Arch. Mech., 48, 1, pp. 143–161, Warszawa 1996

Stiffness loss in laminates with intralaminar cracks
Part I. Two-dimensional modelling

T. LEWIŃSKI and J.J. TELEGA (WARSZAWA)

BY IMPOSING STRESS constraints of Hashin type and corresponding kinematic constraints on the Reissner two-field functional, a new two-dimensional model for the three-layer symmetric laminate is derived. This model is capable of taking into account transverse cracks in the internal layer. These cracks behave according to Signorini's conditions. In Part I the basic properties of the model without and with transverse crack in the internal layer are investigated. The equilibrium problem is governed by a variational inequality involving five kinematic unknowns.

1. Introduction

OWING TO A DISCREPANCY between the values of thermal expansions and elastic moduli of fibres and the matrix, the composite laminates incur transverse microcracking even under relatively low in-plane loadings. The appearing intralaminar cracks go usually across the whole thickness of the layer and are almost equally spaced, cf. GARRETT and BAILEY's [10, Fig. 2] experiments with glass fibre-reinforced polyester; graphite-epoxy patterns in GROVES *et al.* [11, Fig. 3] and cracks in glass-epoxy laminates in HIGHSMITH and REIFSNIDER [19]. At a certain level of loading, the crack patterns attain a saturation state in which the layout of cracks is also nearly uniform, called CDS-characteristic damage state.

The loss of stiffnesses of a laminate can be viewed as a characteristic of the degree of its damage. The main characteristics of the laminate interrelate its stiffnesses with density of the transverse microcracks defined as parameters inversely proportional to the crack spacing. Apparent regularity of cracking patterns warrants the derivation of the effective macro-properties of the laminate from the properties of microcracking. Such approach stipulates formation of special micromechanics models, the point of departure of which is the analysis of stresses in the vicinity of transverse cracks. A review of such models can be found in LEE *et al.* [25], YANG and BOEHLER [58], MCCARTNEY [39] and in TSAI and DANIEL [56], cf. also ABOUDI and BENVENISTE [2], ABRATE [4], THIONNET [55], TENG [54], GAMBAROTTA and LAGOMARSINO [9], KATTAN and VOYIADJIS [20], GUDMUNDSON and ÖSTLUND [12, 13], GUDMUNDSON and ZANG [14].

A successful method, albeit based on simple stress assumptions, of local analysis in the stretching and shearing problems is due to HASHIN [17, 18]. Hashin's approach was followed by new displacement-based approaches of HAN *et al.* [16], HAN and HAHN [15], ABOUDI [1], ABOUDI *et al.* [3] and TSAI and DANIEL [56]. The development of the theory resembles here the progress in the theory of moderately thick plates which was started by stress-based approach of REISSNER [42, 43] as developed by Hencky, Bollé and Mindlin into its displacement-based formulations, cf. REISSNER [45]. Hashin's approach has recently been reconsidered in McCARTNEY [37, 38], where two new relevant models have been discussed: a "generalized plane strain" model (called further GPS) and an "approximate 3-D solution". The range of validity of the former model is similar to that of HASHIN [17], but, as it has been reported in McCARTNEY [38], this model results in the unique formulae for all the effective stiffnesses relevant to the tension problem. The latter model is applicable for tension of laminates with finite width. Hashin's approach is also a basis for NAIRN [40] to define a strain energy release rate due to microcracking. By putting forward a hypothesis that its value is a material constant, Nairn arrived at a relationship between the given load and the crack density, cf. also YALVAÇ *at al.* [57].

The modern tools of homogenization make it possible to perform the process of smearing-out the cracks rigorously. The aim of the present paper is, by using these new tools, to put forward an alternative model of cracked laminates by basing it on HASHIN's [17, 18] – type stress assumptions. The model is constructed by:

i) relaxing HASHIN's [18] stress assumptions by introducing new stress resultant fields, none of them being viewed as directly dependent upon the edge loading;

ii) augmenting them with assumptions on displacements;

iii) constructing a new two-dimensional model of a three-layer symmetric laminate by using a two-field variational principle of REISSNER [43]. Hashin adopted a method of REISSNER [42] but without introducing Lagrange multipliers;

iv) performing the process of smearing-out the transverse cracks in the internal layer by the method of homogenization.

A short outline of this modelling has been announced in LEWIŃSKI and TELEGA [32, 33, 34] and TELEGA and LEWIŃSKI [53].

The hypotheses assumed make it possible to reduce the transverse dimension. In spite of it, they enable us to figure out transverse cracks in the internal layer. Reduction of the transverse dimension implies neglecting the singularities of the transverse distribution of stresses but does not suppress the in-plane distributed singularities of the stress resultants.

The modelling carried out in the present paper is neither a generalization of HASHIN's [17, 18] approach nor that of McCARTNEY's [37, 38] analysis. In the homogenization approach used in the second part [35], the periodicity cell is subjected to macrodeformations $\varepsilon^h_{\alpha\beta}$. These macrofields control deformations in the cell of periodicity. On the contrary, Hashin's approach does not distinguish between problems in macro- and microscale. This author considers a state of stress and deformation in a finite domain of the cracked laminate and the control variables are boundary loads. This does not mean that such an approach is equivalent to the stress method of homogenization, since in the latter case the deformation state in the periodicity cell are controlled by effective stress resultants $N^{\alpha\beta}_h(x)$, $x \in \Omega$ and not by boundary forces. Moreover, in contrast to Hashin's approach, in the modelling proposed the stresses are not directly related to the edge loading. The in-plane stresses $\sigma^{\alpha\beta}$ are expressed in terms of two independent and *a priori* unknown tensors of stress resultants $N^{\alpha\beta}$, $L^{\alpha\beta}$ (see (2.7)), whilst in the Hashin-type models mentioned above N^{11} and N^{12} are viewed as known. Consequently, in the approach presented the number of Lagrangian multipliers (generalized displacements) is greater by three. Only in this manner one can make the model capable of describing a general family of in-plane edge loadings.

Despite these substantial differences between the Hashin-type models and the model presented, some comparisons will be made in [36] inasmuch as appropriate reinterpretation of Hashin's or McCartney's quantities in terms of notions of the homogenization approach is possible.

Regular crack patterns observed in experiments justify the assumption of periodicity of the crack distribution, which enables us to use homogenization methods effectively. A version of this method, developed primarily by SANCHEZ–PALENCIA [47], encompassing Signorini-type cracks that can open or close (and then extended by the present authors [29, 30, 31], TELEGA [51], TELEGA and LEWIŃSKI [52]) makes it possible to describe the opening and closure of the transverse cracks in the internal layer.

In 3D problems of averaging properties of periodic elastic composites, a uniquely determined construction of the homogenized model is implied by solution of the basic cell problem. All other averaging methods can only furnish approximations and their accuracy should be measured by the deviation from the homogenization results, cf. SUQUET [48]. Less clear situation occurs in problems of averaging stiffnesses of plates with a periodic structure. The results of KOHN and VOGELIUS [22, 23, 24] and CAILLERIE [7] prove that a correct starting point should be the three-dimensional model; the method of homogenization results in the Kirchhoff thin plate model whose stiffnesses are determined by the properties of 3D periodicity cells. Similar approach for the case of cracked plates is due to CHACHA and SANCHEZ–PALENCIA [8]. However, if the starting point is two-dimensional, then the following two methods of averaging are useful:

a) the method based on an in-plane scaling of the longitudinal dimensions of the periodicity cells,

b) the method based on simultaneous scaling of all dimensions of cells.

Consequences of both averaging methods for moderately thick plates have been examined in LEWIŃSKI [27, 28] and TELEGA [50]. The (a) method leads to moderately thick plate effective model and applies only when in-plane dimensions of the periodicity cells are much smaller than plate thickness. The (b) method leads to thin plate effective model and does not impose any conditions on the dimensions of periodicity cells. Thus the applicability ranges of both methods are not identical.

In the second part of the paper [35] the consequences of using (a) and (b) scalings applied to a new laminate model will be examined.

Part I constitutes the basis for such considerations. Particularly, the two-dimensional model of an undamaged three-layer laminate will be derived in Sec. 2. In Sec. 3 we assume that the internal layer is weakened by a transverse crack. The variational inequality of type (3.11) will be used as a starting point for the homogenization procedures developed in the second part of the paper [35].

Throughout the whole paper (Part I and II), the following conventions are adopted: small Greek indices (except for ε) assume values 1, 2 while Latin ones (except for h) run over 1, 2, 3. Summation convention holds only for repeated indices at different levels. Index h labels the quantities referred to the homogenized models. Comma implies partial differentiation with respect to x_i, particularly x_α. An arrow ($\rightarrow$) will denote either convergence or replacement, which should not lead to misunderstandings.

2. In-plane deformation of symmetric three-layer laminates of moderate thickness

The aim of this section is to form a new two-dimensional model of a three-layer symmetric laminate, capable of describing the independent in-plane displacements of the faces and of the internal layer. In particular, the cross-ply laminates of the $[0^\circ_m, 90^\circ_n]_s$ class are of such a type.

Consider a symmetric laminate composed of the faces of thickness d and the internal layer of thickness $2c$. The middle plane Ω of the internal layer is parametrized by Cartesian coordinates x_α; $(x_\alpha) = x \in \Omega$. The whole laminate occupies a cylindrical domain $B = \Omega \times (-h, h)$; $h = c + d$. To an arbitrary point $\mathbf{x} \in B$ we assign its coordinates $\mathbf{x} = (x_i) = (x_\alpha, x_3 = z)$, z-axis being perpendicular to the Ω plane.

The lower and upper faces $z = \pm h$ are assumed to be free of loads, whilst the lateral edge surface $S = \Gamma \times (-h, h)$, $\Gamma = \partial\Omega$, is subjected to the tractions $p^i(s, z)$, $s \in \Gamma$, on its part $S_\sigma = \Gamma_\sigma \times (-h, h)$. The remaining part of S, $S_w = \Gamma_w \times (-h, h)$ $(\overline{\Gamma}_w \cup \overline{\Gamma}_\sigma = \Gamma)$ is clamped. For the sake of further simplifications, the loading p^i is assumed to have the following through-the-thickness distribution

$$
p^\alpha(s, z) = \begin{cases} \dfrac{1}{2c}\overline{L}^\alpha(s), & |z| < c, \\ \dfrac{1}{2d}(\overline{N}^\alpha(s) - \overline{L}^\alpha(s)), & \text{otherwise}; \end{cases} \tag{2.1}
$$

$$
p^3(s, z) = \begin{cases} \dfrac{1}{2d}(z - h)\overline{Q}(s), & c < z < h, \\ -\dfrac{z}{2c}\overline{Q}(s), & |z| < c, \\ \dfrac{1}{2d}(z + h)\overline{Q}(s), & -h < z < -c. \end{cases} \tag{2.2}
$$

The loading functions $\overline{N}^{\alpha}$, $\overline{L}^{\alpha}$, $\overline{Q}$ are defined on Γ_{σ}. The body forces are omitted.

The through-the-thickness distribution of elastic compliances has the form

$$D_{ijkl} = \begin{cases} D^{m}_{ijkl}, & |z| < c, \\ D^{f}_{ijkl}, & \text{otherwise}, \end{cases} \tag{2.3}$$

and the components of the tensors $\mathbf{D}^{m}$ and $\mathbf{D}^{f}$ may depend on $\mathbf{x} \in \mathcal{B}$. As it is usually done in most treatments, we assume that the z = const planes are the planes of material symmetry, hence

$$D^{n}_{3\alpha\beta\gamma} = D^{n}_{333\alpha} = 0, \qquad n = m \quad \text{or} \quad f. \tag{2.4}$$

The tensor $\mathbf{D}$ satisfies the usual symmetry condition

$$D_{ijkl} = D_{jikl} = D_{klij}. \tag{2.5}$$

We make further the following assumption

$$(H) \qquad \left| \begin{array}{l} D_{ijkl} \in L^{\infty}(\mathcal{B}); \\ \text{there exists a constant } \; C > 0 \; \text{ such that} \\ D_{ijkl}(x) T^{ij} T^{kl} \geq C |\mathbf{T}|^{2}, \end{array} \right.$$

for a.e. $\mathbf{x} \in \mathcal{B}$ and for each $\mathbf{T} \in \mathbb{E}^{3}_{s}$, where $\mathbb{E}^{3}_{s}$ is the space of symmetric 3×3 matrices and

$$|\mathbf{T}|^{2} = \sum_{i,j=1}^{3} T^{ij} T^{ij}.$$

We recall that throughout this paper only Cartesian coordinate systems are employed, thus we may identify (T^{ij}) with (T_{ij}), etc. Moreover, C possibly with a subscript will denote a positive constant.

Within the three-dimensional framework, the problem of equilibrium of the laminate considered amounts to finding the stress fields $\widetilde{\sigma}^{ij}$ as well as the displacement fields $\widetilde{w}^{i}$ for which the two-field REISSNER [43] functional

$$I(\mathbf{w}, \boldsymbol{\sigma}) = \int_{\mathcal{B}} \left[\frac{1}{2}(w_{\alpha,\beta} + w_{\beta,\alpha})\sigma^{\alpha\beta} + (w_{\alpha,3} + w_{3,\alpha})\sigma^{\alpha 3} + w_{3,3}\sigma^{33} - \frac{1}{2} D_{ijkl}\sigma^{ij}\sigma^{kl} \right] d\mathbf{x} - \int_{S_{\sigma}} p^{i}(s,z) w_{i}(s,z)\, ds\, dz, \tag{2.6}$$

assumes its stationary value at a saddle point, cf. NEČAS and HLAVÁČEK [41].

To prove that the functional I possesses a unique saddle point one can use the ARNOLD and FALK [5] version of BREZZI's theorem [6].

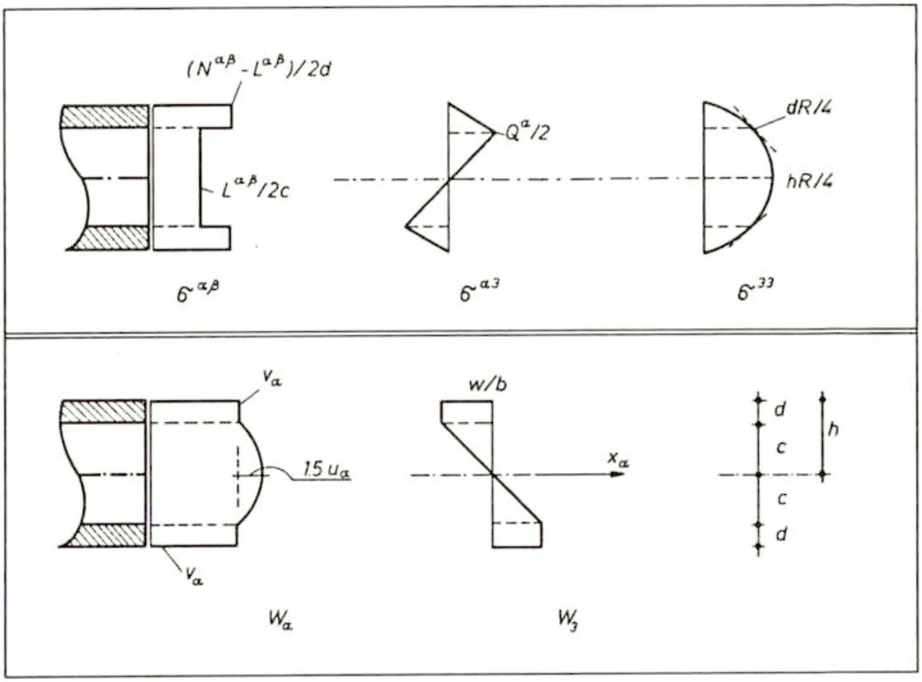

FIG. 1. Stress and displacement assumptions.

To facilitate the envisaged treatment of cracks in the interior layer $|z| < c$, it is thought to be helpful to develop a new two-dimensional laminate model. We base this modelling upon the following stress assumptions (Fig. 1)

$$\sigma^{\alpha\beta}(\mathbf{x}) = \begin{cases} \dfrac{1}{2c} L^{\alpha\beta}(x), & |z| < c, \\ \dfrac{1}{2d}\left[N^{\alpha\beta}(x) - L^{\alpha\beta}(x)\right], & \text{otherwise}; \end{cases} \tag{2.7}$$

$$\sigma^{\alpha 3}(\mathbf{x}) = \begin{cases} \dfrac{1}{2d}(z-h)Q^{\alpha}(x), & c < z < h, \\ -\dfrac{z}{2c}Q^{\alpha}(x), & |z| < c, \\ \dfrac{1}{2d}(z+h)Q^{\alpha}(x), & -h < z < -c; \end{cases} \tag{2.8}$$

$$\sigma^{33}(\mathbf{x}) = \begin{cases} \dfrac{1}{4d}(z-h)^2 R(x), & c < z < h, \\ \dfrac{1}{4c}(-z^2 + ch)R(x), & |z| < c, \\ \dfrac{1}{4d}(z+h)^2 R(x), & -h < z < -c. \end{cases} \tag{2.9}$$

Here $L^{\alpha\beta}$, $N^{\alpha\beta}$, Q^{α}, R are unknown fields defined on Ω. The stress assumptions (2.7)–(2.9) can be viewed as a modification of HASHIN's [17, 18] formulae for stresses in which counterparts of the stress resultants N^{11}, N^{12} were considered as determined by the boundary loads applied, Q^{α} and R being directly expressed in terms of $L^{\alpha\beta}$ by: $Q^{\alpha} = L^{\alpha\beta}_{,\beta}$, $R = -L^{\alpha\beta}_{,\alpha\beta}$. In our approach the stress resultants $N^{\alpha\beta}$ are unknowns of the model. Moreover, we introduce Q^{α} and R as independent stress resultants, similarly as it is usually done in the theory of plates, cf. REISSNER [42, 43].

The stress assumptions (2.7)–(2.9) determine the two-dimensional model to be derived. However, to have all information about the through-the-thickness distribution of the displacements, it is helpful to add the kinematic assumptions:

$$
w_\alpha(\mathbf{x}) = \begin{cases} v_\alpha(x) + \dfrac{3}{2c^2}(c^2 - z^2)u_\alpha(x), & |z| < c, \\ v_\alpha(x), & \text{otherwise}; \end{cases} \tag{2.10}
$$

$$
w_3(\mathbf{x}) = \begin{cases} \dfrac{1}{b}w(x), & c < z < h, \\ \dfrac{z}{c}\dfrac{w(x)}{b}, & |z| < c, \\ -\dfrac{1}{b}w(x), & -h < z < -c; \end{cases} \tag{2.11}
$$

where $b = \dfrac{d}{2} + \dfrac{c}{3}$, cf. Fig. 1. Note that

$$
\int_{-h}^{h} \sigma^{\alpha\beta}\, dz = N^{\alpha\beta}, \qquad \frac{1}{2c}\int_{-c}^{c} w_\alpha\, dz = v_\alpha + u_\alpha . \tag{2.12}
$$

The specific form of the hypotheses (2.7)–(2.11) cannot be cleared up at this stage of the analysis. We mention only that stresses (2.8) and (2.9) satisfy the boundary conditions on the faces

$$
\sigma^{k3}(x, \pm h) = 0, \tag{2.13}
$$

as well as the continuity conditions on the interfaces $z = \pm c$.

Deformations associated with displacements (2.10) and (2.11) are not correlated with stresses (2.7)–(2.9) by constitutive relationships. Modelling based upon the $I(\mathbf{w}, \boldsymbol{\sigma})$ functional admits such a mismatch. Nevertheless, the form of kinematic assumptions (2.10) and (2.11) has no influence on the errors of stress evaluation. The model construction could be based only upon the stress constraints (2.7)–(2.9) and then $(\mathbf{v}, \mathbf{u}, w)$ would occur formally as Lagrangian multipliers. The hypotheses (2.10) and (2.11) endow the multipliers with a physical meaning.

A more accurate evaluation of the displacement fields can be done after solving the problem by integrating the three-dimensional constitutive relationships with stresses given by (2.7)–(2.9). In this manner, displacement representations compatible with stress fields of HASHIN [18] have recently been found by MCCARTNEY [37]; similar ones can be constructed for the (2.7)–(2.9) representations. They could be helpful in forming the displacement-based models, cf. comments in Final Remarks.

Let us now proceed to form the two-dimensional model of the laminate assuming, for the sake of simplicity, that the compliances D_{ijkl} depend on $x \in \Omega$ only. We substitute the expressions (2.7)–(2.11) into the Reissner functional (2.6) and perform z-integration to obtain a new functional J:

$$J(\mathbf{v},\mathbf{u},w;\mathbf{N},\mathbf{L},\mathbf{Q},R) = I(\mathbf{w},\boldsymbol{\sigma}), \tag{2.14}$$

where the fields $\mathbf{w}$, $\boldsymbol{\sigma}$ have the form (2.7)–(2.11). The functional J has the form

$$\begin{aligned} J = \int\limits_{\Omega} \Big[& v_{\alpha,\beta}N^{\alpha\beta} + u_{\alpha,\beta}L^{\alpha\beta} + (u_\alpha - w_{,\alpha})Q^\alpha + w\,R \\ & -W_c(x,\mathbf{N},\mathbf{L},\mathbf{Q},R)\Big]\,dx - \int\limits_{\Gamma_\sigma}\left(\overline{N}^\alpha v_\alpha + \overline{L}^\alpha u_\alpha - \overline{Q}w\right)ds, \end{aligned} \tag{2.15}$$

where the complementary energy reads

$$\begin{aligned} 2W_c = {} & D^N_{\alpha\beta\lambda\mu}N^{\alpha\beta}N^{\lambda\mu} + D^L_{\alpha\beta\lambda\mu}L^{\alpha\beta}L^{\lambda\mu} + 2D^{NL}_{\alpha\beta\lambda\mu}N^{\alpha\beta}L^{\lambda\mu} \\ & + D^Q_{\alpha\beta}Q^\alpha Q^\beta + 2D^{RL}_{\alpha\beta}RL^{\alpha\beta} + 2D^{RN}_{\alpha\beta}RN^{\alpha\beta} + D^R R^2. \end{aligned} \tag{2.16}$$

The generalized compliances $\mathbf{D}^N$, $\mathbf{D}^L$, $\mathbf{D}^{NL}$, $\mathbf{D}^Q$, $\mathbf{D}^{RL}$, $\mathbf{D}^{RN}$, D^R depend on $x \in \Omega$ if D_{ijkl} do, and they are given by

$$\begin{aligned} D^N_{\alpha\beta\lambda\mu} &= \frac{1}{2d}D^f_{\alpha\beta\lambda\mu}, & D^{LN}_{\alpha\beta\lambda\mu} &= -D^N_{\alpha\beta\lambda\mu}, \\ D^L_{\alpha\beta\lambda\mu} &= D^N_{\alpha\beta\lambda\mu} + \frac{1}{2c}D^m_{\alpha\beta\lambda\mu}, & D^Q_{\alpha\beta} &= \frac{2}{3}\left(d\,D^f_{\alpha3\beta3} + c\,D^m_{\alpha3\beta3}\right), \\ D^{RL}_{\alpha\beta} &= -\frac{d}{12}D^f_{\alpha\beta33} + \frac{1}{4}\left(h - \frac{c}{3}\right)D^m_{\alpha\beta33}, & D^{RN}_{\alpha\beta} &= \frac{d}{12}D^f_{\alpha\beta33}, \\ D^R &= \frac{d^3}{40}D^f_{3333} + \frac{c}{8}\left(h^2 - \frac{2}{3}ch + \frac{c^2}{5}\right)D^m_{3333}. \end{aligned} \tag{2.17}$$

Their symmetry properties follow directly from those of D^f_{ijkl} and D^m_{ijkl}. The functional J attains its stationary value if the following relations are satisfied:

i) the equilibrium equations

$$-N^{\alpha\beta}{}_{,\beta} = 0, \qquad -L^{\alpha\beta}{}_{,\beta} + Q^\alpha = 0, \qquad Q^\alpha{}_{,\alpha} + R = 0, \tag{2.18}$$

ii) the constitutive relationships

$$
\begin{aligned}
\varepsilon_{\alpha\beta} &= D^{N}_{\alpha\beta\lambda\mu} N^{\lambda\mu} + D^{NL}_{\alpha\beta\lambda\mu} L^{\lambda\mu} + D^{RN}_{\alpha\beta} R, \\
\gamma_{\alpha\beta} &= D^{NL}_{\alpha\beta\lambda\mu} N^{\lambda\mu} + D^{L}_{\alpha\beta\lambda\mu} L^{\lambda\mu} + D^{RL}_{\alpha\beta} R, \\
w &= D^{RN}_{\alpha\beta} N^{\alpha\beta} + D^{RL}_{\alpha\beta} L^{\alpha\beta} + D^{R} R,
\end{aligned}
\tag{2.19}
$$

$$
\kappa_\alpha = D^{Q}_{\alpha\beta} Q^{\beta}, \tag{2.20}
$$

where the deformation measures introduced above are defined as follows:

$$
\begin{aligned}
\varepsilon_{\alpha\beta} &= \varepsilon_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}), \\
\gamma_{\alpha\beta} &= \gamma_{\alpha\beta}(\mathbf{u}) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \\
\kappa_\alpha &= \kappa_\alpha(\mathbf{u}, w) = u_\alpha - w_{,\alpha},
\end{aligned}
\tag{2.21}
$$

iii) the stress-type boundary conditions along the line Γ_σ

$$
\begin{aligned}
N_n &= \overline{N}_n, \qquad N_\tau = \overline{N}_\tau, \\
L_n &= \overline{L}_n, \qquad L_\tau = \overline{L}_\tau, \qquad Q = \overline{Q}.
\end{aligned}
\tag{2.22}
$$

Here

$$
\begin{aligned}
N_n &= N^{\alpha\beta} n_\alpha n_\beta, \qquad & N_\tau &= N^{\alpha\beta} n_\beta \tau_\alpha, \\
L_n &= L^{\alpha\beta} n_\alpha n_\beta, \qquad & L_\tau &= L^{\alpha\beta} n_\beta \tau_\alpha, \qquad Q = Q^\alpha n_\alpha, \\
\overline{N}_n &= \overline{N}^\alpha n_\alpha, \qquad & \overline{N}_\tau &= \overline{N}^\alpha \tau_\alpha, \quad \text{etc.}
\end{aligned}
\tag{2.23}
$$

Here $\mathbf{n}$ and $\boldsymbol{\tau}$ are unit vectors: outward normal and tangent to the Γ_σ line, respectively.

The constitutive relationships (2.19) and (2.20) can be inverted to the form

$$
\begin{aligned}
N^{\lambda\mu} &= A^{\lambda\mu\alpha\beta}_{v} \varepsilon_{\alpha\beta} + A^{\lambda\mu\alpha\beta}_{vu} \gamma_{\alpha\beta} + A^{\lambda\mu}_{vw} w, \\
L^{\lambda\mu} &= A^{\lambda\mu\alpha\beta}_{vu} \varepsilon_{\alpha\beta} + A^{\lambda\mu\alpha\beta}_{u} \gamma_{\alpha\beta} + A^{\lambda\mu}_{uw} w, \\
R &= A^{\alpha\beta}_{vw} \varepsilon_{\alpha\beta} + A^{\alpha\beta}_{uw} \gamma_{\alpha\beta} + A_{w} w;
\end{aligned}
\tag{2.24}
$$

$$
Q^\alpha = H^{\alpha\beta} \kappa_\beta, \tag{2.25}
$$

that can be viewed as the primal one.

Note that if the equations of equilibrium (2.18) are fulfilled for every $x \in \Omega$, then the stresses σ^{ij} determined by formulae (2.7)–(2.9) satisfy the three- dimensional equilibrium equations $\sigma^{ij}{}_{,j} = 0$ identically for every $(x, z) \in B$. Bearing in mind that these stresses satisfy the boundary conditions (2.13) on the faces we note that, up to the boundary zone along S, the stresses assumed are statically admissible. This property clears up the form of the stress hypotheses (2.7)–(2.9). Therefore the model construction presented here is similar to REISSNER's [43] construction concerning transversely homogeneous plates.

The strong formulation of the equilibrium problem reads:

Find the kinematic and stress fields $\mathbf{v} = (v_\alpha)$, $\mathbf{u} = (u_\alpha)$, w; $\mathbf{N} = (N^{\alpha\beta})$, $\mathbf{L} = (L^{\alpha\beta})$, $\mathbf{Q} = (Q^\alpha)$ and R such that there are satisfied:

- the equilibrium equations (2.18),
- the constitutive relationships (2.24), (2.25),
- the strain – displacements relations (2.21),
- the boundary conditions (2.22) on Γ_σ,
- and the boundary conditions:

$$\mathbf{v} = \mathbf{0}, \qquad \mathbf{u} = \mathbf{0}, \qquad w = 0 \qquad \text{on} \quad \Gamma_w . \tag{2.26}$$

As a prerequisite for the primal variational formulation, we define the space

$$V = \Big\{(\mathbf{v}, \mathbf{u}, w) \mid \mathbf{v} \in H^1(\Omega)^2, \ \mathbf{u} \in H^1(\Omega)^2, \ w \in H^1(\Omega); \\ \mathbf{v} = \mathbf{0}, \ \mathbf{u} = \mathbf{0}, \ w = 0 \ \text{on} \ \Gamma_w\Big\}$$

representing the space of kinematically admissible fields. Let us define the bilinear form

$$a_\Omega(\mathbf{v}, \mathbf{u}, w; \mathbf{v}', \mathbf{u}', w') = \int_\Omega \Big[N^{\alpha\beta}(\mathbf{v}, \mathbf{u}, w)\varepsilon_{\alpha\beta}(\mathbf{v}') + L^{\alpha\beta}(\mathbf{v}, \mathbf{u}, w)\gamma_{\alpha\beta}(\mathbf{u}') \\ + Q^\alpha(\mathbf{u}, w)\kappa_\alpha(\mathbf{u}', w') + R(\mathbf{v}, \mathbf{u}, w)w'\Big]\, dx, \tag{2.27}$$

(where $\mathbf{N}$, $\mathbf{L}$, $\mathbf{Q}$, R depend on $(\mathbf{v}, \mathbf{u}, w)$ according to the relationships (2.21), (2.24) and (2.25)) and the linear form

$$f(\mathbf{v}', \mathbf{u}', w') = \int_{\Gamma_\sigma} \left(\overline{N}^\alpha v'_\alpha + \overline{L}^\alpha u'_\alpha - \overline{Q} w'\right) ds. \tag{2.28}$$

The variational problem in its primal formulation reads:

$$(P_\Omega) \quad \left| \begin{array}{l} \text{Find} \ \ (\mathbf{v}, \mathbf{u}, w) \in V \quad \text{such that} \\ a_\Omega(\mathbf{v}, \mathbf{u}, w; \mathbf{v}', \mathbf{u}', w') = f(\mathbf{v}', \mathbf{u}', w') \quad \forall(\mathbf{v}', \mathbf{u}', w') \in V. \end{array} \right. \tag{2.29}$$

Instead of studying problem P_Ω one can solve a more general one, that of the existence of a saddle point of the Reissner-type functional (2.15). We set

$$\mathfrak{D} = \begin{bmatrix} \mathbf{D}^N & \mathbf{D}^{NL} & \mathbf{D}^{RN} \\ \mathbf{D}^{NL} & \mathbf{D}^L & \mathbf{D}^{RL} \\ \mathbf{D}^{RN} & \mathbf{D}^{RL} & D^R \end{bmatrix}, \qquad \mathfrak{R} = (\mathbf{N}, \mathbf{L}, R), \qquad \mathfrak{R}^T = \begin{bmatrix} \mathbf{N} \\ \mathbf{L} \\ R \end{bmatrix}. \tag{2.30}$$

Thus we may write

$$W_c(x,\mathbf{N},\mathbf{L},\mathbf{Q},R) = \frac{1}{2}(\mathfrak{R}\,\mathfrak{D}\,\mathfrak{R}^T + D^Q_{\alpha\beta}Q^\alpha Q^\beta). \tag{2.31}$$

The properties of the compliance tensor (D_{ijkl}) imply that each element of the matrix $\mathfrak{D}$ belongs to $L^\infty(\Omega)$; similarly $D^Q_{\alpha\beta} \in L^\infty(\Omega)$. Further, the assumption (H) implies that there exists a constant $C > 0$ such that

$$(H_1)\quad \left| \begin{array}{l} \mathbf{K}\mathfrak{D}(x)\mathbf{K}^T \geq C\sum_{\alpha,\beta=1}^2 (A_{\alpha\beta}A_{\alpha\beta} + B_{\alpha\beta}B_{\alpha\beta} + a^2), \\ D^Q_{\alpha\beta}(x)a^\alpha a^\beta \geq \sum_{\alpha=1}^2 a^\alpha a^\alpha\,, \end{array}\right.$$

for a.e. $x \in \Omega$ and for $\mathbf{K} = (\mathbf{A},\mathbf{B},a) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^2$.

To corroborate this statement we write

$$\frac{1}{2}D_{ijkl}(x)\sigma^{ij}\sigma^{kl} = \frac{1}{2}D_{\alpha\beta\lambda\mu}(x)\sigma^{\alpha\beta}\sigma^{\lambda\mu} + D_{\alpha\beta33}(x)\sigma^{\alpha\beta}\sigma^{33} + 2D_{\alpha3\beta3}(x)\sigma^{\alpha3}\sigma^{\beta3} + \frac{1}{2}D_{3333}(x)(\sigma^{33})^2.$$

Because $[D_{ijkl}(x)]$ is positive definite for a.e. $x \in \Omega$, therefore $D_{3333}(x) > 0$ and consequently $D^R(x) > 0$ for a.e. $x \in \Omega$.

Similarly, we have

$$D_{\alpha3\beta3}(x)t^\alpha t^\beta > 0, \qquad \text{for a.e. } x \in \Omega \qquad \text{and for all } \mathbf{t} \in \mathbb{R}^2, \qquad \mathbf{t} \neq \mathbf{0}.$$

Treating $(\sigma^{\alpha3})$ and $\mathbf{Q} \neq \mathbf{0}$ in (2.8) as arbitrary, though sufficiently regular, we write

$$\int_{-h}^{h} D_{\alpha3\beta3}(x)\sigma^{\alpha3}\sigma^{\beta3}\,dz = \left(\int_{-h}^{h} f(z)D_{\alpha3\beta3}(x)\,dz\right)Q^\alpha(x)Q^\beta(x) = D^Q_{\alpha\beta}Q^\alpha(x)Q^\beta(x) > 0, \qquad \text{a.e. } x \in \Omega,$$

where $f(z)$ is inferred from (2.8). Hence the matrix $\mathbf{D}^Q$ is positive definite and the inequality $(H_1)_2$ follows, since for a fixed $x \in \Omega$ at which $\mathbf{Q}(x)$ makes sense one may set $\mathbf{a} = \mathbf{Q}(x) \in \mathbb{R}^2$ and treat $\mathbf{a}$ as an arbitrary element of $\mathbb{R}^2$.

For any sufficiently regular $\boldsymbol{\sigma} = (\sigma^{ij}) \neq \mathbf{0}$ such that $\sigma^{\alpha3} = 0$ $(\alpha = 1,2)$, one has

$$j_0(x,\sigma^{\alpha\beta},\sigma^{33}) := \frac{1}{2}D_{\alpha\beta\lambda\mu}(x)\sigma^{\alpha\beta}\sigma^{\lambda\mu} + D_{\alpha\beta33}(x)\sigma^{\alpha\beta}\sigma^{33} + \frac{1}{2}D_{3333}(x)(\sigma^{33})^2 > 0, \qquad \text{a.e. } x \in \Omega. \tag{2.32}$$

For such a $\boldsymbol{\sigma}$ we write Eqs. (2.7) and (2.9) concisely in the form

$$(\sigma^{\alpha\beta}, \sigma^{33}) = \mathbf{A}(z)\mathfrak{R}^T. \tag{2.33}$$

Hence, on account of (2.32) we obtain

$$\begin{aligned} j_c(x, \mathbf{N}(x), \mathbf{L}(x), R(x)) &:= \int\limits_{-h}^{h} j_0(x, \sigma^{\alpha\beta}(x,z), \sigma^{33}(x,z))\, dz \\ &= \int\limits_{-h}^{h} j_0(x, \mathbf{A}(z)\mathfrak{R}^T(x))\, dz = \frac{1}{2}\mathfrak{R}(x)\mathfrak{D}(x)\mathfrak{R}^T(x) > 0, \qquad \text{a e } x \in \Omega, \end{aligned} \tag{2.34}$$

provided that $\mathfrak{R}(x) \neq \mathbf{0}$.

Consequently, the matrix $\mathfrak{D}$ of the generalized elastic compliances is positive definite and the partial complementary potential j_c is strictly convex, cf. also ROCKAFELLAR's Th. 5.7 [46]. The condition $(H_1)_1$ is thus satisfied.

The regularity of the boundary loading is specified by

$$(H_2) \qquad \overline{N}^{\alpha} \in L^2(\Gamma_\sigma), \qquad \overline{L}^{\alpha} \in L^2(\Gamma_\sigma), \qquad \overline{Q} \in L^2(\Gamma_\sigma).$$

By applying ARNOLD and FALK [5] version of the BREZZI's theorem [6], one can now prove that the functional J possesses a unique saddle point.

3. Modelling transverse cracking in the internal layer

When stretched, the composite laminates undergo interlaminar delaminations, fiber breakage and intralaminar cracking. Only the last mode of damage mentioned is discussed in the present paper. The intralaminar cracks observed are not necessarily straight; they can also be curved or partially angled, cf. GROVES *et al.* [11]. The model proposed in Sec. 2 makes it possible to figure out only cracks, which are straight in the transverse direction.

In the present section we will show how to express the presence of such cracks in terms of the two-dimensional fields of our model. The cracks are allowed to open or close.

Consider the laminate weakened by a crack in the internal layer. The crack surface $S_F = F \times (-c, c)$ is perpendicular to the domain Ω, F being its projection on Ω, cf. Fig. 2. The crack surfaces observed in experiments are parallel to the fibers, cf. GARRETT and BAILEY [10]. In the present paper, however, we shall not associate the shape of the crack directions with anisotropy of the laminae. We assume simply, that the crack geometry is *a priori* given.

Let us extend the arc F so that it divides the domain Ω into two parts Ω_1, Ω_2, cf. Fig. 2. Consequently, the domain B is divided into two subdomains $B_\alpha =$

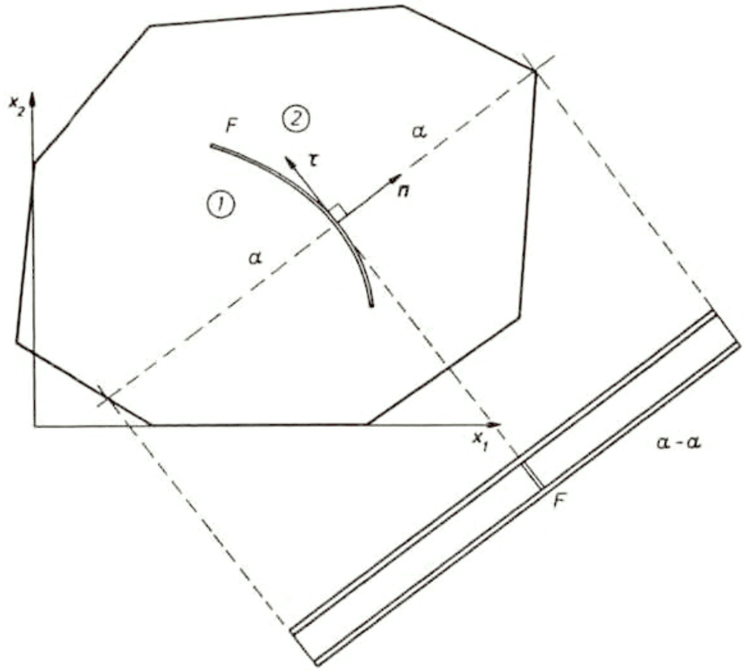

FIG. 2. Laminate with a crack in the internal layer.

$\Omega_\alpha \times (-h, h)$. Let F_{z_0} be the set of points (x, z) such that $x \in F$ and $z = z_0$. Let $\mathbf{n}^z$ be a unit vector at a point (x, z) normal to F_z and such that $\mathbf{n}^z = (n_\alpha, 0)$, where $(n_\alpha) = \mathbf{n}$ is a unit vector at $(x, 0)$ normal to F and directed outward the domain Ω_1. Similarly we define the tangent vector $\boldsymbol{\tau}^z = (\tau_\alpha, 0)$. A field g defined on $\mathcal{B}$ for $|z| < c$ can suffer a jump across the surface S_F:

$[\![g]\!]_{S_F} = g|_2 - g|_1$; here $g|_\alpha$ represents the value of g at the α-th side of S_F, viz. from the side of the domain $\mathcal{B}_\alpha$. Now we can express the contact conditions on the surface S_F of the crack as follows:

$$(3.1) \qquad \begin{aligned} &\sigma_n = \overset{1}{\sigma}_n = \overset{2}{\sigma}_n \le 0, \qquad [\![w_n]\!]_{S_F} \ge 0, \qquad \sigma_n [\![w_n]\!]_{S_F} = 0, \\ &\overset{1}{\sigma}_{n\tau} = \overset{2}{\sigma}_{n\tau} = 0, \qquad \overset{1}{\sigma}_{nz} = \overset{2}{\sigma}_{nz} = 0, \end{aligned}$$

where

$$(3.2) \qquad \begin{aligned} &\overset{\delta}{\sigma}_n = \sigma^{\alpha\beta}{}_{|\delta} n_\alpha n_\beta, \qquad \overset{\delta}{\sigma}_{n\tau} = \sigma^{\alpha\beta}{}_{|\delta} n_\alpha \tau_\beta, \\ &\overset{\delta}{\sigma}_{nz} = \sigma^{\alpha 3}{}_{|\delta} n_\alpha, \qquad w_n = \sum_\alpha w_\alpha n_\alpha, \end{aligned}$$

and

$$\sigma^{\alpha k} = \sigma^{\alpha k}(x, z), \qquad w_\alpha = w_\alpha(x, z).$$

In the Signorini-type conditions (3.1) the friction between crack lips is obviously neglected. Thus the jump $[\![w_\tau]\!]_{S_F}$ assumes arbitrary values.

According to the stress hypotheses (2.7)–(2.9) one readily finds

$$
\sigma_n = \begin{cases} \dfrac{1}{2c} L_n\,, & |z| < c, \\ \dfrac{1}{2d}(N_n - L_n), & \text{otherwise}; \end{cases}
\tag{3.3}
$$
$$
\sigma_{n\tau} = \begin{cases} \dfrac{1}{2c} L_\tau\,, & |z| < c, \\ \dfrac{1}{2d}(N_\tau - L_\tau), & \text{otherwise}; \end{cases}
$$

$$
\sigma_{nz} = \begin{cases} \dfrac{z-h}{2d} Q, & c < z < h, \\ -\dfrac{z}{2c} Q, & |z| < c, \\ \dfrac{z+h}{2d} Q, & -h < z < -c, \end{cases}
\tag{3.4}
$$

where

$$
\begin{aligned}
L_n &= L^{\alpha\beta} n_\alpha n_\beta\,, \qquad N_n = N^{\alpha\beta} n_\alpha n_\beta\,, \qquad L_\tau = L^{\alpha\beta} n_\alpha \tau_\beta\,, \\
N_\tau &= N^{\alpha\beta} n_\alpha \tau_\beta\,, \qquad Q = Q^\alpha n_\alpha\,.
\end{aligned}
\tag{3.5}
$$

In view of (2.10) and (3.3), the conditions $(3.1)_{1,2}$ can be rewritten in the form:

$$
L_n = \overset{1}{L}_n = \overset{2}{L}_n \le 0, \qquad [\![u_n]\!] \ge 0, \qquad L_n [\![u_n]\!] = 0, \qquad \overset{1}{L}_\tau = \overset{2}{L}_\tau = 0 \quad \text{on } F.
\tag{3.6}
$$

Now $[\![\cdot]\!]$ denotes a jump on F and

$$
\overset{\delta}{L}_n = L^{\alpha\beta}|_\delta n_\alpha n_\beta\,, \qquad \overset{\delta}{L}_\tau = L^{\alpha\beta}|_\delta n_\alpha \tau_\beta
\tag{3.7}
$$

are values associated with δ-th side of F; $\delta = 1, 2$. The fields $\mathbf{v}$, w are assumed not to suffer jumps on F, the jump $[\![u_\tau]\!]$ being unconstrained.

The stress assumptions (2.7)–(2.9) do not allow for constructing a two-dimensional approximation of condition $(3.1)_2$.

In the domains of the exterior layers ($c < z < h$, $-h < z < -c$) all the fields are assumed as continuous. Hence the fields N_n, N_τ, Q do not suffer jumps on F.

The strong formulation of the equilibrium problem of the plate with a crack amounts to finding the fields $(\mathbf{v}, \mathbf{u}, w)$ satisfying:

- the equilibrium equations (2.18) for $x \in \Omega \backslash F$,
- the constitutive relations (2.24) and (2.25) for $x \in \Omega \backslash F$,
- the boundary conditions (2.26) and (2.22),
- Signorini-type conditions (3.6) on F.

Let us pass now to the variational formulation of the problem governed by (2.18), (2.24) and (2.25), (2.26) and (2.22) along with (3.6). The set of kinematically admissible fields has the form

$$\mathbb{K} = H_{\Gamma_w}(\Omega)^2 \times K(\Omega\backslash F) \times H_{\Gamma_w}(\Omega), \tag{3.8}$$

where

$$H_{\Gamma_w}(\Omega) = \left\{ \mathbf{v} \in H^1(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma_w \right\}, \tag{3.9}$$

$$K(\Omega\backslash F) = \left\{ \mathbf{u} \in H^1(\Omega\backslash F)^2 \mid \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_w \quad \text{and} \quad [\![u_n]\!] \geq 0 \quad \text{on} \quad F \right\}. \tag{3.10}$$

The variational problem reads:

$$(P_{\Omega\backslash F}) \left| \begin{array}{l} \text{Find} \quad (\mathbf{v}, \mathbf{u}, w) \in \mathbb{K} \quad \text{such that} \\ a_{\Omega\backslash F}(\mathbf{v}, \mathbf{u}, w; \mathbf{v}', \mathbf{u}' - \mathbf{u}, w') \geq f(\mathbf{v}', \mathbf{u}' - \mathbf{u}, w') \;\; \forall (\mathbf{v}', \mathbf{u}', w') \in \mathbb{K}. \end{array} \right. \tag{3.11}$$

The bilinear form $a_{\Omega\backslash F}(.,.)$ is defined by Eq. (2.27) with integration now over $\Omega\backslash F$.

By applying Th. 2.1 of KINDERLEHRER and STAMPACCHIA [21, Chap. II] we will prove that the variational inequality (3.11) admits a unique solution. In fact, in our case the set $\mathbb{K}$ is convex and closed and the linear form f, given by (2.28), is continuous in the space

$$V(\Omega\backslash F) = H_{\Gamma_w}(\Omega)^2 \times H_{\Gamma_w}(\Omega\backslash F)^2 \times H_{\Gamma_w}(\Omega),$$

and also on $L^2(\Omega)^2 \times L^2(\Omega\backslash F)^2 \times L^2(\Omega)$. It remains to verify that the bilinear form $a_{\Omega\backslash F}$ is coercive. Toward this end we set

$$\mathfrak{A} = \mathfrak{D}^{-1} = \begin{bmatrix} \mathbf{A}_v & \mathbf{A}_{vu} & \mathbf{A}_{vw} \\ \mathbf{A}_{vu} & \mathbf{A}_u & \mathbf{A}_{uw} \\ \mathbf{A}_{vw} & \mathbf{A}_{uw} & A_w \end{bmatrix}, \qquad \mathbf{H} = [H^{\alpha\beta}] = (\mathbf{D}^Q)^{-1}. \tag{3.12}$$

The explicit form of the stiffness matrix $\mathfrak{A}$ can be found by using the Fenchel conjugate of $j_c(x,.,.,.)$, i.e.

$$\begin{aligned} j_1(x, \boldsymbol{\varepsilon}, \boldsymbol{\gamma}, r) &:= j_c^*(x, \boldsymbol{\varepsilon}, \boldsymbol{\gamma}, r) \\ &= \sup\left\{ N^{\alpha\beta}\varepsilon_{\alpha\beta} + L^{\alpha\beta}\gamma_{\alpha\beta} + Rr - j_c(x, \mathbf{N}, \mathbf{L}, R) \mid (\mathbf{N}, \mathbf{L}, R) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R} \right\} \\ &= \frac{1}{2}\mathbf{E}\mathfrak{A}(x)\mathbf{E}^T, \end{aligned} \tag{3.13}$$

where $\mathbf{E} = (\boldsymbol{\varepsilon}, \boldsymbol{\gamma}, r) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}$. The properties of the compliance matrices $\mathfrak{D}$ and $\mathbf{D}^Q$ imply that a positive constant $C_1 > 0$ exists such that for a.e. $x \in \Omega$

$$\mathbf{E}\mathfrak{A}(x)\mathbf{E}^T \geq C_1\left(|\boldsymbol{\varepsilon}|^2 + |\boldsymbol{\gamma}|^2 + r^2\right), \qquad H^{\alpha\beta}(x)a_\alpha a_\beta \geq C_1|\mathbf{a}|^2, \tag{3.14}$$

for all $\mathbf{E} = (\boldsymbol{\varepsilon}, \boldsymbol{\gamma}, r) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^2$. By taking now into account (3.14), we obtain, for any $(\mathbf{v}', \mathbf{u}', w') \in V(\Omega \backslash F)$

$$a_{\Omega\backslash F}(\mathbf{v}', \mathbf{u}', w'; \mathbf{v}', \mathbf{u}', w') \geq \widetilde{C}_1 \Big(\|\boldsymbol{\varepsilon}(\mathbf{v}')\|^2_{0,\Omega} + \|\boldsymbol{\gamma}(\mathbf{u}')\|^2_{0,\Omega\backslash F} + \|w'\|^2_{0,\Omega} + \|\nabla w' - \mathbf{u}'\|^2_{0,\Omega} \Big) \geq C_1 \left(\|\mathbf{v}'\|^2_{1,\Omega} + \|\mathbf{u}'\|^2_{1,\Omega\backslash F} + \|w'\|^2_{1,\Omega} \right),$$

because Korn's inequality still holds for the domain $\Omega \backslash F$, cf. SANCHEZ–PALENCIA [47]. Consequently, the problem $P_{\Omega\backslash F}$ admits a unique solution $(\mathbf{v}, \mathbf{u}, w) \in \mathbb{K}$. It is worth noting that unique solvability is preserved in the practically important case, when F intersects the boundary Γ of Ω, cf. also CHACHA and SANCHEZ–PALENCIA [8]. The subdomains Ω_α ($\alpha = 1, 2$), however, have to be sufficiently regular.

4. Final remarks

The two-dimensional model of an elastic three-layer laminate, proposed in Sec. 2, has been shown to readily include transverse intralaminar cracks in the internal layer. In the last case, the equilibrium problem, written in the variational (weak) form, is described by variational inequality (3.11), provided that friction is precluded. In Part II of the paper [35], just this variational inequality will be of fundamental importance for modelling the macroscopic behaviour of laminates with periodically distributed transverse cracks in the internal layer.

The starting point of the modelling performed in Part I are stress-displacement assumptions (2.7)–(2.9). The modelling can be repeated without imposing the displacement fields if one appropriately interprets the Lagrangian multipliers, similarly as it was done by REISSNER [42, 43] in the case of bending problems. Then, however, the distribution of displacements $w_k(x, z)$ across the thickness cannot be uniquely recovered, which would result in ambiguities in expressing conditions on the crack lips in the internal layer. The model construction of Sec. 2 can be based upon displacement asssumptions only, provided that they are sufficiently "rich" to figure out the influence of the stresses σ^{k3}. Such displacement fields can be found in MCCARTNEY [37]. On the other hand, they cannot be too complicated, since the variational method based on displacement approach would only increase the grade of complexity of the model – a known dilemma in plate bending modelling – and the energy-inconsistent approaches, like that of LEVINSON [26] would here be impractical. One of the simplest possible forms of such assumptions has been adopted by HAN *et al.* [16], HAN and HAHN [15] and YOUNG and BOEHLER [58, Eq. (31)]. Subsequent steps of the modelling would be similar. Nevertheless, the conclusions drawn from the bending theory of plates are such that the theories based upon stress assumptions lead directly to well-assessed stiffnesses, while the same accuracy is difficult to achieve via the displacement-based models unless artificial "correction factors" are adopted.

Acknowledgement

The authors were supported by the State Committee for Scientific Research through the grants No 300199101 and No 3 P404 013 06.

References

1. J. Aboudi, *Stiffness reduction of cracked solids*, Engng. Fract. Mech., **26**, 637–650, 1987.
2. J. Aboudi and Y. Benveniste, *The effective moduli of cracked bodies in plane deformations*, Engng. Fract. Mech., **26**, 171–184, 1987.
3. J. Aboudi, S.W. Lee and C.T. Herakovich, *Three-dimensional analysis of laminates with cross cracks*, J. Appl. Mech. Trans. ASME, **55**, 389–397, 1988.
4. S. Abrate, *Impact on laminate composite materials*, Appl. Mech. Rev., **44**, 155–190, 1991.
5. D.N. Arnold and R.S. Falk, *Well-posedness of the fundamental boundary value problems for constrained anisotropic elastic materials*, Arch. Rat. Mech. Anal., **98**, 143–165, 1987.
6. F. Brezzi, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, RAIRO, Anal. Numer., **8**, 129–154, 1974.
7. D. Caillerie, *Thin elastic and periodic plates*, Math. Meth. Appl. Sci., **6**, 159–191, 1984.
8. D. Chacha and E. Sanchez-Palencia, *Overall behaviour of elastic plates with periodically distributed fissures*, Asympt. Anal., **5**, 381–396, 1992.
9. L. Gambarotta and S. Lagomarsino, *A microcrack damage model for brittle materials*, Int. J. Solids Struct., **30**, 177–198, 1977.
10. K.W. Garrett and J.E. Bailey, *Multiple transverse fracture in 90° cross-ply laminates of a glass fibre-reinforced polyester*, J. Mater. Sci., **12**, 157–168, 1977.
11. S.E. Groves, C.E. Harris, A.L. Highsmith, D.H. Allen and R.G. Norvell, *An experimental and analytical treatment of matrix cross-ply laminates*, Exper. Mech., **27**, 73–79, 1987.
12. P. Gudmundson and S. Östlund, *First order analysis of stiffness reduction due to matrix cracking*, J. Compos. Mater., **26**, 1009–1030, 1992.
13. P. Gudmundson and S. Östlund, *Numerical verification of a procedure for calculation of elastic constants in microcracking composite laminates*, J. Compos. Mater., **26**, 2480–2492, 1992.
14. P. Gudmundson and W. Zang, *An analytic model for thermoelastic properties of composite laminates containing transverse matrix cracks*, Int. J. Solids Struct., **30**, 3211–3231, 1993.
15. Y.M. Han and H.T. Hahn, *Ply cracking and property degradations of symmetric balanced laminates under general in-plane loading*, Compos. Sci. Technol., **35**, 377–397, 1989.
16. Y.M. Han, H.T. Hahn and R.B. Croman, *A simplified analysis of transverse ply cracking in cross-ply laminates*, Compos. Sci. Technol., **31**, 165–177, 1988.
17. Z. Hashin, *Analysis of cracked laminates: a variational approach*, Mech. Mater., **4**, 121–136, 1985.
18. Z. Hashin, *Analysis of orthogonally cracked laminates under tension*, J. Appl. Mech. Trans. ASME, **54**, 872–879, 1987.
19. A.L. Highsmith and K.L. Reifsnider, *Stiffness reduction mechanisms in composite materials*, [in:] Damage in Composite Materials, K.L. Reifsnider [Ed.], ASTM STP, 775, 103–117, 1982.
20. P.I. Kattan and G.Z. Voyiadjis, *Micromechanical modelling of damage in uniaxially loaded unidirectional fibre-reinforced composite laminae*, Int. J. Solids Struct., **30**, 19–36, 1993.
21. D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York 1980.
22. R.V. Kohn and M. Vogelius, *A new model for thin plates with rapidly varying thickness*, Int. J. Solids Struct., **20**, 333–350, 1984.
23. R.V. Kohn and M. Vogelius, *A new model for thin plates with rapidly varying thickness, II. A convergence proof*, Q. Appl. Math., **43**, 1–22, 1985.
24. R.V. Kohn and M. Vogelius, *A new model for thin plates with rapidly varying thickness. III. Comparison of different scalings*, Q. Appl. Math., **44**, 35–48, 1986.

25. J.-W. LEE, D.H. ALLEN and C.E. HARRIS, *Internal state variable approach for predicting stiffness reductions in fibrous laminated composites with matrix cracks*, J. Compos. Mater., **23**, 1273–1291, 1989.

26. M. LEVINSON, *An accurate simple theory of the statics and dynamics of elastic plates*, Mech. Res. Comm., **7**, 343–350, 1980.

27. T. LEWIŃSKI, *Effective models of composite periodic plates. I. Asymptotic solution; II. Simplifications due to symmetries; III. Two-dimensional approaches*, Int. J. Solids Struct., **27**, 1155–1172, 1173–1184, 1185–1203, 1991.

28. T. LEWIŃSKI, *Homogenizing stiffnesses of plates with periodic structure*, Int. J. Solids Struct., **29**, 309–326, 1992.

29. T. LEWIŃSKI and J.J. TELEGA, *Asymptotic method of homogenization of fissured elastic plates*, J. Elasticity, **19**, 37–62, 1988.

30. T. LEWIŃSKI and J.J. TELEGA, *Homogenization of fissured Reissner-like plates. I. Method of two-scale asymptotic expansions; III. Some particular cases and an illustrative example*, Arch. Mech., **40**, 97–117; 295–303, 1988.

31. T. LEWIŃSKI and J.J. TELEGA, *Homogenization and effective properties of plates weakened by partially penetrating fissures: asymptotic analysis*, Int. J. Engng. Sci., **29**, 1129–1155, 1991.

32. T. LEWIŃSKI and J.J. TELEGA, *Stiffness loss of cracked laminates*, [in:] Proc. of the Xth Polish Conference "Computer Methods in Mechanics" Świnoujście, Poland, E. BIELEWICZ [Ed.], May 1991, pp. 477–483, 1991.

33. T. LEWIŃSKI and J.J. TELEGA, *Assessing stiffness loss of cracked laminates. The homogenization method*, ZAMM, **72**, T161–T164, 1992.

34. T. LEWIŃSKI and J.J. TELEGA, *Effective properties of cracked cross-ply laminates*, Proc. 7th Int. Symp. on Continuum Models of Discrete Systems, Paderborn, Germany, Materials Sci. Forum, **123–125**, pp. 515–524, 1993.

35. T. LEWIŃSKI and J.J. TELEGA, *Stiffness loss in laminates with intralaminar cracks. Part II. Periodic distribution of cracks and homogenization*, Arch. Mech., **48**, 1, 163–190, 1996.

36. T. LEWIŃSKI and J.J. TELEGA, *Stiffness loss of laminates with aligned intralaminar cracks, Part I. Macroscopic constitutive relations, Part II. Comparisons*, Arch. Mech., [in press].

37. L.N. MCCARTNEY, *Theory of stress transfer in* $0° - 90° - 0°$ *cross-ply laminate containing a parallel array of transverse cracks*, J. Mech. Phys. Solids, **40**, 27–68, 1992.

38. L.N. MCCARTNEY, *The prediction of cracking in biaxially loaded cross-ply laminates having brittle matrices*, Composites, **24**, 84–92, 1993.

39. L.N. MCCARTNEY, *Analytical micromechanical models for the prediction of multiple cracking in composites*, J. de Physique, **3**, 7, 1637–1646, 1993.

40. J.A. NAIRN, *The strain energy release rate of composite microcracking: a variational approach*, J. Compos. Mater., **23**, 1106–1129, 1989.

41. J. NEČAS and I. HLAVAČEK, *Mathematical theory of elastic and elasto-plastic bodies. An introduction*, Elsevier, Amsterdam 1981.

42. E. REISSNER, *The effect of transverse shear deformation on the bending of elastic plates*, J. Appl. Mech., Trans. ASME, **12**, A69–A77, 1945.

43. E. REISSNER, *On a variational theorem in elasticity*, J. Math. Phys., **29**, 90–95, 1950.

44. E. REISSNER, *A twelfth order theory of transverse bending of transversely isotropic plates*, ZAMM, **63**, 285–289, 1983.

45. E. REISSNER, *Reflections on the theory of elastic plates*, Appl. Mech. Rev., **38**, 1453–1464, 1985.

46. R.T. ROCKAFELLAR, *Convex analysis*, Princeton University Press., Princeton 1970.

47. E. SANCHEZ-PALENCIA, *Non-homogeneous media and vibration theory*, Springer Verlag, Berlin 1980.

48. P. SUQUET, *Une méthode duale en homogénéisation: application aux milieux élastiques*, J. de Méc. Théor. Appl., Numero spécial, pp. 79–98, 1982.

49. J.J. TELEGA, *Homogenization of fissured elastic solids in the presence of unilateral conditions and friction*, Comput. Mech., **6**, 109–127, 1990.

50. J.J. TELEGA, *Justification of refined scaling of stiffnesses of Reissner plates with fine periodic structure*, Math. Models Meth. Appl. Sci., **2**, 375–406, 1992.

51. J.J. TELEGA, *Homogenization and effective properties of plates weakened by partially penetrating fissures: convergence and duality*, Math. Modell. Num. Anal., **27**, 421–456, 1993.

52. J.J. Telega and T. Lewiński, *Homogenization of fissured Reissner-like plates. II. Convergence*, Arch. Mech., **40**, 119–134, 1988.

53. J.J. Telega and T. Lewiński, *Stiffness loss of cross-ply laminates with intralaminar cracks*, MECAMAT 93; Int. Seminar on Micromechanics of Materials, Editions Eyrolles, Paris, pp. 317–326, 1993.

54. H. Teng, *Effective longitudinal shear modulus of a unidirectional fiber composite containing interfacial cracks*, Int. J. Solids Struct., **29**, 1581–1595, 1992.

55. A. Thionnet, *Prévision d'endommagement sous chargements quasi-statiques et cycliques des structures composites stratifieés*, Note Technique 1991-4, Office National d'Études et de Recherches Aérospatiales, 1991.

56. C.-L. Tsai and I.M. Daniel, *The behaviour of cracked cross-ply composite laminates under shear loading*, Int. J. Solids Struct., **29**, 3251–3267, 1992.

57. S. Yalvaç, L.D. Yats and D.G. Wetters, *Transverse ply cracking in toughened and untoughened graphite/epoxy and graphite/polycyanate cross-ply laminates*, J. Compos. Mater., **25**, 1653–1667, 1991.

58. W. Yang and J.-P. Boehler, *Micromechanics modelling of anisotropic damage in cross-ply laminates*, Int. J. Solids Struct., **29**, 1303–1328, 1992.

WARSAW UNIVERSITY OF TECHNOLOGY
CIVIL ENGINEERING FACULTY
INSTITUTE OF STRUCTURAL MECHANICS
and
POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received June 2, 1995.