and formulae for the effective stiffnesses do not depend upon ϱ_α. They apply indeed to the case when ϱ_α are very small, which is a consequence of the in-plane scaling $Z_\alpha \to \varepsilon l_\alpha$, $h \to h$ (an arrow means replacement).

4. Thin laminate with transverse cracks of high density Model (h_0, l_0)

On the basis of the results of the previous section, a model will be derived suitable for describing densely distributed transverse cracks in the internal layer of very thin laminates. Notation (h_0, l_0) means that $2h \ll \operatorname{diam} \Omega$ and $l_\alpha \ll 2h$.

In the conventional engineering analysis of statics of thin laminates, the longitudinal displacements w_α are usually treated as uniform through the thickness and the influence of the stress σ^{33} is neglected. Such an approximation, justifiable for thin laminates, corresponds to the following assumptions

$$u_\alpha^0 = 0, \qquad R_0 = 0. \tag{4.1}$$

Note that only at the macrolevel the graph $w_\alpha(z)$ is assumed to be uniform; it is not stipulated that $u^1_\alpha = 0$.

Let us substitute $\mathbf{u}' = \mathbf{0}$ and $w' = 0$ into Eq. (3.14). Then we arrive at the variational equation

$$(4.2)\qquad \int_\Omega N_h^{\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}')dx = \int_{\Gamma_\sigma} \overline{N}^\alpha v'_\alpha\, ds, \qquad \forall\, \mathbf{v}' \in H_{\Gamma_w}(\Omega)^2.$$

The field w^h involved in the constitutive relation

$$(4.3)\qquad N_h^{\lambda\mu} = A_v^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta} + A_{vu}^{\lambda\mu\alpha\beta}\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle + A^{\lambda\mu}_{vw}w^h,$$

can be eliminated by means of the equation

$$(4.4)\qquad R_h = \langle R_0\rangle = 0, \qquad A^{\alpha\beta}_{vw}\varepsilon^h_{\alpha\beta} + A^{\alpha\beta}_{uw}\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle + A_w w^h = 0,$$

to obtain the homogenized constitutive relations in the form of KACHANOV [6], cf. also SAYERS and KACHANOV [13], KACHANOV [7] and HORII and SAHASAKMONTRI [5].

$$(4.5)\qquad N_h^{\lambda\mu} = A_1^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta} - A_2^{\lambda\mu\alpha\beta}\varepsilon^F_{\alpha\beta}.$$

"Crack deformation measures" $\varepsilon^F_{\alpha\beta}$ are defined by

$$(4.6)\qquad \varepsilon^F_{\alpha\beta} = -\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle,$$

or, equivalently

$$(4.7)\qquad \varepsilon^F_{\alpha\beta} = \frac{1}{2|Y|}\int_F \left([\![u^1_\alpha]\!]N_\beta + [\![u^1_\beta]\!]N_\alpha\right) ds.$$

We can also write

$$(4.8)\qquad \varepsilon^F_{\alpha\beta} = \frac{1}{|Y|}\int_F \left[[\![u^1_N]\!]N_\alpha N_\beta + \frac{1}{2}[\![u^1_T]\!](T_\alpha N_\beta + T_\beta N_\alpha)\right] ds,$$

since

$$u^1_\alpha = u^1_N N_\alpha + u^1_T T_\alpha, \qquad u^1_N = u^1_1 N_1 + u^1_2 N_2 \qquad \text{and} \qquad u^1_T = u^1_1 T_1 + u^1_2 T_2.$$

The stiffnesses involved in Eq. (4.5) are given by

$$(4.9)\qquad A_1^{\alpha\beta\lambda\mu} = A_v^{\alpha\beta\lambda\mu} - A^{\alpha\beta}_{vw}A^{\lambda\mu}_{vw}(A_w)^{-1}, \qquad A_2^{\alpha\beta\lambda\mu} = A_{vu}^{\alpha\beta\lambda\mu} - A^{\alpha\beta}_{vw}A^{\lambda\mu}_{uw}(A_w)^{-1}.$$

The stiffnesses $A_1^{\alpha\beta\lambda\mu}$ characterize effective properties of the uncracked laminate, the moduli $A_2^{\alpha\beta\lambda\mu}$ represent damage moduli of the cracked laminate.

To be consistent, we should appropriately reformulate the local problem (P^1_{loc}). We put $R_0 = 0$, where R_0 is given by Eq. $(3.8)_3$, and $\gamma^h_{\alpha\beta} = 0$ according to $(4.1)_1$. Hence

$$A^{\alpha\beta}_{vw}\left[\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\right] + A^{\alpha\beta}_{uw}\gamma^y_{\alpha\beta}(\mathbf{u}^1) + A_w w^h = 0. \tag{4.10}$$

Therefore, according to (3.8), one finds

$$\begin{aligned} N_0^{\lambda\mu} &= A_1^{\lambda\mu\alpha\beta}\left[\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\right] + A_2^{\lambda\mu\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}^1), \\ L_0^{\lambda\mu} &= A_3^{\lambda\mu\alpha\beta}\left[\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\right] + A_4^{\lambda\mu\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}^1), \end{aligned} \tag{4.11}$$

where

$$A_3^{\lambda\mu\alpha\beta} = A_2^{\alpha\beta\lambda\mu}, \qquad A_4^{\alpha\beta\lambda\mu} = A_u^{\alpha\beta\lambda\mu} - A^{\alpha\beta}_{uw}A^{\lambda\mu}_{uw}(A_w)^{-1}. \tag{4.12}$$

Thus the local problem assumes the form of the problem $(\overline{P}^1_{\text{loc}})$ in which $n^{\lambda\mu}$ and $l^{\lambda\mu}$ are interpreted in the following way:

$$\begin{aligned} n^{\lambda\mu} &= A_1^{\lambda\mu\alpha\beta}\varepsilon^y_{\alpha\beta}(\mathbf{v}^1) + A_2^{\lambda\mu\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}^1), \\ l^{\lambda\mu} &= A_3^{\lambda\mu\alpha\beta}\varepsilon^y_{\alpha\beta}(\mathbf{v}^1) + A_4^{\lambda\mu\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}^1), \end{aligned} \tag{4.13}$$

while $n_0^{\lambda\mu}$ and $l_0^{\lambda\mu}$ assume the form

$$n_0^{\lambda\mu} = A_1^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta}, \qquad l_0^{\lambda\mu} = A_3^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta}. \tag{4.14}$$

The new local problem thus obtained will be referred to as (P^0_{loc}). It is *equivalent* to the following minimization problem:

PROBLEM $\widetilde{P}^0_{\text{loc}}$

$$\left| \begin{aligned} &\text{For } \boldsymbol{\varepsilon}^h \in \mathbb{E}_s \text{ find} \\ &\mathcal{V}_h(x, \boldsymbol{\varepsilon}^h) := \inf\left\{ \frac{1}{2|Y|}\int_{Y\setminus F}\left[\boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}), \boldsymbol{\gamma}^y(\mathbf{u})\right]\mathfrak{A}_1\left[\boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}), \boldsymbol{\gamma}^y(\mathbf{u})\right]^T dy \right. \\ &\qquad\qquad \left. \mid\ (\mathbf{v}, \mathbf{u}) \in H^1_{\text{per}}(Y)^2 \times K_{YF} \right\}, \end{aligned} \right. \tag{4.15}$$

provided that the matrix

$$\mathfrak{A}_1 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} = \mathfrak{A}_1^T \tag{4.16}$$

is positive. Then the problem $(\widetilde{P}^0_{\mathrm{loc}})$ is a convex one, since $\mathbf{A}_3 = \mathbf{A}_2^T$. To assess the solvability of $(\widetilde{P}^0_{\mathrm{loc}})$, we assume that $\mathfrak{A}_1$ is positive definite, i.e.

$$\left| \begin{array}{l} \exists \ C > 0 \ \text{ such that for a.e. } \ x \in \Omega \ \text{ and for all } \ \mathbf{e}_1, \ \mathbf{e}_2 \in \mathbb{E}^2_s \\ [\mathbf{e}_1, \mathbf{e}_2]\mathfrak{A}_1(x)[\mathbf{e}_1, \mathbf{e}_2]^T \geq C(|\mathbf{e}_1|^2 + |\mathbf{e}_2|^2). \end{array} \right. \tag{4.17}$$

For

$$(\mathbf{v}, \mathbf{u}) \in \widetilde{H}^1_{\mathrm{per}}(Y)^2 \times \widetilde{K}_{YF} = \left\{ (\mathbf{v}, \mathbf{u}) \in H^1_{\mathrm{per}}(Y)^2 \times K_{YF} \mid \int_Y \mathbf{v}(y)\, dy = 0, \quad \int_{Y \setminus F} \mathbf{u}(y)\, dy = 0 \right\},$$

the minimization problem in (4.15) is coercive, provided that F is regular and does not separate Y into two disjoint subdomains like in Fig. 3. As previously, the unique solution is denoted by $(\mathbf{v}^1, \mathbf{u}^1)$. It is determined up to a constant vector as an element of $H^1_{\mathrm{per}}(Y)^2 \times K_{YF}$.

The homogenized constitutive relationship remains hyperelastic:

$$N_h^{\alpha\beta} = \frac{\partial \mathcal{V}_h}{\partial \varepsilon^h_{\alpha\beta}}, \tag{4.18}$$

where the elastic effective potential has the form

$$\mathcal{V}_h = \langle N_0^{\alpha\beta}[\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)] + L_0^{\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle / 2. \tag{4.19}$$

Now let us examine the effective potential $\mathcal{V}_h$. Its properties can be verified similarly as those of U_h, cf. Subsec. 3.3 of the present paper. Therefore we will only summarize the main properties, assuming that condition (4.17) is satisfied:

(i) $\mathcal{V}_h(x, .)$ is a strictly convex function of class C^1 in the space $\mathbb{E}^2_s$.

(ii) There exist constants $C_1 > C_0 > 0$ such that for a.e. $x \in \Omega$

$$C_0|\mathbf{e}|^2 \leq \mathcal{V}_h(x, \mathbf{e}) \leq C_1|\mathbf{e}|^2 \qquad \forall \ \mathbf{e} \in \mathbb{E}^2_s. \tag{4.20}$$

The right-hand side inequality in (4.20) is obvious since each element of the matrix $\mathfrak{A}_1$ belongs to $L^\infty(\Omega)$. The left-hand one also readily follows

$$\begin{aligned} \mathcal{V}_h(x, \mathbf{e}) &\geq \frac{C}{2|Y|} \int_Y |\mathbf{e} + \boldsymbol{\varepsilon}^y(\mathbf{v}^1)|^2 dy + \frac{C}{2|Y|} \int_{Y \setminus F} |\boldsymbol{\gamma}^y(\mathbf{u}^1)|^2 dy \\ &\geq \frac{C}{2|Y|} \int_Y |\mathbf{e} + \boldsymbol{\varepsilon}^y(\mathbf{v}^1)|^2 dy \geq \frac{C}{2|Y|}|\mathbf{e}|^2 = C_0|\mathbf{e}|^2. \end{aligned} \tag{4.21}$$

It is worth noting that on account of the assumption $\boldsymbol{\gamma}^h = \mathbf{0}$, these two properties of the potential $\mathcal{V}_h$ are satisfied also when F divides Y into two disjoint subdomains. Macroscopic constitutive relation (4.18) is a by-product of (i) while (4.19) is implied by (4.15) and (4.11). Variational equation (4.2) along with (4.15) constitutes a new homogenized problem (P_h^0). Due to the property (ii), the last problem admits a unique solution.

Let us substitute $v'_\alpha = v^1_\alpha$ into Eq. (3.36), $u'_\alpha = 2u^1_\alpha$ and then $u'_\alpha = 0$ into inequality (3.37). On combining these relations with (4.19) one can reduce the expression for $\mathcal{V}_h$ to the form

$$\mathcal{V}_h = N_h^{\alpha\beta}(\boldsymbol{\varepsilon}^h)\varepsilon^h_{\alpha\beta}/2, \tag{4.22}$$

consistent with (3.23) and the simplifications assumed. Let us observe that the assumptions (4.1) are, in general, contradictory. They imply equality $w^0 = 0$ which cannot be satisfied simultaneously with equation $R_h = 0$. Such internal contradictions are inevitable in constructing engineering models of thin plates.

5. Thin laminate weakened by transverse cracks of arbitrary density Model (h_0, l)

In this section we derive formulae for assessing stiffness loss of a thin three-layer laminate with transverse cracks in the internal layer. No limitations concerning crack spacing will be imposed. The abbreviation (h_0, l) means that $h \ll \operatorname{diam}\Omega$ and l_α are arbitrary.

The homogenization process will be based upon a scaling according to which all characteristic length scales of the model of the laminate are viewed as small parameters. These length scales represent both the transverse and longitudinal dimensions of the periodicity cell of the original laminate considered as a three-dimensional structure.

5.1. Family of ε-problems

The following quantities

$$c,\ d,\ h,\ b,\ Z_1,\ Z_2 \tag{5.1}$$

are internal length scales of the model of Sec. 2. We shall assume that all these parameters depend upon a small parameter ε. In this context the following replacement is natural

$$c \to \varepsilon c, \quad d \to \varepsilon d, \quad h \to \varepsilon h, \quad b \to \varepsilon b, \quad Z_\alpha \to \varepsilon l_\alpha, \quad F \to \varepsilon F. \tag{5.2}$$

If ε tends to zero, the thickness of the laminate also diminishes to zero. To compensate for this degeneracy we scale the loading

$$\overline{N}^\beta \to \varepsilon \overline{N}^\beta, \qquad \overline{L}^\beta \to \varepsilon L^\beta, \qquad \overline{Q} \to \overline{Q}. \tag{5.3}$$

Instead of scaling the loading we could scale the elastic moduli (cf. CIARLET [2], CAILLERIE [I.7]), which seems to be even more esoteric.

The length scales scaling (5.2) implies the following scaling of the stiffnesses involved in the constitutive relationships (I.2.24) and (I.2.25)

$$
(5.4)\qquad
\begin{gathered}
\left(A_v^{\alpha\beta\lambda\mu},\ A_{vu}^{\alpha\beta\lambda\mu},\ A_u^{\alpha\beta\lambda\mu}\right) \to \left(\varepsilon A_v^{\alpha\beta\lambda\mu},\ \varepsilon A_{vu}^{\alpha\beta\lambda\mu},\ \varepsilon A_u^{\alpha\beta\lambda\mu}\right),\\
\left(A_{vw}^{\alpha\beta},\ A_{uw}^{\alpha\beta}\right) \to \left(\frac{1}{\varepsilon}A_{vw}^{\alpha\beta},\ \frac{1}{\varepsilon}A_{uw}^{\alpha\beta}\right),\qquad H^{\alpha\beta} \to \frac{1}{\varepsilon}H^{\alpha\beta},\qquad A_w \to \frac{1}{\varepsilon^3}A_w\,.
\end{gathered}
$$

Scaling (5.2) concerns all dimensions of the three-dimensional periodicity cell. That is why this scaling will be referred to as the space scaling, although the problem itself is posed as a two-dimensional one.

Let us re-define the bilinear form of the problem consistently with the scaling (5.4) of the stiffnesses:

$$
(5.5)\qquad
\begin{aligned}
&b_{\Omega^\varepsilon}\left(\mathbf{v}^\varepsilon,\mathbf{u}^\varepsilon,w^\varepsilon;\mathbf{v}',\mathbf{u}',w'\right)\\
&\quad= \int_{\Omega^\varepsilon}\Big[N_\varepsilon^{\alpha\beta}\left(\mathbf{v}^\varepsilon,\mathbf{u}^\varepsilon,w^\varepsilon\right)\varepsilon_{\alpha\beta}(\mathbf{v}') + L_\varepsilon^{\alpha\beta}\left(\mathbf{v}^\varepsilon,\mathbf{u}^\varepsilon,w^\varepsilon\right)\gamma_{\alpha\beta}(\mathbf{u}')\\
&\qquad + Q_\varepsilon^{\alpha}\left(\mathbf{u}^\varepsilon,w^\varepsilon\right)\kappa_\alpha(\mathbf{u}',w') + R_\varepsilon\left(\mathbf{v}^\varepsilon,\mathbf{u}^\varepsilon,w^\varepsilon\right)w'\Big]\,dx.
\end{aligned}
$$

The constitutive relations become

$$
(5.6)\qquad
\begin{aligned}
N_\varepsilon^{\lambda\mu} &= \varepsilon A_v^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon) + \varepsilon A_{vu}^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}^\varepsilon) + \frac{1}{\varepsilon}A_{vw}^{\lambda\mu}w^\varepsilon,\\
L_\varepsilon^{\lambda\mu} &= \varepsilon A_{vu}^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon) + \varepsilon A_u^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}^\varepsilon) + \frac{1}{\varepsilon}A_{uw}^{\lambda\mu}w^\varepsilon,\\
R_\varepsilon &= \frac{1}{\varepsilon}A_{vw}^{\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon) + \frac{1}{\varepsilon}A_{uw}^{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}^\varepsilon) + \frac{1}{\varepsilon^3}A_w w^\varepsilon,
\end{aligned}
$$

$$
(5.7)\qquad Q_\varepsilon^{\alpha} = \frac{1}{\varepsilon}H^{\alpha\beta}\kappa_\beta(\mathbf{u}^\varepsilon,w^\varepsilon).
$$

The equilibrium problem reads:

$$
(5.8)\qquad (P^2)_{\omega^\varepsilon}\quad
\left|\quad
\begin{aligned}
&\text{Find}\ \ (\mathbf{v}^\varepsilon,\mathbf{u}^\varepsilon,w^\varepsilon)\in\mathbb{K}(\Omega^\varepsilon)\ \ \text{such that}\\
&b_{\Omega^\varepsilon}(\mathbf{v}^\varepsilon,\mathbf{u}^\varepsilon,w^\varepsilon;\ \mathbf{v}',\mathbf{u}'-\mathbf{u}^\varepsilon,w') \ge g^\varepsilon(\mathbf{v}',\mathbf{u}'-\mathbf{u}^\varepsilon,w')\\
&\qquad\qquad\qquad\qquad\forall\ (\mathbf{v}',\mathbf{u}',w')\in\mathbb{K}(\Omega^\varepsilon),
\end{aligned}
\right.
$$

where

$$
(5.9)\qquad g^\varepsilon(\mathbf{v}',\mathbf{u}',w') = \int_{\Gamma_\sigma}\left(\varepsilon\overline{N}^\alpha v'_\alpha + \varepsilon\overline{L}^\alpha u'_\alpha - \overline{Q}w'\right)ds.
$$

The solution to problem $(P^2_{\Omega^\varepsilon})$ has been denoted similarly to the solution of the problem $(P^1_{\Omega^\varepsilon})$. This ambiguity should not lead to any misunderstanding. For a fixed $\varepsilon > 0$ the problem $(P^2_{\Omega^\varepsilon})$ admits a unique solution $(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon) \in \mathbb{K}(\Omega^\varepsilon)$. Due to the presence of singular terms this result is not straightforward. Details are given in our separate paper [14].

5.2. Asymptotic solution

In this section a formal asymptotic procedure will be applied to find the main terms of the solution to the problem $(P^2_{\Omega^\varepsilon})$. A rigorous justification of this method via the theory of epi-convergence and dual homogenization is addressed to in the paper by TELEGA and LEWIŃSKI [14].

The solution to the problem $(P^2_{\Omega^\varepsilon})$ will be looked for in the following form:

$$v^\varepsilon_\alpha = v^0_\alpha(x) + \varepsilon v^1_\alpha(x,y) + \varepsilon^2 v^2_\alpha(x,y) + \ldots, \tag{5.10}$$

$$u^\varepsilon_\alpha = \varepsilon u^1_\alpha(x,y) + \varepsilon^2 u^2_\alpha(x,y) + \ldots, \tag{5.11}$$

$$w^\varepsilon = \varepsilon^2 w^2(x,y) + \varepsilon^3 w^3(x,y) + \ldots, \qquad y = x/\varepsilon. \tag{5.12}$$

The trial fields are expanded similarly

$$v'_\alpha = v'^0_\alpha(x) + \varepsilon v'^1_\alpha(x,y) + \varepsilon^2 v'^2_\alpha(x,y) + \ldots, \tag{5.13}$$

$$u'_\alpha = \varepsilon u'^1_\alpha(x,y) + \varepsilon^2 u'^2_\alpha(x,y) + \ldots, \tag{5.14}$$

$$w' = \varepsilon^2 w'^2(x,y) + \varepsilon^3 w'^3(x,y) + \ldots, \qquad y = x/\varepsilon. \tag{5.15}$$

It is assumed that

$$\begin{aligned} &v^0_\alpha,\ v'^0_\alpha \in H_{\Gamma_w}(\Omega),\\ &v^1_\alpha(x,.),\ v'^1_\alpha(x,.),\ w^2(x,.),\ w'^2x,.) \in H^1_{\mathrm{per}}(Y),\\ &\mathbf{u}^1(x,.),\ \mathbf{u}'^1(x,.) \in K_{YF}. \end{aligned} \tag{5.16}$$

The deformation measures associated with the kinematic fields (5.10)–(5.12) are

$$\begin{aligned} \varepsilon_{\alpha\beta}(\mathbf{v}^\varepsilon) &= \varepsilon_{\alpha\beta}(\mathbf{v}^0) + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)|_{y=x/\varepsilon} + 0(\varepsilon),\\ \gamma_{\alpha\beta}(\mathbf{u}^\varepsilon) &= \gamma^y_{\alpha\beta}(\mathbf{u}^1)|_{y=x/\varepsilon} + 0(\varepsilon),\\ \kappa_\alpha(\mathbf{u}^\varepsilon, w^\varepsilon) &= \varepsilon\kappa^y_\alpha(\mathbf{u}^1, w^2)|_{y=x/\varepsilon} + 0(\varepsilon^2), \end{aligned} \tag{5.17}$$

where

$$\kappa^y(\mathbf{u}^1, w^2) = u^1_\beta - \frac{\partial w^2}{\partial y_\beta}. \tag{5.18}$$

The remaining symbols have been already introduced in Sec. 3.2.

The stress resultants associated with the kinematics (5.10)–(5.12) are

$$
\begin{aligned}
N_\varepsilon^{\alpha\beta} &= \varepsilon N_0^{\alpha\beta} + 0(\varepsilon^2), \qquad & L_\varepsilon^{\alpha\beta} &= \varepsilon L_0^{\alpha\beta} + 0(\varepsilon^2),\\
Q_\varepsilon^{\alpha} &= Q_0^{\alpha} + 0(\varepsilon), & R_\varepsilon &= \frac{1}{\varepsilon} R_0 + 0(1),
\end{aligned} \tag{5.19}
$$

where the rescaled stress resultants assume the form

$$
\begin{aligned}
N_0^{\lambda\mu} &= A_v^{\lambda\mu\alpha\beta}\left[\varepsilon_{\alpha\beta}^h + \varepsilon_{\alpha\beta}^y(\mathbf{v}^1)\right] + A_{vu}^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}^1) + A_{vw}^{\lambda\mu} w^2,\\
L_0^{\lambda\mu} &= A_{vu}^{\lambda\mu\alpha\beta}\left[\varepsilon_{\alpha\beta}^h + \varepsilon_{\alpha\beta}^y(\mathbf{v}^1)\right] + A_{u}^{\lambda\mu\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}^1) + A_{uw}^{\lambda\mu} w^2,\\
R_0 &= A_{vw}^{\alpha\beta}\left[\varepsilon_{\alpha\beta}^h + \varepsilon_{\alpha\beta}^y(\mathbf{v}^1)\right] + A_{uw}^{\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}^1) + A_w w^2,
\end{aligned} \tag{5.20}
$$

$$
Q_0^\alpha = H^{\alpha\beta}\kappa_\beta^y(\mathbf{u}^1, w^2), \tag{5.21}
$$

and $\varepsilon_{\alpha\beta}^h = \varepsilon_{\alpha\beta}(\mathbf{v}^0)$. Once more we note that quantities introduced in this section are frequently denoted by the same letters as their counterparts of Sec. 3.2, but they do not coincide with them.

By using the relations (5.17)–(5.21) one can express the bilinear form (5.5) as follows:

$$
\begin{aligned}
b_{\Omega^\varepsilon}(\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon, w^\varepsilon; \mathbf{v}', \mathbf{u}', w') = \varepsilon \int_{\Omega^\varepsilon} \Big\{ N_0^{\alpha\beta}\left[\varepsilon_{\alpha\beta}(\mathbf{v}'^0) + \varepsilon_{\alpha\beta}^y(\mathbf{v}'^1)\right]&\\
+ L_0^{\alpha\beta}\gamma_{\alpha\beta}^y(\mathbf{u}'^1) + Q_0^\alpha \kappa_\alpha^y(\mathbf{u}'^1, w'^2) + R_0 w'^2 \Big\} dx + 0(\varepsilon^2).&
\end{aligned} \tag{5.22}
$$

On the other hand, the linear form (5.9) assumes the form

$$
g^\varepsilon(\mathbf{v}', \mathbf{u}', w') = \varepsilon \int_{\Gamma_\sigma} \overline{N}^\alpha v_\alpha'^0 \, ds + 0(\varepsilon^2). \tag{5.23}
$$

Our aim is to determine the main part of the solution to the problem $(P^2_{\Omega^\varepsilon})$, see (5.8), in which the bilinear and linear forms are given by (5.22) and (5.23), respectively. At the first stage we put

$$
\mathbf{v}' = \pm \mathbf{v}'^0(x), \qquad \mathbf{u}' = \mathbf{0}, \qquad w' = 0, \tag{5.24}
$$

and next divide both sides of (5.8) by ε and pass with ε to zero. We arrive at the variational equation

$$
\int_\Omega N_h^{\alpha\beta}\varepsilon_{\alpha\beta}(\mathbf{v}'^0)dx = \int_{\Gamma_\sigma} \overline{N}^\alpha v_\alpha'^0 \, ds, \qquad \text{where} \quad N_h^{\alpha\beta} = \langle N_0^{\alpha\beta} \rangle. \tag{5.25}
$$

Let us return now to inequality (5.8), in which the left and right-hand sides are given by (5.22), (5.23). On dividing its both sides by ε, passing to zero with ε and taking account of (5.25) one obtains the following variational inequality:

$$\begin{aligned}(5.26)\qquad &\int_{\Omega} \langle N_0^{\alpha\beta}\varepsilon^y_{\alpha\beta}(\mathbf{v}'^1) + L_0^{\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}'^1 - \mathbf{u}^1)\\ &+Q_0^{\alpha}\left(u'^1_{\alpha} - u^1_{\alpha} - \frac{\partial w'^2}{\partial y_\alpha}\right) + R_0 w'^2\rangle dx \geq 0, \qquad \forall\,(\mathbf{v}',\mathbf{u}',w') \in \mathbb{K}(\Omega^{\varepsilon}).\end{aligned}$$

Now we put

$$\begin{aligned}&\mathbf{v}'^1(x,y) = \pm\mathbf{v}'(y)\varphi(x), \qquad w'^2(x,y) = \pm w'(y)\psi(x),\\ (5.27)\qquad &\mathbf{u}'^1(x,y) = \mathbf{u}^1(x,y) + \chi(x)\left[\mathbf{u}'(y) - \mathbf{u}^1(x,y)\right], \qquad 0 \leq \chi \leq 1,\\ &\varphi,\psi,\chi \in \mathcal{D}(\Omega), \qquad (\mathbf{v}',\mathbf{u}',w') \in \mathbb{K}_{YF}.\end{aligned}$$

In the standard manner we find the set consisting of two variational equations and a variational inequality that constitutes the basic cell problem:

$$(5.28)\qquad (P^2_{\text{loc}})\quad \left|\begin{array}{l}\text{Find } (\mathbf{v}^1,\mathbf{u}^1,w^2) \in \mathbb{K}_{YF} \text{ such that}\\ \langle N_0^{\alpha\beta}\varepsilon^y_{\alpha\beta}(\mathbf{v}')\rangle = 0,\\ \langle L_0^{\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}' - \mathbf{u}^1) + Q_0^{\alpha}(u'_\alpha - u^1_\alpha)\rangle \geq 0,\\ \left\langle R_0 w' - Q_0^{\alpha}\dfrac{\partial w'}{\partial y_\alpha}\right\rangle = 0, \qquad \forall\,(\mathbf{v}',\mathbf{u}',w') \in \mathbb{K}_{YF}.\end{array}\right.$$

The rescaled stress resultants depend on $(\mathbf{v}^1,\mathbf{u}^1,w^2)$ according to the relationships (5.20) and (5.22).

The local problem (P^2_{loc}) is equivalent to the following minimization problem:

$$(\tilde{P}^2_{\text{loc}})\quad\left|\begin{array}{l}\text{Find}\\ \inf\left\{\dfrac{1}{|Y|}\displaystyle\int_{Y\setminus F} j\left(x,\boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}),\boldsymbol{\gamma}^y(\mathbf{u}),\boldsymbol{\kappa}^y(\mathbf{u},w),w\right)dy \mid (\mathbf{v},\mathbf{u},w) \in \mathbb{K}_{YF}\right\},\end{array}\right.$$

where $\boldsymbol{\varepsilon}^h \in \mathbb{E}^2_s$ and

$$j(x,\boldsymbol{\varepsilon},\boldsymbol{\gamma},\boldsymbol{\kappa},r) = j_1(x,\boldsymbol{\varepsilon},\boldsymbol{\gamma},\boldsymbol{\kappa},r) + \frac{1}{2}H^{\alpha\beta}(x)\kappa_\alpha\kappa_\beta,$$
$$(\boldsymbol{\varepsilon},\boldsymbol{\gamma},\boldsymbol{\kappa},r) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}^2 \times \mathbb{R}.$$

The properties of the microscopic elastic stored energy function j, readily inferred from (I.3.14), imply that a solution $(\mathbf{v}^1,\mathbf{u}^1,w^2) \in \mathbb{K}_{YF}$ to $(\tilde{P}^2_{\text{loc}})$ and hence to (P^2_{loc}), exists and is such that $\mathbf{v}^1$ and $\mathbf{u}^1$ are unique up to constant vectors while w^2 is unique up to a constant.

Now we are ready to formulate the homogenized problem:

(P_h^2) | Find $\mathbf{v}^0 \in H_{\Gamma_w}(\Omega)$ such that the variational equation $(5.25)_1$ holds, where $N_h^{\alpha\beta}$ are given by $(5.25)_2$ and$(5.20)_1$, and the fields $(\mathbf{v}^1, \mathbf{u}^1, w^2)$ appearing in the constitutive relation $(5.25)_2$ depend on the tensor $\varepsilon^h_{\alpha\beta} = \varepsilon_{\alpha\beta}(\mathbf{v}^0)$ according to the implicit relation determined by the problem (P^2_{loc}).

REMARK 5.1. Unlike the results based upon the in-plane scaling, the problem (P^2_{loc}) is sensitive to the change of the coefficients $\varrho_\alpha = l_\alpha/2h$. The space scaling (5.2) preserves the relations: l_α/d, l_α/c for each ε. Consequently, the homogenized constitutive relation $N_h^{\alpha\beta}(\boldsymbol{\varepsilon}^h)$ will depend on the crack spacing measured with respect to the laminate thickness.

5.3. Hyperelastic potential. Well-posedness of the problem (P_h^2)

The constitutive relationship $(5.25)_2$ can be rewritten by introducing the homogenized potential

$$(5.29)\qquad W_h(x, \boldsymbol{\varepsilon}^h) = \frac{1}{|Y|} \int\limits_{Y \setminus F} j\left(x, \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^y(\mathbf{v}^1),\ \boldsymbol{\gamma}^y(\mathbf{u}^1),\ \boldsymbol{\kappa}^y(\mathbf{u}^1, w^2),\ w^2\right) dy$$

$$= \frac{1}{2}\left\langle N_0^{\alpha\beta}\left[\varepsilon^h_{\alpha\beta} + \varepsilon^y_{\alpha\beta}(\mathbf{v}^1)\right] + L_0^{\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}^1) + Q_0^\alpha \kappa^y_\alpha(\mathbf{u}^1, w^2) + R_0 w^2\right\rangle.$$

One can prove that

$$(5.30)\qquad N_h^{\alpha\beta} = \frac{\partial W_h}{\partial \varepsilon^h_{\alpha\beta}}.$$

The proof follows the lines of the demonstration of the property (ii) characterizing the effective potential U_h, see Sec. 3.3 and, in particular, formula (3.25). More precisely, the basic properties of W_h are specified by:

(a) $W_h(x, .)$ $(\mathbf{e}_1 \in \mathbb{E}^2_s)$ is strictly convex and of class C^1.

(b) There exist constants $C_1 > C_0 > 0$ such that for a.e. $x \in \Omega$

$$C_0|\mathbf{e}|^2 \le W_h(x, \mathbf{e}) \le C_1|\mathbf{e}|^2 \qquad \text{for all} \quad \mathbf{e} \in \mathbb{E}^2_s.$$

We observe that formulae (4.18), (5.30) and the properties (a) and (b) just stated are preserved when F divides the basic cell Y into two disjoint subdomains. The property (b) of the effective potential W_h ensures unique solvability of the homogenized problem (P_h^2), i.e. its well-posedness, provided that the length of

Γ_w is greater than zero. Note that the homogenized potential (5.29) assumes a simple form

$$W_h(x, \boldsymbol{\varepsilon}^h) = \frac{1}{2} N_h^{\alpha\beta}(\boldsymbol{\varepsilon}^h)\varepsilon^h_{\alpha\beta}\,. \tag{5.31}$$

To prove it, let us substitute $\mathbf{v}' = \mathbf{v}^1$, $w' = w^2$, $\mathbf{u}'^1 = 2\mathbf{u}^1$ and then $\mathbf{u}'^1 = \mathbf{0}$ into (5.28). Upon adding the equalities obtained in this manner one finds

$$\left\langle N_0^{\alpha\beta}\varepsilon^y_{\alpha\beta}(\mathbf{v}^1) + L_0^{\alpha\beta}\gamma^y_{\alpha\beta}(\mathbf{u}^1) + Q_0^{\alpha}\kappa^y_{\alpha}(\mathbf{u}^1, w^2) + R_0 w^2\right\rangle = 0, \tag{5.32}$$

and (5.31) follows.

5.4. Kachanov's form of the homogenized constitutive relations

Let us focus our attention on the homogenized constitutive relation $(5.25)_2$. Considering $(5.20)_1$ and recalling that $\mathbf{v}^1 \in H^1_{\text{per}}(Y)^2$ one obtains

$$N_h^{\lambda\mu} = A_v^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta} + A_{vu}^{\lambda\mu\alpha\beta}\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle + A_{vw}^{\lambda\mu}\langle w^2\rangle. \tag{5.33}$$

We show below that the formula above can be rearranged to a new one depending only on $\varepsilon^h_{\alpha\beta}$ and $\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle$. Indeed, let us substitute $w' = \text{const}$ into variational equation $(5.28)_3$. One finds

$$\langle R_0\rangle = 0. \tag{5.34}$$

Taking into account relation $(5.20)_3$, one can reformulate Eq. (5.33) to the form

$$N_h^{\lambda\mu} = A_1^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta} + A_2^{\lambda\mu\alpha\beta}\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle, \tag{5.35}$$

similar to the previous formula (4.5): the tensors $A_\sigma^{\alpha\beta\lambda\mu}$ are determined by Eqs. (4.9). Formula (5.35) can be written as follows

$$N_h^{\lambda\mu} = A_1^{\lambda\mu\alpha\beta}\varepsilon^h_{\alpha\beta} - A_2^{\lambda\mu\alpha\beta}\varepsilon^F_{\alpha\beta}\,, \tag{5.36}$$

where

$$\varepsilon^F_{\alpha\beta} = -\langle\gamma^y_{\alpha\beta}(\mathbf{u}^1)\rangle. \tag{5.37}$$

These quantities will be referred to as crack deformation measures, similarly to the quantities (4.6) of the model (h_0, l_0). One can represent them in the form (4.7), (4.8). Thus we conclude that the knowledge of the relations $[\![u^1_N(\boldsymbol{\varepsilon}^h)]\!]$, $[\![u^1_T(\boldsymbol{\varepsilon}^h)]\!]$ suffices for the determination of the homogenized constitutive relationship (5.36).

The constitutive relationship (5.36), along with the representation (4.7) of (5.37), have assumed the form of KACHANOV [6]. The crack deformation measures $\varepsilon^F_{\alpha\beta}$ turn out to play the role of internal state variables. In contrast to Kachanov's phenomenological approach, formulae (5.36) and (5.37) have not been proposed but rigorously derived. The internal state variables are directly connected with macrodeformation fields ($\varepsilon^h_{\alpha\beta}$) through the local problem, cf. also TELEGA [I.49]. The tensor $\mathbf{A}_2$ may be called a "damage moduli tensor". For the investigation of damage in laminates, ALLEN *at al.* [1] use Kachanov's concept. According to their analysis $\mathbf{A}_1 = -\mathbf{A}_2$. The formulae (4.9) do not warrant such an identification.

5.5. Characteristics of the model (h_0, l)

Let us list the main characteristics of the model (h_0, l). Within its framework, the analysis is decomposed into macro and micro- (or local) levels.

At the local level:

• the axial stresses in the external and internal layers are assumed to depend only upon the in-plane coordinates,

• the equilibrium equations are satisfied exactly,

• the interface conditions for both stresses and displacements are satisfied exactly,

• the stress-strain relations are satisfied in an average sense by requiring that the Reissner functional expressed in terms of stress resultants and generalized displacemets attain a saddle point at the solution,

• the local elasticity problem is reduced to solving the set (5.28) of two variational equalities and one inequality. In the case of cracks going parallely through the whole laminate, this problem is reduced to solving a set of three ordinary differential equations (see [9] for details).

At the macro-level:

• equilibrium equations involve the in-plane stress resultants as in the conventional plane-stress description; hence the equilibrium equations are satisfied in an average sense,

• boundary conditions are formulated as in the plane-stress model, in an average sense,

• stress-strain relations are nonlinear and their form expresses the unilateral cracking effects at the local (micro) level.

5.6. The (h_0, l_0) laminate model as a limiting case of the model (h_0, l) when $\varrho_\alpha \to 0$

We shall now prove that the thin laminate model (h_0, l_0) of Sec. 4 can be derived from the (h_0, l) model by passing to zero with ϱ_α; $\varrho_\alpha = l_\alpha/2h$. Let us introduce non-dimensional coordinates

$$\xi_\alpha = y_\alpha/l_1\,, \qquad \xi = (\xi_1, \xi_2) \in \Sigma, \qquad \Sigma = (0,1) \times (0, \xi_0), \qquad \xi_0 = l_2/l_1\,.$$

In the sequel, a re-defined cell of periodicity Σ and the transverse dimensions h, c, d will be held fixed; averaging over Σ being denoted by $\langle\cdot\rangle_0$.

Consider the consequences of passing to zero with ϱ_1 (then also ϱ_2 tends to zero). On introducing the variables ξ_α into (P^2_{loc}), muliplying both sides of $(5.28)_3$ by $(l_1)^2$, taking into account the relations (5.21) and passing to zero with ϱ_1 (h is held fixed), one obtains the following variational equation

$$\left\langle H^{\alpha\beta}\frac{\partial w^2}{\partial\xi_\alpha}\,\frac{\partial w'}{\partial\xi_\beta}\right\rangle_0 = 0, \tag{5.38}$$

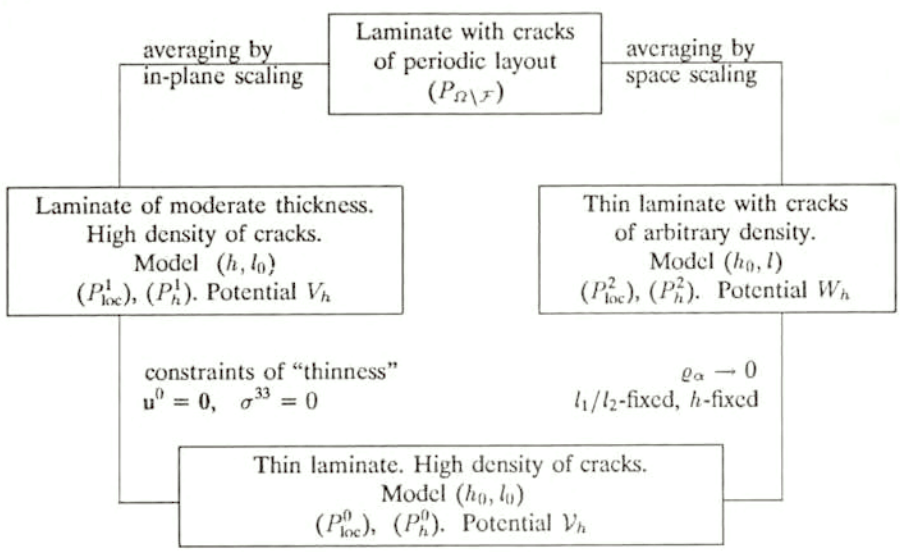
satisfied for all $w'\in H^1_{\text{per}}(\Sigma)$. Hence $w^2 = \text{const}$ and consequently $w^2 = \langle w^2\rangle_0$.

According to (5.34) $\langle R_0\rangle_0 = 0$ and we can eliminate w^2 from the relations (5.20) to arrive at relations (4.11). On multiplying Eqs. $(5.28)_{1,2}$ by l_1 and passing to zero with ϱ_1, one arrives at an equality and inequality of the form $(3.18)_1$ and $(3.18)_2$, respectively. Thus we obtain the problem (P^0_{loc}) formulated in Sec. 4. Averaging equation $(4.11)_1$ results in the homogenized constitutive relation (5.35). Thus we obtain the problem (P^0_h) in an alternative manner. In particular, the ϱ_α-independent potential $\mathcal{V}_h$ defined by Eq. (4.22) turns out to be a limit of the ϱ_α-dependent potential W_h (cf. (5.31)) as $\varrho_\alpha\to 0$, viz.:

$$\mathcal{V}_h = \lim_{\varrho_\alpha\to 0} W_h(\varrho_1,\varrho_2), \qquad l_1/l_2 - \text{ fixed.} \tag{5.39}$$

Two ways of deriving the model (h_0, l_0) are outlined in Diagram 1.

Diagram 1.



6. Final remarks

In our accompanying paper [9], the case of cracks going along straight lines is examined in detail. The analytical results are compared with the available experimental data and with theoretical predictions of HASHIN [I.17], ABOUDI [I.1], ALLEN *et al.* [1] and MCCARTNEY [I.37, I.38].

Partially angled and curved cracks observed in experiments, as well as delamination (cf. GROVES *et al.* [I.11]), could not easily be accounted for by two-dimensional laminate models. Curved cracks result, in general, in transverse asymmetry, thus coupling the membrane and bending effects. Such effects could be considered within the framework of the three-dimensional local problems, similar to that derived by CHACHA and SANCHEZ–PALENCIA [I.8].

Acknowledgement

The authors were supported by the State Committee for Scientific Research through the grants No 300199101 and No 3 P404 013 06.

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Received June 2, 1995.