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The application of the boundary element method to the study of 2D subsonic lifting flow past smooth profiles

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In this paper, using the Imai–Lamla–Iacob method based on the asymptotic expansion of the complex velocity potential with respect to Chaplygin's number M_0, we study the subsonic steady 2D lifting flow around arbitrary smooth obstacles. Reducing the investigation of the compressible flow to a sequence of boundary value problems for the Laplace equation, we use the boundary element method in order to calculate the distribution of fluid speed and pressure coefficients on the obstacle. A comparison between the results obtained numerically and analytically for the circular obstacle shows a very good agreement.

Notations

$\mathbf{v} = (u, v)$	velocity,
V	local flow speed,
V_0	flow speed at infinity,
c_0	speed of the sound corresponding to null flow speed,
$M_0 = V_0/c_0$	Chaplygin's number,
γ	ratio of specific heats,
ϱ	density of fluid,
ϱ_0	density corresponding to null flow speed,
φ	potential of the velocity,
ψ	stream function,
Ψ	perturbation stream function,
$f = \varphi + i\psi$	complex potential,
f_0, f_1	terms of the asymptotic expansion of the complex potential,
Ψ_0	perturbation stream function for the incompressible approximation,
(x, y)	Cartesian coordinates,
$z = x + iy$	complex variable,
α	angle of attack,
$\mathbf{n} = (n_x, n_y)$	unit inward normal on the obstacle,
$\mathbf{s}$	unit tangent on the obstacle,
s	arc length on the obstacle,
Γ	obstacle,
ℓ	length of Γ,
Γ_j	panels on the obstacle,
ℓ_j	length of Γ_j,
(x_j, y_j)	control points (nodes) on Γ,
(ξ, η)	current variable on Γ,

$\mathbf{v}_0 = (u_0, v_0)$ velocity for the incompressible approximation,
k_0 circulation corresponding to the incompressible approximation,
C_p pressure coefficient,
C_L lift coefficient,

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2},$$

$$r_i = \sqrt{(x_i-\xi)^2 + (y_i-\eta)^2}.$$

1. Introduction

THE PAPER is devoted to the numerical study of 2d potential subsonic steady flow of ideal fluids past smooth obstacles. We utilise herein the boundary element method which is more economical from the computational point of view than the domain-type methods like the finite-element method, finite-difference method etc.

The boundary element (or panel) method is widely and currently utilised for studying the potential incompressible flow past obstacles, because the equations governing the flow are linear and one can obtain representations of the solutions involving only integrals on the boundary of the domain. For the nonlinear equations governing the compressible flow such integral representations are not known.

In order to avoid this inconvenience, we shall use an approximate method conceived by I. IMAI [8], E. LAMLA [9] and improved by C. IACOB [7]. In the framework of this method, we consider the asymptotic expansion of the complex potential (and implicitly of the complex velocity) with respect to Chaplygin's number M_0.

For the first approximation corresponding to the incompressible flow, one has to solve, using the boundary element technique, the Neumann problem for Laplace's equation. For the second approximation, one utilises again the integral representation for the harmonic functions, but the boundary conditions depend on the results of the previous approximation and so on. In this way the nonlinear boundary value problem was replaced by a sequence of linear problems.

Until now the Imai-Lamla-Iacob approximate method was utilised especially for obtaining analytical results concerning the flow past obstacles [2, 5, 10, 11].

In order to establish if this method is satisfactory, some comparisons with other methods were performed. G. VOICULESCU-PLESI [11] compared the analytical results obtained for the circulation-free flow past the elliptical obstacle with the results obtained by I. FILIMON [4], who used Chaplygin's approximate hodograph method. A. DUCARU-DRAGA [2] studied the subsonic flow with circulation around the circular obstacle, both by means of the Imai-Lamla-Iacob method and the finite-element method. In all cases it is observed that the results obtained by means of Imai-Lamla-Iacob method and the results obtained by means of other methods are very close to each other.

In the present paper, using the first and second approximations, we investigate the subsonic flow with circulation past arbitrary smooth obstacles. Numerical results (tangential velocity and pressure coefficients in the control points)are obtained for the circular obstacle.

In the framework of the second asymptotic approximation, we know the analytical expression of the tangential velocity for the subsonic compressible flow with circulation past the circular obstacle [1]. The comparisons between the values of the pressure coefficients calculated analytically and by means of the boundary element method show a very good agreement, as we can observe from Fig. 1.

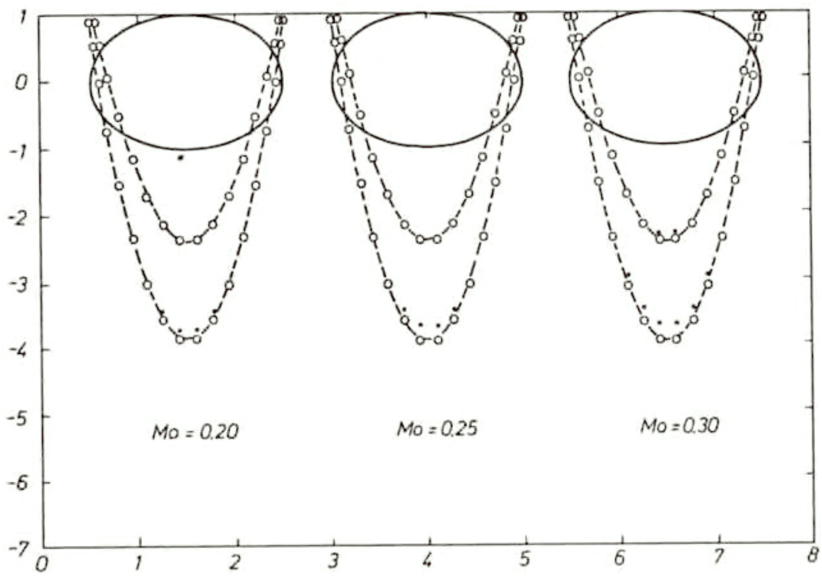

FIG. 1. Calculated (o) and analytical (– –) chordwise coefficient. • • • pressure coefficient for incompressible flow.

2. Imai–Lamla–Iacob approximate method

Following C. IACOB [7] we shall present the method conceived by I. IMAI [8] and E. LAMLA [8] for investigating the subsonic circulation-free flow and adjusted by C. Iacob for the study of the flow with circulation past obstacles.

From the equation of continuity

$$\operatorname{div}(\varrho \mathbf{v}) = 0 \tag{2.1}$$

it follows that there exists a function $\psi(x, y)$ (the stream function) so that

$$u = \frac{\varrho_0}{\varrho}\frac{\partial \psi}{\partial y}, \qquad v = -\frac{\varrho_0}{\varrho}\frac{\partial \psi}{\partial x}. \tag{2.2}$$

We consider the irrotational flow, i.e. there exists a function $\varphi(x,y)$ (the velocity potential) such that

(2.3) $$u = \frac{\partial\varphi}{\partial x}, \qquad v = \frac{\partial\varphi}{\partial y}.$$

Introducing the operators,

(2.4) $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and the complex potential

(2.5) $$f = \varphi + i\psi$$

we get from (2.2)–(2.5)

(2.6) $$\frac{\partial f}{\partial \overline{z}} = \frac{\varrho_0 - \varrho}{\varrho_0 + \varrho}\frac{\partial \overline{f}}{\partial \overline{z}}.$$

Taking into account that for the isentropic flow,

$$\varrho = \varrho_0\left(1 - \frac{\gamma - 1}{2}\frac{V^2}{V_0^2}M_0^2\right)^{1/(\gamma-1)},$$

it results

(2.7) $$\frac{\varrho_0 - \varrho}{\varrho_0 + \varrho} = \frac{M_0^2}{4}\frac{V^2}{V_0^2} + M_0^4(\ldots).$$

Expressing the local speed of the fluid as follows

(2.8) $$V^2 = \left(\frac{\partial f}{\partial z} + \frac{\partial \overline{f}}{\partial z}\right)\left(\frac{\partial f}{\partial \overline{z}} + \frac{\partial \overline{f}}{\partial \overline{z}}\right),$$

we deduce from (2.6)–(2.8)

(2.9) $$\frac{\partial f}{\partial \overline{z}} = \left[\frac{M_0^2}{4V_0^2}\left(\frac{\partial f}{\partial z} + \frac{\partial \overline{f}}{\partial z}\right)\left(\frac{\partial f}{\partial \overline{z}} + \frac{\partial \overline{f}}{\partial \overline{z}}\right) + M_0^4(\ldots)\right]\frac{\partial \overline{f}}{\partial \overline{z}}.$$

Considering the asymptotic expansion of f with respect to M_0^2

(2.10) $$f(z,\overline{z}) = f_0(z,\overline{z}) + M_0^2 f_1(z,\overline{z}) + M_0^4 f_2(z,\overline{z}) + \ldots$$

we obtain from (2.9), (2.10), equating the coefficients of the same powers of M_0

$$\frac{\partial f_0}{\partial \overline{z}} = 0, \tag{2.11}$$

$$\frac{\partial f_1}{\partial \overline{z}} = \frac{1}{4V_0^2}\left(\frac{\partial f_0}{\partial z} + \frac{\partial \overline{f}_0}{\partial z}\right)\left(\frac{\partial f_0}{\partial \overline{z}} + \frac{\partial \overline{f}_0}{\partial \overline{z}}\right)\frac{\partial \overline{f}_0}{\partial \overline{z}}, \tag{2.12}$$

and so on.

From (2.4), (2.5) and (2.11) it follows that f_0 is an analytical function depending only on z. We shall consider $f_0(z)$ as the complex potential of the incompressible flow past the given obstacle.

From (2.11) and (2.12) it results

$$\frac{\partial f_1}{\partial \overline{z}} = \frac{1}{4V_0^2}\frac{df_0}{dz}\left(\frac{d\overline{f}_0}{d\overline{z}}\right)^2 \tag{2.13}$$

whence, integrating with respect to $\overline{z}$, we get

$$f_1(z,\overline{z}) = \varphi_1 + i\psi_1 = \frac{1}{4V_0^2}\frac{df_0}{dz}\overline{\int_{z_1}^{z}\left(\frac{df_0}{dz}\right)^2 dz} + \frac{1}{4}g(z), \tag{2.14}$$

where $g(z)$ is an analytic function. Similarly, we can calculate $f_2(z,\overline{z})$ and so on, but in this paper we shall use only the second approximation (i.e. we deal only with f_0 and f_1). From the relation

$$u - iv = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial y} = \frac{\partial f}{\partial z} + \frac{\partial \overline{f}}{\partial z} \tag{2.15}$$

and from (2.9)–(2.14), we deduce

$$u - iv = \frac{df_0}{dz} + \frac{M_0^2}{4}\left[\frac{1}{V_0^2}\frac{d^2 f_0}{dz^2}\overline{\int_{z_1}^{z}\left(\frac{df_0}{dz}\right)^2 dz} + \frac{1}{V_0^2}\left(\frac{df_0}{dz}\right)^2\frac{d\overline{f}_0}{d\overline{z}} + \frac{dg}{dz}\right] + M_0^4(\ldots). \tag{2.16}$$

In the sequel we shall neglect the terms of order M_0^4.

The conditions that we impose are:

- at infinity,

$$\lim_{\infty}(u - iv) = V_0 e^{-i\alpha}, \qquad \lim_{\infty}(u_0 - iv_0) = V_0 e^{-i\alpha}, \tag{2.17}$$

• on the solid boundaries,

(2.18) $$\psi|_{\Gamma} = q_0 + \frac{M_0^2}{4} q_1 , \qquad \psi_0|_{\Gamma} = q_0 ,$$

(i.e. the solid boundaries are streamlines) and

(2.19) $$un_x + vn_y|_{\Gamma} = 0, \qquad u_0 n_x + v_0 n_y|_{\Gamma} = 0$$

(the slip condition).

3. A boundary element approach to investigation of the incompressible lifting flow past smooth obstacles

Considering the flow uniform at infinity and incompressible, the behaviour of the stream function is

(3.1) $$\psi_0 = V_0(y\cos\alpha - x\sin\alpha) + \frac{k_0}{2\pi i}\ln\sqrt{x^2+y^2} + O\left(\frac{1}{\sqrt{x^2+y^2}}\right),$$

since $\sqrt{x^2+y^2} \Rightarrow \infty$.

The perturbation stream function for the incompressible flow

(3.2) $$\Psi_0(x,y) = \psi_0(x,y) - V_0(y\cos\alpha - x\sin\alpha)$$

behaves at infinity as follows:

(3.3) $$\Psi_0 = \frac{k_0}{2\pi i}\ln\sqrt{x^2+y^2} + O\left(\frac{1}{\sqrt{x^2+y^2}}\right), \qquad \sqrt{x^2+y^2} \Rightarrow \infty .$$

From (3.3) it results, via Plemelj's formula, that the harmonic function $\Psi_0(x,y)$ has for $(x,y) \in \Gamma$ the integral representation

(3.4) $$\frac{1}{2}\Psi_0(x,y) = \frac{1}{2\pi}\int_{\Gamma}\left(\frac{\partial\Psi_0}{\partial n}(\xi,\eta)\ln\frac{1}{r} - \Psi_0(\xi,\eta)\frac{\partial}{\partial n}\ln\frac{1}{r}\right)ds,$$

where (ξ,η) is the current variable on Γ, ds is the element of arc length, $\partial/\partial n$ is the inward normal derivative and $r^2 = (x-\xi)^2 + (y-\eta)^2$.

In the boundary element approach, the airfoil is approximated by a piecewise linear curve consisting of N panels Γ_j, $j = 1, N$; the extremes (nodes or control points) are found on the actual airfoil. For the i-th node Eq. (3.4) becomes

(3.5) $$\frac{1}{2}\Psi_0(x_i,y_i) = \frac{1}{2\pi}\int_{\Gamma}\left(\frac{\partial\Psi_0}{\partial n}(\xi,\eta)\ln\frac{1}{r_i} - \Psi_0(\xi,\eta)\frac{\partial}{\partial n}\ln\frac{1}{r_i}\right)ds$$

with $r_i = \sqrt{(x_i-\xi)^2 + (y_i-\eta)^2}$.

Denoting by (x_j, y_j) and (x_{j+1}, y_{j+1}) the extremes of the panel Γ_j, the current point $(\xi, \eta) \in \Gamma_j$ may be represented as follows

$$(3.6) \qquad (\xi, \eta) = \frac{1-\sigma}{2}(x_j, y_j) + \frac{1+\sigma}{2}(x_{j+1}, y_{j+1}), \qquad \sigma \in [-1, 1].$$

For Ψ_0 and $\partial\Psi_0/\partial n$ on Γ_j we consider the linear interpolation

$$(3.7) \qquad \Psi_0(\xi, \eta) = \frac{1-\sigma}{2}\Psi_0(x_j, y_j) + \frac{1+\sigma}{2}\Psi_0(x_{j+1}, y_{j+1}), \qquad \sigma \in [-1, 1],$$

$$(3.8) \qquad \frac{\partial\Psi_0}{\partial n}(\xi, \eta) = \frac{1-\sigma}{2}\,\frac{\partial\Psi_0}{\partial n}(x_j, y_j) + \frac{1+\sigma}{2}\,\frac{\partial\Psi_0}{\partial n}(x_{j+1}, y_{j+1}), \qquad \sigma \in [-1, 1].$$

For smooth obstacles, the approximation of Ψ_0 and $\partial\Psi_0/\partial n$ by piecewise constant functions gives also good results; we prefer, however, the linear interpolation because it is more general and it allows for the implementation of the Kutta-Joukovsky condition in the case of airfoils with sharp trailing edge [6].

The linear element on Γ_j is

$$(3.9) \qquad ds = \sqrt{dx^2 + dy^2} = \frac{\ell_j}{2}d\alpha,$$

where

$$(3.10) \qquad \ell_j = \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2}$$

is the length of Γ_j.

On Γ_j we have also,

$$(3.11) \qquad \frac{\partial}{\partial n}\ln\frac{1}{r_i} = \frac{-\left(\frac{1+\sigma}{2}x_{j+1} + \frac{1-\sigma}{2}x_j - x_i\right)n_x^j}{r_{ji}^2} + \frac{-\left(\frac{1+\sigma}{2}y_{j+1} + \frac{1-\sigma}{2}y_j - y_i\right)n_y^j}{r_{ji}^2}$$

with

$$(3.12) \qquad n_x^j = \frac{y_j - y_{j+1}}{\ell_j},$$

$$(3.13) \qquad n_y^j = \frac{x_{j+1} - x_j}{\ell_j},$$

$$(3.14) \qquad r_{ji}^2 = \left(\frac{1+\sigma}{2}x_{j+1} + \frac{1-\sigma}{2}x_j - x_i\right)^2 + \left(\frac{1+\sigma}{2}y_{j+1} + \frac{1-\sigma}{2}y_j - y_i\right)^2.$$

The contour Γ_j being closed, the subscript $N+1$ is identified with 1 and the subscript 0 is identified with N.

From (3.6)–(3.14) we deduce that equation (3.5) may be written as follows:

$$(3.15)\qquad \sum_{j=1}^{N}\frac{\partial \Psi_0}{\partial n}(x_j,y_j)G_{ji}-\sum_{j=1}^{N}\Psi_0(x_j,y_j)H_{ji}=0,\qquad i=1,N,$$

where

$$(3.16)\qquad G_{ji}=g_{ji}^{(1)}+g_{ji}^{(2)},\qquad i,j=1,N,$$

$$(3.17)\qquad g_{ji}^{(1)}=\frac{-\ell_j}{8\pi}\int_{-1}^{1}(1-\sigma)\ln r_{ji}\,d\sigma,$$

$$(3.18)\qquad g_{ji}^{(2)}=\frac{-\ell_{j-1}}{8\pi}\int_{-1}^{1}(1+\sigma)\ln r_{(j-1)i}\,d\sigma,$$

$$(3.19)\qquad H_{ji}=h_{ji}^{(1)}+h_{ji}^{(2)}+\frac{1}{2}\delta_{ji},$$

$$(3.20)\qquad h_{ji}^{(1)}=\frac{-\ell_j}{8\pi}\int_{-1}^{1}(1-\sigma)\left[\frac{\left(\frac{1+\sigma}{2}x_{j+1}+\frac{1-\sigma}{2}x_j-x_i\right)n_x^j}{r_{ji}^2}+\frac{\left(\frac{1+\sigma}{2}y_{j+1}+\frac{1-\sigma}{2}y_j-y_i\right)n_y^j}{r_{ji}^2}\right]d\alpha,$$

$$(3.21)\qquad h_{ji}^{(2)}=\frac{-l_{j-1}}{8\pi}\int_{-1}^{1}(1-\sigma)\left[\frac{\left(\frac{1+\sigma}{2}x_{j+1}+\frac{1-\sigma}{2}x_j-x_i\right)n_x^j}{r_{(j-1)i}^2}+\frac{\left(\frac{1+\sigma}{2}y_{j+1}+\frac{1-\sigma}{2}y_j-y_i\right)n_y^j}{r_{(j-1)i}^2}\right]d\alpha.$$

The integrals (3.20), (3.21) may be computed analytically using the relations

$$(3.22)\qquad I_1=\int_{-1}^{1}\frac{d\sigma}{a\sigma^2+b\sigma+c}=\frac{2}{\sqrt{-\Delta}}\operatorname{atan}\frac{\sqrt{-\Delta}}{c-a},\qquad \Delta=b^2-4ac,$$

$$(3.23)\qquad I_2=\int_{-1}^{1}\frac{\sigma\,d\sigma}{a\sigma^2+b\sigma+c}=\frac{-b}{a\sqrt{-\Delta}}\operatorname{atan}\frac{\sqrt{-\Delta}}{c-a}+\frac{1}{2a}\ln\frac{a+b+c}{a-b+c}.$$

For calculating (3.17), (3.18) we use the formulas

$$(3.24)\qquad I_3 = \int_{-1}^{1} \ln(a\sigma^2 + b\sigma + c)\,d\sigma = \ln[(a+b+c)(a-b+c)] - 4 + bI_2 + 2cI_1\,,$$

$$(3.25)\qquad I_4 = \int_{-1}^{1} \sigma \ln(a\sigma^2 + b\sigma + c)\,d\sigma$$

$$= \frac{1}{2a}\left[(a+b+c)\ln(a+b+c) - (a-b+c)\ln(a-b+c) - 2b\right] - \frac{b}{2a} I_3\,.$$

For $i = j-1,\ j,\ j+1$ we get for the singular integrals occurring in (3.17), (3.18)

$$(3.26)\qquad g_{jj}^{(1)} = \frac{\ell_j}{8\pi}(3 - 2\ln \ell_j),$$

$$(3.27)\qquad g_{j(j+1)}^{(1)} = \frac{\ell_j}{8\pi}(1 - 2\ln \ell_j),$$

$$(3.28)\qquad g_{jj}^{(2)} = \frac{\ell_{j-1}}{8\pi}(3 - 2\ln \ell_{j-1}),$$

$$(3.29)\qquad g_{j(j-1)}^{(2)} = \frac{\ell_{j-1}}{8\pi}(1 - 2\ln \ell_{j-1}).$$

We notice also that the singular integrals occurring in (3.20), (3.21) vanish. From (3.2) and the streamline condition (2.18) it follows that

$$(3.30)\qquad \Psi_0|_{\Gamma} = q_0 - V_0(y\cos\alpha - x\sin\alpha).$$

From (3.15), (3.30) and from the relations

$$(3.31)\qquad \sum_{j=1}^{N} H_{ji} = 1, \qquad i = 1, N$$

we obtain the algebraic system of equations

$$(3.32)\qquad -q_0 + \sum_{j=1}^{N} \frac{\partial \Psi_0}{\partial n}(x_j, y_j) G_{ji} = -V_0 \sum_{j=1}^{N} (y_j\cos\alpha - x_j\sin\alpha) H_{ji}\,, \qquad i = 1, N.$$

The system (3.32) consists of N linear equations for $N+1$ unknows q_0 and $\dfrac{\partial \Psi_0}{\partial n}(x_j, y_j)$, $j = 1, N$.

We may establish the $N+1$-th equation imposing a prescribed value to the circulation

$$(3.33)\qquad \int_{\Gamma} \frac{\partial \Psi_0}{\partial n} ds = k_0\,.$$

Equation (3.33) may be transformed by discretization into

$$\sum_{j=1}^{N} \frac{\partial \Psi_0}{\partial n}(x_j, y_j)(\ell_j + \ell_{j+1}) = 2k_0 . \tag{3.34}$$

It is more convenient to reduce the number of unknowns imposing a prescribed value to the velocity in a certain point on the airfoil; it is usual to impose the zero value to the velocity in the vicinity of the trailing edge, i.e.

$$\frac{\partial \Psi_0}{\partial n}(x_1, y_1) = -V_0(n_y^1 \cos\alpha - n_x^1 \sin\alpha). \tag{3.35}$$

In the last case the circulation is calculated *a posteriori* using (3.34).

The tangential velocity in the nodes (x_j, y_j), $j = 1, N$ may be computed by means of the relation

$$\mathbf{v}_0 \cdot \mathbf{s} = \frac{\partial \psi_0}{\partial n}(x_j, y_j) = \frac{\partial \Psi_0}{\partial n}(x_j, y_j) + V_0(n_y^j \cos\alpha - n_x^j \sin\alpha), \qquad j = 1, N. \tag{3.36}$$

After computing $\frac{\partial \Psi_0}{\partial n}(x_j, y_j)$, the components of the velocity are obtained using the relations

$$u_0(x_j, y_j) = \frac{\partial \psi_0}{\partial n}(x_j, y_j) n_y^j , \tag{3.37}$$

$$v_0(x_j, y_j) = -\frac{\partial \psi_0}{\partial n}(x_j, y_j) n_x^j . \tag{3.38}$$

4. The study of the second approximation of the compressible lifting flow past a smooth obstacle by the boundary element method

Taking into account that the harmonic functions $u_0(x, y)$ and $v_0(x, y)$ behave at infinity as follows:

$$u_0(x, y) = V_0 \cos\alpha + O\left(\frac{1}{\sqrt{x^2 + y^2}}\right), \qquad x^2 + y^2 \Rightarrow \infty, \tag{4.1}$$

$$v_0(x, y) = V_0 \sin\alpha + O\left(\frac{1}{\sqrt{x^2 + y^2}}\right), \qquad x^2 + y^2 \Rightarrow \infty, \tag{4.2}$$

we obtain the integral representations (for $(x, y) \in \Gamma$):

$$\begin{aligned} &\frac{1}{2}(u_0(x, y) - V_0 \cos\alpha) \\ &\quad = \frac{1}{2\pi} \int_{\Gamma} \left(\frac{\partial u_0}{\partial n}(\xi, \eta) \ln\frac{1}{r} - (u_0(\xi, \eta) - V_0 \cos\alpha) \frac{\partial}{\partial n} \ln\frac{1}{r} \right) ds, \end{aligned} \tag{4.3}$$

$$\int_{\Gamma} \frac{\partial u_0}{\partial n}(\xi, \eta)\, ds = 0;$$

(4.4)
$$\frac{1}{2}(v_0(x,y) - V_0 \sin\alpha) = \frac{1}{2\pi}\int\limits_{\Gamma}\left(\frac{\partial v_0}{\partial n}(\xi,\eta)\ln\frac{1}{r} - (v_0(\xi,\eta) - V_0\sin\alpha)\frac{\partial}{\partial n}\ln\frac{1}{r}\right)ds,$$
$$\int\limits_{\Gamma}\frac{\partial v_0}{\partial n}(\xi,\eta)ds = 0.$$

From (4.3) we obtain by discretization the algebraic system

(4.5)
$$a + \sum_{j=1}^{N}\frac{\partial u_0}{\partial n}(x_j,y_j)G_{ji} = \sum_{j=1}^{N}\left[u_0(x_j,y_j) - V_0\cos\alpha\right]H_{ji}\,, \qquad i = 1, N,$$
$$\sum_{j=1}^{N}\frac{\partial u_0}{\partial n}(x_j,y_j)(\ell_j + \ell_{j+1}) = 0.$$

The unknowns are $\frac{\partial u_0}{\partial n}(x_j, y_j)$ and the control variable a, which must be zero.

Similarly, from (4.4) we obtain the system

(4.6)
$$b + \sum_{j=1}^{N}\frac{\partial v_0}{\partial n}(x_j,y_j)G_{ji} = \sum_{j=1}^{N}\left[v_0(x_j,y_j) - V_0\sin\alpha\right]H_{ji}\,, \qquad i = 1, N,$$
$$\sum_{j=1}^{N}\frac{\partial v_0}{\partial n}(x_j,y_j)(\ell_j + \ell_{j+1}) = 0,$$

whose unknowns are $\frac{\partial v_0}{\partial n}(x_j, y_j)$, $j = 1, N$ and the control variable b which must be zero.

After obtaining $\frac{\partial v_0}{\partial n}(x_j, y_j)$ and $\frac{\partial u_0}{\partial n}(x_j, y_j)$ we may calculate

(4.7)
$$\frac{\partial u_0}{\partial x}(x_j,y_j) = \frac{\partial u_0}{\partial n}(x_j,y_j)n_y^j + \frac{\partial v_0}{\partial n}(x_j,y_j)n_x^j\,,$$

(4.8)
$$\frac{\partial v_0}{\partial x}(x_j,y_j) = \frac{\partial u_0}{\partial n}(x_j,y_j)n_x^j - \frac{\partial v_0}{\partial n}(x_j,y_j)n_y^j\,,$$

whence we may obtain

(4.9)
$$\frac{d^2 f_0}{dz^2}(z_j) = \frac{\partial u_0}{\partial x}(x_j,y_j) - i\frac{\partial v_0}{\partial x}(x_j,y_j), \qquad z_j = x_j + iy_j\,, \quad j = 1, N.$$

To calculate the velocity distribution over the obstacle by means of Eq. (3.16), we have to find out the analytic function $g(z)$. First of all we notice that, according to the behaviour of $f_0(z)$ at infinity,

$$f_0(z) = V_0 e^{-i\alpha} z + \frac{k_0}{2\pi i} \ln z + \frac{a_1}{z} + \dots , \tag{4.10}$$

it follows that $\int\limits_{z_1}^{z} \left(\frac{df_0}{dz}\right)^2 dz$ is a multi-valued function and

$$\int\limits_{\Gamma} \left(\frac{df_0}{dz}\right)^2 dz = 2k_0 V_0 e^{-i\alpha} \tag{4.11}$$

(we consider the integration along Γ in the positive sense).

Since $u - iv$ and df_0/dz are single-valued functions, it follows from (2.16) that dg/dz is a multi-valued function. We shall choose for $g(z)$ the expression

$$g(z) = -\frac{e^{i\alpha} k_0}{V_0 \pi i} \ln \frac{z}{z_1} \frac{df_0}{dz} + h(z). \tag{4.12}$$

The function

$$P(z, \overline{z}) = \overline{\int\limits_{z_1}^{z} \left(\frac{df_0}{dz}\right)^2 dz} - \frac{V_0 e^{i\alpha} k_0}{\pi i} \ln \frac{z}{z_1} \tag{4.13}$$

is single-valued.

Using the function $P(z, \overline{z})$ and the relation (3.12), we deduce from (2.14) and (2.16) that

$$f_1(z, \overline{z}) = \frac{1}{4} \left(\frac{1}{V_0^2} \frac{df_0}{dz} P(z, \overline{z}) + h(z) \right), \tag{4.14}$$

$$u - iv = \frac{df_0}{dz} + \frac{M_0^2}{4} \left[\frac{1}{V_0^2} \frac{d^2 f_0}{dz^2} P(z, \overline{z}) + \frac{1}{V_0^2} \left(\frac{df_0}{dz}\right)^2 \frac{d\overline{f}_0}{d\overline{z}} - \frac{e^{i\alpha} k_0}{V_0 \pi i} \frac{1}{z} \frac{df_0}{dz} + \frac{dh}{dz} \right]. \tag{4.15}$$

From the relations (2.17), (4.1), (4.2) concerning the behaviour at infinity of $u - iv$ and df_0/dz, we deduce that

$$\lim_{z \to \infty} \frac{dh}{dz} = -V_0 e^{-i\alpha}, \tag{4.16}$$

whence we obtain (for the analytic function $h(z)$) the expansion

$$h(z) = -V_0 e^{-i\alpha} z + \frac{k_1}{2\pi i} \ln z + O\left(\frac{1}{z}\right). \tag{4.17}$$

Introducing the function

$$\widetilde{h}(z) = J(x, y) + iI(x, y) = h(z) + V_0 e^{-i\alpha} z \tag{4.18}$$

we may represent the harmonic function $I(x,y)$ on Γ as follows

$$\frac{1}{2}I(x,y) = \int_{\Gamma} \left(\frac{\partial I}{\partial n}(\xi,\eta)\ln\frac{1}{r} - I(\xi,\eta)\frac{\partial}{\partial n}\ln\frac{1}{r}\right) ds. \tag{4.19}$$

From (4.14), (4.18) and the streamline condition (2.18) we deduce

$$I(x,y) = q_1 + \beta(x,y), \tag{4.20}$$

with

$$\beta(x,y) = -\mathrm{Im}\left[\frac{1}{V_0^2}\frac{df_0}{dz}P(z,\overline{z})\right] + V_0(y\cos\alpha - x\sin\alpha). \tag{4.21}$$

Discretizing (4.19) we obtain the algebraic system

$$-q_1 + \sum_{j=1}^{N}\frac{\partial I}{\partial n}(x_j,y_j)G_{ji} = \sum_{j=1}^{N}\beta(x_j,y_j)H_{ji}, \qquad i = 1, N. \tag{4.22}$$

$\beta(x_j, y_j)$ is computed numerically using the relation

$$\begin{aligned}\beta(x_j,y_j) = {}& V_0(y_j\cos\alpha - x_j\sin\alpha)\\ & -\frac{1}{V_0^2}\mathrm{Im}\left[(u_0(x_j,y_j) - iv_0(x_j,y_j))P(z_j,\overline{z}_j)\right];\end{aligned} \tag{4.23}$$

$$\begin{aligned}P(z_j,\overline{z}_j) = {}& \frac{1}{3}\sum_{l=1}^{j-1}[(x_{l-1} - x_l + i(y_{l-1} - y_l)]\\ & \cdot\Big[(u_0(x_{l+1},y_{l+1}) - iv_0(x_{l+1},y_{l+1}))^2 + (u_0(x_l,y_l) - iv_0(x_l,y_l))^2\\ & +(u_0(x_{l+1},y_{l+1}) - iv_0(x_{l+1},y_{l+1}))(u_0(x_l,y_l) - iv_0(x_l,y_l))\Big]\\ & -\frac{V_0e^{i\alpha}k_0}{\pi i}\ln\frac{z_j}{z_1}, \qquad j = 2, N,\\ P(z_1,\overline{z}_1) = {}& 0.\end{aligned} \tag{4.24}$$

Relation $(4.24)_1$ was deduced from (4.13) and from the linear interpolation of $u_0(x,y)$ and $v_0(x,y)$ on Γ.

The system (4.22) consists of N equations for $N+1$ unknowns q_1 and $\frac{\partial I}{\partial n}(x_j,y_j)$, $j = 1, N$. The most usual way to reduce the number of unknowns is to consider a prescribed value of the velocity in a certain point. It is natural to impose the zero value for the velocity at the same point (x_1,y_1) where $u_0 - iv_0$ vanishes. We have therefore $\frac{dh}{dz}(z_1) = 0$, whence by virtue of (4.18)

$$\frac{\partial I}{\partial n}(x_1,y_1) = \mathrm{Im}\left[(n_x^1 + in_y^1)\frac{\partial\widetilde{h}}{\partial z}\right] - V_0(n_y^1\cos\alpha - n_x^1\sin\alpha). \tag{4.25}$$

From (4.15), (4.18) and the slip condition (2.19) we deduce on Γ:

$$
\begin{aligned}
(4.26)\qquad 0 &= \operatorname{Re}\left[(u-iv)(n_x+in_y)\right]\\
&= -\frac{M_0^2}{4}\operatorname{Re}\left[(n_x+in_y)\left(\frac{1}{V_0^2}\frac{d^2f_0}{dz^2}P(z,\overline{z})+\frac{1}{V_0^2}\left(\frac{df_0}{dz}\right)^2\frac{d\overline{f}_0}{d\overline{z}}\right.\right.\\
&\qquad\left.\left.-\frac{k_0e^{i\alpha}}{V_0\pi iz}\frac{df_0}{dz}+\frac{d\widetilde{h}}{dz}-V_0e^{-i\alpha}\right)\right]
\end{aligned}
$$

whence, since

$$
(4.27)\qquad \operatorname{Re}\left[(n_x+in_y)\frac{d\widetilde{h}}{dz}\right]=\frac{dJ}{dn},
$$

it follows that

$$
\begin{aligned}
(4.28)\qquad &\frac{\partial J}{\partial n}(x_j,y_j)\\
&= -\operatorname{Re}\left\{(n_x^j+in_y^j)\left[\frac{1}{V_0^2}\left(\frac{\partial u_0}{\partial x}(x_j,y_j)-i\frac{\partial v_0}{\partial x}(x_j,y_j)\right)P(z_j,\overline{z}_j)\right.\right.\\
&\qquad+\frac{1}{V_0^2}(u_0(x_j,y_j)-iv_0(x_j,y_j))^2(u_0(x_j,y_j)+iv_0(x_j,y_j))\\
&\qquad\left.\left.-\frac{k_0(u_0(x_j,y_j)-iv_0(x_j,y_j))e^{i\alpha}}{V_0\pi iz_j}+V_0e^{-i\alpha}\right]\right\}.
\end{aligned}
$$

From (4.25) and (4.28) we obtain

$$
\begin{aligned}
(4.29)\qquad \frac{d\widetilde{h}}{dz}(z_j) &= \frac{\partial J}{\partial x}(x_j,y_j)+i\frac{\partial I}{\partial x}(x_j,y_j)\\
&= \frac{\partial J}{\partial n}(x_j,y_j)n_x^j+\frac{\partial I}{\partial n}(x_j,y_j)n_y^j+i\left(\frac{\partial J}{\partial n}(x_j,y_j)n_y^j-\frac{\partial I}{\partial n}(x_j,y_j)n_x^j\right).
\end{aligned}
$$

From (4.15), (4.18) we may determine the complex velocity at the control points

$$
\begin{aligned}
(4.30)\qquad u(x_j,y_j)-iv(x_j,y_j) &= u_0(x_j,y_j)-iv_0(x_j,y_j)\\
&\quad+\frac{M_0^2}{4}\left[\frac{1}{V_0^2}(u_0(x_j,y_j)-iv_0(x_j,y_j))^2(u_0(x_j,y_j)+iv_0(x_j,y_j))\right.\\
&\quad+\frac{d\widetilde{h}}{dz}(z_j)+V_0e^{-i\alpha}-\frac{k_0e^{i\alpha}}{V_0\pi iz_j}(u_0(x_j,y_j)-iv_0(x_j,y_j))\\
&\quad\left.+\frac{1}{V_0^2}\left(\frac{\partial u_0}{\partial x}(x_j,y_j)-i\frac{\partial v_0}{\partial x}(x_j,y_j)\right)P(z_j,\overline{z}_j)\right],
\end{aligned}
$$

and afterwards the speed of the fluid over the obstacle

$$V(x_j, y_j) = \sqrt{u(x_j, y_j)^2 + v(x_j, y_j)^2}\,. \tag{4.31}$$

5. Physical validity of the results

Since the Imai - Lamla - Iacob method is based on the asymptotic expansion with respect to M_0, the accuracy of the results increases when $M_0 \Rightarrow 0$. For investigating the compressibility effects, we are interested in working with great values of M_0, but the actual method imposes some restrictions on M_0.

As we could see, the study of the compressible flow past an obstacle was reduced to a series of boundary value problems for Laplace's equation. This method is not suitable for the supersonic compressible flow which is governed by hyperbolic partial differential equations; we have therefore to request the local speed of the fluid not to exceed the value of the speed of sound. For the isentropic flow, the speed of the sound depends on the speed of the fluid as follows:

$$c^2 = c_0^2 - \frac{\gamma - 1}{2} V^2 \tag{5.1}$$

or equivalently,

$$c^2 \frac{M_0^2}{V_0^2} = 1 - \frac{\gamma - 1}{2} M_0^2 \frac{V^2}{V_0^2}\,. \tag{5.2}$$

Imposing that $V \leq c$, we obtain from (5.2)

$$\frac{V^2}{V_0^2} M_0^2 \frac{\gamma + 1}{2} \leq 1 \tag{5.3}$$

and particularly,

$$\frac{V_{\max}^2}{V_0^2} M_0^2 \frac{\gamma + 1}{2} \leq 1. \tag{5.4}$$

The relation (5.4) is *a posteriori* condition which has to be checked after calculating the distribution of the velocity around the obstacle by the Imai - Lamla - Iacob method.

We shall assign to the ratio of specific heats the value $\gamma = 1.4$.

6. Numerical results

We shall investigate the flow past the circular obstacle

$$z = e^{i\theta}, \qquad \theta \in [0, 2\pi]. \tag{6.1}$$

We approximate the circle by the poligonal contour $\Gamma = \bigcup_{j=1}^{36} \Gamma_j$. The extremes of the panel Γ_j are the points

$$(x_j, y_j) = (\cos\theta_j, \sin\theta_j), \qquad (x_{j+1}, y_{j+1}) = (\cos\theta_{j+1}, \sin\theta_{j+1}) \tag{6.2}$$

with

$$\theta_j = \frac{(2j-3)\pi}{36}, \qquad j = 1, N. \tag{6.3}$$

We impose the zero value to the velocity at the point

$$z_1 = e^{i\theta_1} \tag{6.4}$$

and we consider the angle of attack $\alpha = 0$.

Using the boundary element method we compute the flow speed in the control points, and then the pressure coefficients

$$C_p = \frac{2\left(1 - \frac{\gamma-1}{2}M_0^2\right)}{\gamma M_0^2}\left[\frac{\left(1 - \frac{\gamma-1}{2}M_0^2\frac{V^2}{V_0^2}\right)^{\gamma/(\gamma-1)}}{\left(1 - \frac{\gamma-1}{2}M_0^2\right)^{\gamma/(\gamma-1)}} - 1\right]. \tag{6.5}$$

Expanding the pressure coefficient with respect to Chaplygin's number, we get

$$C_p = 1 - \frac{V^2}{V_0^2} + \frac{M_0^2}{4}\left(1 - \frac{V^2}{V_0^2}\right)^2 + M_0^4(\ldots). \tag{6.6}$$

On the circular obstacle, the analytical expression of the flow speed [1] is known,

$$V = 2V_0|\sin\theta - \sin\theta_1|\left[1 + \frac{M_0^2}{12}(1 - 6\cos 2\theta - 20\sin\theta\sin\theta_1 + 4\sin^2\theta_1)\right] \tag{6.7}$$

whence it follows that

$$C_p = 1 - 4(\sin\theta - \sin\theta_1)^2\left[1 + \frac{M_0^2}{6}(1 - 6\cos 2\theta - 20\sin\theta\sin\theta_1 + 4\sin^2\theta_1)\right] + \frac{M_0^2}{4}\left[(1 - 4(\sin\theta - \sin\theta_1)^2\right]^2. \tag{6.8}$$

Comparisons between the pressure coefficients in the control points, obtained by means of the boundary element method and by means of the analytical formula (6.8) are performed in Fig. 1 for various values of Chaplygin's number.

We introduce the lift coefficient

$$C_L = \frac{1}{\ell}\int_{\Gamma} C_p n_y \, ds, \tag{6.9}$$

where ℓ is the length of Γ. We can compute the lift coefficient numerically using the formulas

$$\ell = \sum_{j=1}^{N} \ell_j, \tag{6.10}$$

$$C_L = \frac{1}{2\ell}\sum_{j=1}^{N}(C_p(x_j, y_j) + C_p(x_{j+1}, y_{j+1}) n_y^j \ell_j. \tag{6.11}$$

For the circular obstacle, from (6.9) and (6.8) we get

$$C_L = -4\sin\theta_1 - M_0^2 \sin\theta_1 \left(\frac{8}{3} + \frac{4}{3}\sin^2\theta_1\right). \tag{6.12}$$

To conclude, we give some values for the lift coefficient obtained by means of formulas (6.12) and (6.11) for various values of M_0:

M_0	0	0.20	0.25	0.30
C_L (numerical)	0.3584	0.3611	0.3660	0.3703
C_L (analytical)	0.3486	0.3580	0.3632	0.3696

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