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On the incremental plastic work and related aspects of invariance – Part I

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Summary. Taking for granted that the free energy function is invariant under a change of a finite strain measure and/or the reference configuration, Hill's transformation rules for selected fundamental constitutive quantities (such as tangent elastic modulus, plastic increments of total strain and work conjugate stress, the work of work-conjugate stress, the work expended in the plastic part of incremental strain etc.) are derived in a manner different from that of Hill. On this background distinguished by Hill [6] subtle aspects of invariance in mechanics of elastic plastic solids are discussed. It is shown that the plastic part of the increment of elastic strain energy (when taken with reverse sign) defines the true invariant incremental plastic work which in general is not equal to the work expended in the plastic part of the strain increment. It plays the role of a potential for the plastic part of the increment of work-conjugate stress. This fundamental fact has not found proper account in the literature. The analytical interrelations between two apparently different theoretical frameworks, Hill-Rice (fixed reference configuration) and Eckart-Mandel (mobile unloaded configuration) are discussed showing their equivalence. Since the transformation rules are complex in the general 3D case, the first part of the paper illustrates instructively the discussed aspects in a 1D situation (simple tension or simple extension).

1 Introduction

In mechanics of finite elastic-plastic deformations the partitioning of the strain rate (or, equivalently, the total strain increment) into elastic and plastic parts is not an invariant concept (cf., e.g., [8] and [10]). That is, in general, the physical dimension change of the material element corresponding to the plastic part of the strain increment is not independent of the choice of strain measure and reference configuration [16]. As a consequence, neither the product of the work-conjugate stress and the *plastic part* of incremental strain nor the product of stress and *incremental permanent strain* do, in general, coincide with *incremental true plastic work*, say dW^p. The latter is commonly understood as the incremental total work of straining from an elastic-plastic state to a neighboring elastic-plastic state, reduced by the difference between the elastic strain energies at those two elastic-plastic states. One of the main purposes of this paper is to derive the general relation between dW^p and other definitions of incremental plastic work. The other objective is to derive the transformation formula that links different measures of incremental plastic strain. To this end, one has to go beyond the standard frame of the description of a material behavior, where a single specified strain measure is adopted. In fact, there are a lot of possible choices of six parameters that taken together determine the current shape of a material element undergoing a homogeneous

deformation [6]. The particular choice may be identified with a generalized coordinate system in a 6-dimensional abstract space of parallelepipeds. In most cases a change of the generalized coordinates (use of the new strain measure or/and new reference configuration) entails the change of the value of the physical quantity that is employed while describing the constitutive property of a material. The question is: what are the transformation rules linking the "old" and "new" values, and which notions and quantities are invariant. The foundations for systematic analysis of this type of invariance of the constitutive objects has been developed by Hill [2], [5], [6], [8], and later extended to thermoelasticity in [7] and thermoplasticity in [10]. The constitutive object is any quantity (e.g., stress, incremental plastic work, hardening modulus etc.) that enters a constitutive relation and possesses a definite transformation rule under the change of the generalized coordinate system. The transformation rules need not be linear nor homogeneous. The analysis of the invariance furnishes transformation rules for new objects, what enables full discussion of their significance. The elements of that type of analysis are also employed here. In addition the connections between the two most popular theoretical frameworks, Hill-Rice [9] and Eckart-Mandel [1], [11] are discussed showing their formal equivalence. Special attention is focused on the interrelations existing between different measures of incremental plastic work. No assumptions concerning the specific inelastic behavior of an elastic-plastic material are made.

As the transformation rules have rather involved character in the general case, the discussion of selected aspects of invariance is divided into two parts. The present first part aims at the elementary illustration of the analysis on the example of simple tension (or compression). Without any substantial changes it can also be applied to simple extension (or contraction, e.g. compression of a material in a rigid cylinder). Some presented formulas may be of interest in experimental mechanics. Most conclusions drawn here (including those concerning the materials with intrinsic elastic properties insensitive to prior plastic straining) will be shown in Part II to be valid in general 3D situations.

2 1D illustration of the Hill-Rice theoretical framework

2.1 Preliminaries

a. Strain measures, work-conjugate stresses and their increments

i) Suppose that a thin bar of mass m_0 under the action of a uniaxial load P (tension or compression) deforms uniformly, at constant temperature, like a typical elastic-plastic material, as illustrated in Fig. 1.

Let in the initial state (and at the initial configuration) its length, cross-sectional area and the mass density be respectively $l_0, A_0, \varrho_0 = m_0/V_0$ ($V_0 = A_0 l_0$). Under action of the force $P(t)$ the dimensions of the bar change, and suppose that at time t in the actual state (current configuration) the corresponding quantities are $l(t)$, $A(t)$ and $\varrho(t) = m_0/V(t)$ ($V(t) = A(t)l(t)$). Assuming that the behavior of the bar material during abrupt unloading is purely elastic, but not necessarily linear or the same after different preloading (paths $A - a$, $B - b$ etc. in Fig. 1), one can associate with its actual state at time t the *conceptual instantaneous natural state* (and corresponding instantaneous stress-free unloaded configuration), i.e., the state which the bar would have attained at time t, had the force P been instantaneously reduced to zero. In Fig. 1 the superscript "*" is used to denote the dimensions of the bar and its mass density in the instantaneously unloaded configuration.

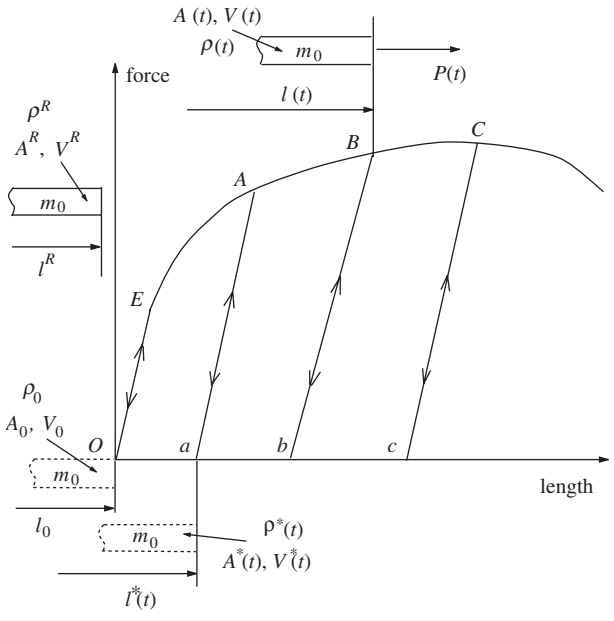

Fig. 1. Load-displacement curve of an elastic-plastic bar

In the range of large deformations, the family of load-displacement curves such as those shown in Fig. 1 can be mapped onto many plausible strain-stress planes. The very concept of strain employs two configurations of the bar: the current configuration and any fictitious reference one that can be chosen at will. In Fig. 1 the length, cross-section area and mass density of the bar in an arbitrary fixed reference configuration are denoted by l^R, A^R and $\varrho^R = m_0/V^R$ ($V^R = A^R l^R$), respectively. In particular, this configuration may be identified with initial, actual or the conceptual unloaded configuration. In the uniaxial situation no rigid body rotation of the longitudinal axis of a bar is admitted. The other factor that can be chosen at will is the strain measure. In systematic analysis it is convenient to distinguish the following general class of strain measures [3], [4], [8]:

$$q = f[U(t)], \quad U(t) = \frac{l(t)}{l^R}, \tag{1}$$

where the smooth and monotone function $f(z)$ is called *scale function* which satisfies the conditions $f(1) = 0, f'(1) = 1$, and U is the stretch of the bar. The most commonly used scale functions belong to the particular family

$$f(z) = \frac{z^{2n} - 1}{2n}, \tag{2}$$

where n can take any value.

ii) The strain increment that gives rise to the notion of strain-rate is defined by the elementary total differential of Eq. (1):

$$dq = f'(U)dU, \quad dq = \frac{1}{l^R} f'[l(t)/l^R]dl(t), \quad f'(U) \equiv \frac{df}{dU}, \tag{3.1–3}$$

from which it follows that in the uniaxial case only the increment of the logarithmic strain ($q = \varepsilon = (l - l_R)/l_R, f(z) = \ln z, n = 0$) is independent of the reference length l^R of the bar.

iii) Denote by dW an increment of total specific work (work per unit of mass). The specific [10] and standard [3] stress measures (p and $\bar{p}$, respectively) work-conjugate to the selected strain measures q are defined by

$$dW = p\,dq = \frac{1}{\varrho^R}\bar{p}\,dq = \frac{1}{m_0}P(t)dl(t), \quad \bar{p} = \varrho^R p. \tag{4}$$

They differ by an unimportant constant multiplier (mass density of a bar in the chosen reference configuration).

By inserting Eq. (3.2) into Eq. (4) we find

$$p = \frac{l^R P(t)}{m_0 f'[U(t)]} = \frac{\tau}{Uf'(U)}, \quad \bar{p} = \frac{P(t)}{A^R f'[U(t)]} = \frac{\bar{\tau}}{Uf'(U)} \tag{5.1,2}$$

$$\tau = \frac{Pl(t)}{m_0} = \frac{1}{\varrho(t)}\sigma(t), \quad \bar{\tau} = \varrho^R\tau = \frac{\varrho^R}{\varrho}\sigma = \frac{Pl}{A^R l^R}, \quad \sigma = \frac{P(t)}{A(t)}. \tag{5.3–5}$$

Here σ and $\bar{\tau}$ are the usual Cauchy's and Kirchhoff stress [6] and τ is the specific counterpart of $\bar{\tau}$. Note that the experimental determination of the values of work-conjugate stresses p and $\bar{p}$ *does not require the knowledge of the actual cross-section area* $A(t)$ *of the bar*.

Remark. The determination of a value of Cauchy's (true) stress in simple tension/compression requires the knowledge of $A(t)$ which is usually not measured. Therefore, in experimental mechanics, in the situation when the reference configuration coincides with the initial one ($l^R = l_0, A^R = A_0$), one frequently adopts the "postulate of material incompressibility" $A(t) = A_0 l_0/l(t)$. In such a case $P/A(t) = Pl/A_0 l_0$. Thus, the so determined stress is in fact the Kirchhoff stress $\bar{\tau}$.

When the strain measure belongs to the family (2), p can be expressed explicitly in terms of τ and q:

$$p = (2nq+1)^{-1}\tau = \frac{Pl}{m_0}\left(\frac{l^R}{l}\right)^{2n} = \tau U^{-2n}. \tag{6}$$

It can readily be seen that τ in the considered simple situation is the specific stress work-conjugate to the logarithmic strain $q = \varepsilon_L$ ($n = 0$). Within the accuracy of a rigid-body rotation this holds also in a three-dimensional situation, provided that an elastic-plastic material is isotropic at every state (cf., e.g., [6] and [14]). The work-conjugate to engineering strain ($n = 1/2, q = \varepsilon$) is the specific nominal stress $p = P/(A^R\varrho^R)$.

By treating l or U as a parameter in Eqs. (1) and (5), the possible experimental curve $P(l)$ such as shown in Fig. 1 can be mapped into the fixed $p - q$ plane (Fig. 2). The form of image curves depends, of course, on the chosen scale function and reference configuration (reference bar length l^R and reference cross-section A^R), cf. Fig. 1.

iv) The increments of τ, $\bar{\tau}$ and the work-conjugate stress p are:

$$d\bar{\tau} = \varrho^R d\tau = \frac{\varrho^R}{\varrho}\left[d\sigma - \frac{d\varrho}{\varrho}\sigma\right], \tag{7.1}$$

$$dp = \frac{l^R}{m_0 f'(U)}\left[dP(t) - P(t)\frac{f''(U)}{f'(U)}dU\right]$$
$$= \frac{1}{Uf'(U)}\left[d\tau - \frac{\tau(f' + Uf'')dU}{Uf'}\right], \tag{7.2}$$

and from this formulas one readily deduces the usual engineering necking criterion (substitute $dP = 0$ and $dq = f'(U)dU$ into Eq. (7.2), and use Eq. (5.1),

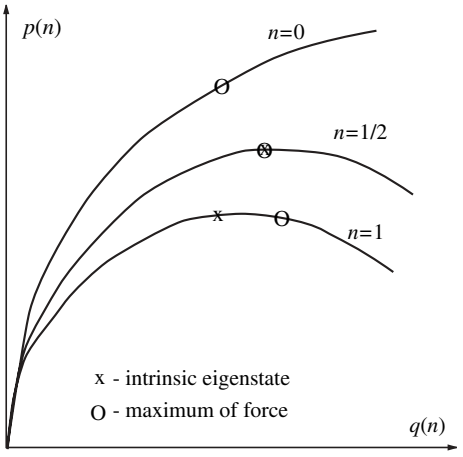

Fig. 2. Characteristics states

$$dp = -\frac{f''}{(f')^2}\, p\, dq. \tag{8}$$

When expressed in terms of the work-conjugate couple associated with the family (2), the above criterion takes the form

$$dp = \frac{1-2n}{1+2nq}\, p\, dq. \tag{9}$$

It supports the familiar observation that the limiting point ($dP = 0$) is placed on the decreasing portion of the $p - q$ *tension* curve whenever $n > 1/2$ (Fig. 2).

b. Relations between plastic increments of conjugate variables and specific free energy

The concise framework for the structure of the constitutive law for elastic-plastic materials has been developed by Hill and Rice [9] and Rice [17]. Some basic concepts of this framework, with slight modifications, are recalled beneath, in the context of uniaxial stress state.

i) Any accessible state of the bar is described by the current value of the selected strain measure q (or by its work-conjugate stress p) and a number of suitably defined internal state parameters. If a specific structural model of inelastic behaviour of the bar is employed, the internal variables describe the microstructural rearrangement of its constituent elements. For the purpose of this paper the internal variables need not be specified and will be collectively denoted by H.

All H are presumed to be constant along those segments of loading paths where the behavior is purely elastic. In particular, all states corresponding to a point on the elastic line such as $A^* - B$ in Fig. 3 are associated with the same natural state (point A^*) and have the same fixed value of H. The change in the cross-section area of the bar could be described by an appropriate constitutive law for the volume change. The basic concept concerning the definition of invariant plastic work, however, may be illustrated without explicit reference to such a relation.

Following Hill and Rice [9] it is convenient to single out the independent d^p-differential and

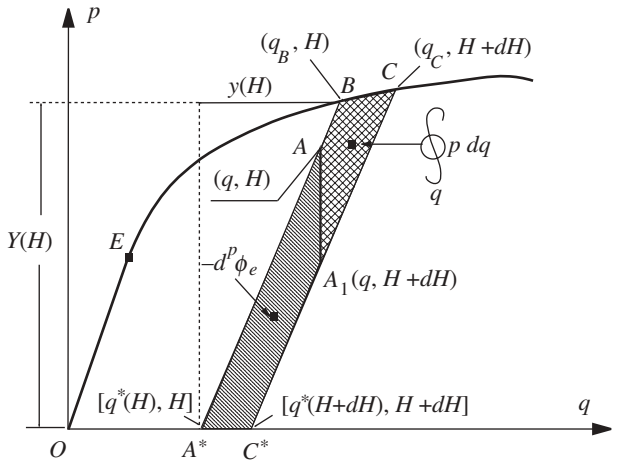

Fig. 3. Work-interpretation of basic invariants

$$
\begin{aligned}
d^p\mathcal{A}(q,H) &= \mathcal{A}(q,H+dH) - \mathcal{A}(q,H),\\
\delta\mathcal{A}(q,H) &= \mathcal{A}(q+\delta q,H) - \mathcal{A}(q,H),\\
d^p\mathcal{A}(p,H) &= \mathcal{A}(p,H+dH) - \mathcal{A}(p,H),\\
\delta\mathcal{A}(p,H) &= \mathcal{A}(p+\delta p,H) - \mathcal{A}(p,H).
\end{aligned}
\tag{10}
$$

The differential $d^p\mathcal{A}$ is interpreted as the isothermal "plastic part" of the total incremental change in $\mathcal{A}$. For rate-independent plastic materials the parameters H may change only along those path segments of a process that lie on the yield surface. Therefore, d^p-variations of any quantity can be effected plastically only by a strain or stress cycle process that is associated with an infinitesimal change in H. δ-differential represents the infinitesimal change in $\mathcal{A}$ in the course of an elastic process.

ii) In the course of isothermal processes of loading the value of the specific (per unit of mass) free energy ϕ of an elastic-plastic material element depends on its actual shape and H, $\phi = \phi(q,H)$. The fundamental equation of state is

$$d\phi(q,H) = p(q,H)dq + d^p\phi(q,H). \tag{11}$$

Hence

$$p = \frac{\partial\phi(q,H)}{\partial q}. \tag{12}$$

The term $d^p\phi(q,H)$ represents (isothermal) incremental internal work expended for micro-structural changes. We shall now restrict the class of considered models of material behavior to such for which there exists the unique solution $q^*(H)$ of the equation

$$\left.\frac{\partial\phi}{\partial q}\right|_{q^*(H)} = 0. \tag{13}$$

This class includes also models of elastic-ideally damaging bodies ($q^* = 0$ for all H). When the initial configuration is taken as reference one ($l^R = l_0$) the quantity q^* usually gets the status of *permanent strain*. The stress-free state of a material in the initial configuration is here assumed

to be the thermodynamical reference state – an ideal state where $\phi = \phi_0 = \text{const}$, $H = 0$ and $\partial\phi/\partial H = 0$. If the isothermal curves of elastic behavior ($A^* - B, C^* - C$ in Fig. 3) are known for all H, it is possible to construct a family of elastic strain energies $\phi_e = \phi_e(q,H)$. The specific free energy ϕ of the bar can be expressed as the sum of ϕ_0, ϕ_e and the part of free energy $\phi_s(H)$ that is independent of q (the free energy that is stored in the material in the course of prior plastic straining)

$$\Delta\phi = \phi - \phi_0 = \phi_e(q,H) + \phi_s(H) \quad \Delta\phi|_{q=q^*(H)} = \phi_s(H). \tag{14}$$

By definition of ϕ_e, we have

$$p = \frac{\partial\phi_e(q,H)}{\partial q}, \quad \phi_e|_{q=q^*} = 0, \quad p(q^*,H) = \left.\frac{\partial\phi_e}{\partial q}\right|_{q=q^*} = 0. \tag{15.1–3}$$

Equation (15.1) describes the family of elastic responses and ϕ_e is represented by the area under the segment of the elastic curve linking a point (q,H) and a point $(q^*(H),H)$ (cf. Fig. 3). The determination of stored energy requires independent measurements [13].

We shall also employ the usual total complementary energy $-\psi_T(p,H)$ and elastic complementary energy $-\psi_e(p,H)$,

$$\psi_T(p,H) = [\phi_e(q,H) - pq]_{q=q(p,H)}, \quad \psi_e(p,H) = \psi_T(p,H) + pq^*(H), \tag{16.1,2}$$

and express the inverse of Eq. (15.1) in the form

$$q = -\frac{\partial\psi_T}{\partial p} = -\frac{\partial\psi_e(p,H)}{\partial p} + q^*(H), \quad \frac{\partial\psi_e}{\partial p} = \psi_e = 0 \quad \text{for } p = 0. \tag{17.1,2}$$

In the case of a rate-independent material the domain of elastic behavior is bounded by a yield curve (see Fig. 1)

$$F(q,H) \equiv q - q^*(H) - y(H) = 0 \quad \text{or} \quad F(p,H) \equiv p - Y(H) = 0, \tag{18.1,2}$$

where

$$Y(H) = \left[\frac{\partial\phi_e}{\partial q}\right]_{q=q^*+y(H)}$$

and Eq. (18.1) defines the domain of determination of the function ϕ_e. In Fig. 3 the initial yield limit $Y(0)$ is marked by the letter E, and the line $E - B - C$ represents the yield curve in the $p - q$ plane.

iii) Applying d- and δ- differentials to Eqs. (15) and (17), we have

$$dp = \frac{1}{\varrho^R} L\, dq + d^p p, \quad dq = \varrho^R M dp + d^p q, \tag{19.1,2}$$

$$\delta p = \frac{1}{\varrho^R} L\, \delta q, \quad \delta q = \varrho^R M \delta p \text{ (elastic domain)}. \tag{20}$$

Here L and M are tangent elastic modulus and compliance measured in the elastic domain and at its border (18):

$$\frac{1}{\varrho^R} L = \frac{\partial p}{\partial q} = \frac{\partial^2\phi}{\partial q^2} = \frac{\partial^2\phi_e}{\partial q^2}, \quad \varrho^R M = \frac{\partial q}{\partial p} = -\frac{\partial^2\psi_T}{\partial p^2} = -\frac{\partial^2\psi_e}{\partial p^2}. \tag{21}$$

In general, they depend on q (or p) and H. The second terms occurring on the right-hand sides of Eq. (19) define the plastic part $d^p p$ of the conjugate stress increment and the plastic part $d^p q$ of total strain increment (cf. (10)),

$$d^p p = p(q,H+dH) - p(q,H) = \frac{\partial d^p \phi_e}{\partial q},$$
$$d^p q = q(p,H+dH) - q(p,H) = \frac{\partial d^p \psi_C}{\partial p} = -\frac{\partial d^p \psi_e}{\partial p} + dq^*(H). \tag{22.1,2}$$

The basic identities following from the existence of dual potentials and connections (15), (17) are obtained by appropriate differentiations of Eqs. (15.1) and (16.1) with account of Eqs. (21)–(22):

$$LM = 1, \quad d^p \phi_e = d^p \psi_T|_{p=p(q,H)} = [d^p \psi_e - p dq^*(H)]_{p=p(q,H)}, \tag{23.1,2}$$

and the calculation of the partial derivative of Eq. (23.2) with respect to q shows the known connection between $d^p p$ and $d^p q$:

$$d^p p = -\frac{1}{\varrho^R} L \, d^p q \tag{24}$$

2.2 The transformation rules

a. Two basic transformation rules-Prototypes of invariants

i) Suppose that two observers take different scale functions (f_1 and f) and different reference configurations to describe the deformation state of a bar. The corresponding strains q_1 and q are

$$q_1 = f_1[l(t)/l_1^R] = f_1(U_1) \quad \text{and} \quad q = f[l(t)/l^R] = f(U), \tag{25}$$

where $U_1 = l^R U / l_1^R$. The elimination of U between the above two equations yields the symbolic relation between q_1 and q that concerns also the permanent strains ($l^R/l_1^R = 1$ for common reference configuration)

$$q_1 = q_1(q), \quad q_1^*(H) = q_1[q^*(H)]. \tag{26.1,2}$$

When f_1 and f belong to the family (2), the relation (26) becomes

$$q_1 = \frac{1}{2n_1}\Big[(1+2nq)^{\frac{n_1}{n}}(l^R/l_1^R)^{2n_1} - 1\Big], \tag{27.1}$$

$$q = \frac{1}{2n}\Big[(1+2n_1 q_1)^{\frac{n}{n_1}}(l_1^R/l^R)^{2n} - 1\Big], \tag{27.2}$$

where n_1 and n are exponents defining two different scale functions.

The inverse relation (27.2) is found directly from (27.1) by interchanging the ordered quantities (q, n, l^R) and (q_1, n_1, l_1^R).

Suppose also that the observers use the elements of the theoretical framework presented in Sect. 2.1.b to describe the properties of the same physical solid. One of the basic aims of the algebra of constitutive objects is to find the relations between values of the quantities (defined in Sect. 2.1.b) measured independently by the two observers. The relations are referred to as the transformation rules.

ii) Since the infinitesimal increments of dq_1 and dq describe the same change in the length of the bar, they are connected by

$$dq_1 = Q dq, \quad dq = Q^{-1} dq_1, \quad Q = \frac{dq_1(q)}{dq} > 0, \quad Q^{-1}Q = 1. \tag{28.1–4}$$

For the common reference configuration $l_1^R = l^R$, the relation (26.1) between q and q_1 has the properties:

$$q_1|_{q=0} = 0, \quad Q|_{q=0} = 1. \tag{29}$$

The transformation rules for δq and dq are the same, of course. For the class (27) the explicit form of Q is

$$Q(q) = \frac{\Gamma_1^0(q_1)|_{q_1=q_1(q)}}{\Gamma^0(q)} \quad \text{or} \quad Q(q_1) = \frac{\Gamma_1^0(q_1)}{\Gamma^0(q)|_{q=q(q_1)}}, \tag{30}$$

where

$$2\Gamma^0(q) = 1 + 2nq, \quad 2\Gamma_1^0(q_1) = 1 + 2n_1 q_1. \tag{31}$$

During any infinitesimal strain cycle the net increment of a strain vanishes, no matter whether the process is associated with the infinitesimal change in H or not. Since according to Eq. (28) $dq = 0 \Rightarrow dq_1 = 0$ one concludes that the notion of the strain cycle is the invariant concept.

iii) To derive the transformation formula for other incremental constitutive quantities, Hill [6] regards the variation of the total external work $p\,dq$ as the only prototype of the invariant. We here additionally take for granted that the (isothermal) incremental internal work $d^p\phi$ expended for microstructural changes is also invariant. This work when taken with inverse sign, $-d^p\phi$, represents the incremental mechanical energy dissipation, provided that rate effects are negligible [16]. It is obvious that these two work-type quantities cannot depend on the manner the external shape of the material element is described. Thus, each term of the differential form (11) is regarded as the prototype of an invariant:

$$p_1 dq_1 = p\,dq, \quad d^p\phi^{(1)}(q_1, H) = d^p\phi(q, H). \tag{32.1,2}$$

By combining Eqs. (28.1) and (32.1) one obtains the transformation rule for work-conjugate stresses [6]:

$$p_1 = \frac{\bar{p_1}}{\varrho_1^R} = Q^{-1}p = Q^{-1}\frac{\bar{p}}{\varrho^R}. \tag{33}$$

The transformation rule for dq is the contravariant of the p-rule. Since $p = 0$ implies $p_1 = 0$, the notion of a stress cycle that *starts and terminates at stress-free state* is the invariant concept.

We have (cf. Eq. (14)) $d^p\phi = d^p\phi_e + d^p\phi_s(H)$ and the question arises whether each term of this additive decomposition is individually invariant. To show that it is the case, fix (q, H) and dH, and consider the strain cycle $A - B - C - A_1$ along the route $A - A^* - B - C - C^* - A_1$ as illustrated in Fig. 3. The result is

$$-d^p\phi_e(q, H) = \oint_{p=0} p\,dq - \oint_q p\,dq. \tag{34}$$

It holds for all (q, H) in the elastic domain and on its border (yield surface). The symbols $q = q^*(H)$ and $q^*(H + dH)$ occurring in Fig. 3 denote the permanent strains in stress-free states $(p = 0, H)$ and $(p = 0, H + dH)$, respectively. They are roots of algebraic equations $\partial\phi(q, H)/\partial q = 0$ and $\partial\phi(q, H + dH)/\partial q = 0$. Thus, as expected, the invariance of incremental total work (32.1) implies the invariance of the plastic increment of elastic strain energy $d^p\phi_e$. Our additional requirement mentioned above is equivalent to the statement that the incremental change $d^p\phi_s(H) = d\phi_s$ of the stored free energy is also invariant. This equivalence can be expressed still in different way. Denote by $du_s(H)$ the incremental change of the stored

internal energy effected by a stress-cycle initiated from a stress-free state at reference temperature $\vartheta = \vartheta_0$. From the first law of thermodynamics it follows that

$$du_s(H) = u_s(H + dH) - u_s(H) = \oint\limits_{p=0} p\, dq - DQ, \tag{35}$$

where DQ is the incremental heat transferred to surrounding. Both sides of Eq. (35) are invariants since DQ cannot depend on a manner of the description of the material element shape.

Denote $s_s(H)$ the so-called "configurational entropy", i.e., the difference between the values of entropy in a stress-free state and in the thermodynamical reference state. Since

$$d\phi_s = du_s - \vartheta_0 ds_s(H), \tag{36}$$

our additional requirement mentioned above is equivalent to the statement that both $du_s(H)$ and $ds_s(H)$ are invariants. Accordingly the corresponding free energy functions and elastic strain energy functions are interrelated by

$$\phi^{(1)}[q_1(q), H] = \phi(q, H), \quad \phi_e^{(1)}[q_1(q), H] = \phi_e(q, H), \tag{37}$$

provided that both observers adopt the same stress-free thermodynamic reference state.

Remark. Hill [6] presented the following work interpretation for $d^p\phi$. Calculate the total mechanical work in the course of an arbitrary strain cycle, such as $A - B - C - A_1$ shown in Fig. 3:

$$\oint\limits_q p\, dq = [\phi(q, H + dH) - \phi(q, H)]^A + \int\limits_B^C (p\, dq - d\phi) \tag{38}$$

and choose $d\phi = p\, dq$ along BC (the segment of a route on the yield surface where the change of internal state occurs) to get (Eq. (2.33) in Hill [6])

$$\oint\limits_q p\, dq = d^p\phi(q, H). \tag{39}$$

We accept neither this interpretation nor some of the conclusions following from it (e.g. concerning the Iliuszyn postulate) discussed in Hill [6]. In the case of metallic solids the configurational entropy $s_s(H)$ is usually regarded as negligible (cf., e.g., [15]) so that $du_s \simeq d\phi_s$, and du_s can be measured using Eq. (35). According to the experimental evidence (cf., e.g., [12] and [13]) $du_s \approx d\phi_s$ is a small fraction of $\oint_{p=0} p\, dq$. Indeed from the combination of Eqs. (34) and (39) one gets

$$d^p\phi - d^p\phi_e = d^p\phi_s = \phi_s(H + dH) - \phi_s(H) = \oint\limits_{p=0} p\, dq. \tag{40}$$

The whole net work done in a stress cycle is accumulated in the bar in the form of the stored energy. In fact, by adopting $d\phi = p\, dq$, Hill [6] neglects the incremental energy dissipated along $B - C$.

The acceptable counterparts of Eqs. (39) and (40) satisfying the requirement of second law of thermodynamics are

$$\oint\limits_q p\, dq \geq d^p\phi(q, H) \Leftrightarrow d^p\phi_s = \phi_s(H + dH) - \phi_s(H) \leq \oint\limits_q p\, dq - d^p\phi_e(q, H) = \oint\limits_{p=0} p\, dq, \tag{41}$$

where the equality holds when $dH = 0$. In the course of the real processes the energy stored due to the cold work cannot exceed the plastic work done. This is a known implication of the second law of thermodynamics concerning an elastic-plastic continuum. The implication is independent of the choice of strain measure and/or reference configuration.

b. Transformation rules for other quantities

i) The transformation rules for all others thermostatic quantities occurring in Eqs. (19) and (20) can be found by direct differentiation of Eq. (37). In particular, the partial differentiation of Eq. (32.2) with respect to q_1 at constant H shows that the plastic increment $d^p p$ of the conjugate stress transforms like p:

$$d^p p_1 = Q^{-1} d^p p. \tag{42}$$

Hence, using Eqs. (19), (20), (24) and (28) we have

$$\delta q d^p p = -\delta p d^p q = \delta q dp - \delta p dq, \quad dq d^p p = (d^p p - dp) d^p q, \tag{43.1,2}$$

which are 1D examples of Hill's bilinear differential invariants. The work-interpretation of the invariant (43.1) follows from Eqs. (34) and (22.1). Since

$$\delta \left(\oint_{p=0} p \, dq \right) = 0,$$

the application of the δ-differential to Eq. (34) gives

$$\delta \left(\oint_{q} p \, dq \right) = \delta [d^p \phi_e(q, H)] = \delta (d^p \phi) = \delta q d^p p. \tag{44}$$

Note that to provide this interpretation we did not use the controversial equality (38), contrary to Hill [6].

The transformation rule for the tangent elastic modulus and its inverse follows from the calculation of the second derivative of Eq. (37) with respect to q_1, and use of Eq. (28),

$$\frac{L_1}{\varrho_1^R} = Q^{-2} \left(\frac{L}{\varrho^R} - p\Gamma \right), \quad \varrho_1^R M_1 = Q^2 \varrho^R M (1 - \varrho_R M p \Gamma)^{-1}, \tag{45.1,2}$$

where

$$\Gamma = Q^{-1} \frac{dQ}{dq} = \frac{dQ}{dq_1}. \tag{46}$$

When the group of admissible q is restricted to Eq. (27) the strain function Γ becomes

$$\Gamma = \frac{2(n_1 - n)}{1 + 2nq}, \tag{47}$$

such that $\Gamma = 0$ for $n_1 = n$. In general, $\Gamma = 0$ ($Q = \text{const.}$) provided that $f_1 = f$ and $f'(U) \equiv df/dU$ is a homogeneous function. This property does not hold in a 3D situation (except for $n = \pm 1$ in the family (2)).

Furthermore, the combination of the identity (24) (applied to both (p_1, q_1) and (p, q)) with use of Eqs. (42) and (45.1) results in the transformation rule for the plastic increment of strain

$$d^p q_1 = Q \, d^p q (1 - \varrho^R M p \Gamma)^{-1}. \tag{48}$$

Finally, calculate the d- and δ- differential of Eq. (33) to see that the rules for the stress increments are

$$dp_1 = Q^{-1}(dp - \Gamma p\, dq), \quad \delta p_1 = Q^{-1}(\delta p - \Gamma p \delta q). \tag{49.1,2}$$

The first relation could also be obtained by combining Eq. (42) with the state equations (19) and use of (45)–(48), whereas the relation (49.2) follows also from the multiplication of Eq. (45) by δq_1.

Remark. All transformation rules presented in this Subsection are 1D counterparts of 3D formulas derived by Hill (e.g. [6] and [8]). Hill aims at generality and displays the importance of the formulas (49) $[(p\, dq \equiv \text{invariant}) \Rightarrow (33) \Rightarrow (49)]$ also in the situation when the potential ϕ does not exist. In fact, from Eqs. (49) (that hold for any set of independent differentials) one can derive remaining transformation rules. We have here took for granted that Eq. (32.2) is also the prototype of the invariant and have emphasized the invariance of the elastic strain energy function and the stored free energy. This is necessary for a meaningful discussion concerning the definition of the invariant plastic work (Sect. 2.3). The invariance of the free energy (37) is also utilized in Sect. 3 to show the correlation between the theoretical framework of Sect. 2.1.b and other common approaches where the mobile stress-free configuration is regarded as the reference one. This is merely a small supplement to the outstanding constribution of Hill in this field.

ii) Let us briefly recall some conclusions following from the presented transformation rules.

In general Q is not constant ($\Gamma \neq 0$), therefore $dp_1 \neq Q^{-1}dp$ on account of Eq. (49). Hence, a closed infinitesimal cycle in p ($dp = 0$) does not necessarily close the infinitesimal cycle in p_1 (i.e., in general $dp = 0$ does not imply $dp_1 = 0$) until it starts at a stress-free state. In this sense we can say that the notion of an infinitesimal cycle in work-conjugate stress is not the invariant concept under the change of strain measure and reference configuration [6]. The same variance property concerns also the stress rate.

Denote by L^{e-p} the tangent modulus measured during active plastic yielding,

$$dp = \frac{1}{\varrho^R} L^{e-p} dq \Leftrightarrow d^p p = -\frac{1}{\varrho^R}(L - L^{e-p})dq. \tag{50.1,2}$$

According to Eq. (49) the transformation rule for L^{e-p} is the same as for L (cf. Eq. (45)). The vanishing of L_1^{e-p} for a particular choice of the work-conjugate couple $(q_1, p_1 \neq 0)$ does not imply vanishing of the modulus L^{e-p} for other choice, say (q, p)

$$L_1^{e-p} = 0 \Rightarrow \frac{1}{\varrho^R} L^{e-p} = p\Gamma. \tag{51}$$

In a 3D situation the states of a material where strain can change while stress is stationary are termed by Hill [8] the intrinsic eigenstates (cf. Fig. 2). If $\varrho_1^R dp_1/dq_1 = L_1^{e-p} = 0$ for the exponent n_1 in the family (27) then this intrinsic eigenstate will manifest itself at a point of the $p - q$ curve where (cf. (47))

$$\frac{dp}{dq} = \frac{1}{\varrho^R} L^{e-p} = \frac{2(n_1 - n)}{1 + 2nq} p \left(\frac{dp}{dq} > 0 \quad \text{if } n_1 > n \right). \tag{52}$$

In particular, in the 1D case when $n_1 = 1/2$ the intrinsic eigenstate corresponds to a point of maximum force (cf. Fig. 2).

Since according to Eq. (48) $d^p q$ does not transform like dq, the change in the length of a bar corresponding to $d^p q$ depends on the choice of the strain measure. Hence, the additive partitioning of incremental strain (cf. (19.2)) into elastic and plastic parts is also not an invariant concept [8]. The other consequence of Eq. (48) is discussed in Sect. 2.5.

The application of Eq. (42) to Eq. (50.2) shows that the relative modulus $L - L^{e-p}$ has the quite pleasant (different from L) transformation rule

$$\frac{1}{\varrho_1^R}(L_1 - L_1^{e-p}) = \frac{1}{\varrho^R} Q^{-2}(L - L^{e-p}). \tag{53}$$

In this sense the notion of a "relative tangent modulus" is the invariant concept [6], [8] (L and L^{e-p} link two independent pairs of differentials).

The immediate consequence of Eqs. (33) and (48) is that the work p expended on the plastic part of an incremental strain

$$(dW)_p = p\, d^p q \tag{54}$$

is not invariant, i.e., $(dW)_p \neq p_1\, d^p q_1 = (dW)_p^1$. The resulting transformation rule is

$$(dW)_p^1 = [1 - \varrho^R M p \Gamma]^{-1} (dW)_p. \tag{55}$$

2.3 Invariant incremental plastic work

Return to the strain cycle $A - B - C - A_1$ shown in Fig. 3. When (q,H) is on the yield surface (points A and B in Fig. 3 coincide) the second term occurring on the right-hand side of Eq. (34) vanishes to first order, and within this accurary one gets

$$-d^p \phi_e(q,H) = \oint_{p=0} p\, dq. \tag{56}$$

The exact relation between $d^p \phi_e$ and the right-hand side of Eq. (56) one obtains by considering the stress-cycle $A^* - B - C - C^*$ (Fig. 3) that starts at a natural state A^* $(p = 0, H)$ and ends at a neighboring natural state C^* $(p = 0, H + dH)$. Since

$$\int_{A^*}^{B} p\, dq = \phi[q(B),H] - \phi_s(H) = \phi_e[q(B),H],$$

$$\int_{B}^{C} p\, dq = \phi_e[q(C),H + dH] - \phi_e[q(B),H] - \int_{B}^{C} d^p \phi_e,$$

$$\int_{C}^{C^*} p\, dq = -\phi_e[q(C),H + dH],$$

the net total work done in such a cycle can be identified with an actual invariant plastic work

$$W = W^p(C^*,A^*) = \oint_{p=0} p\, dq = -\int_B^C d^p \phi_e. \tag{57}$$

The interpretation (57) enables to define the *invariant incremental plastic work* (incremental irreversible mechanical work) dW^p as

$$dW^p = -d^p \phi_e(q,H) = -\frac{\partial \phi_e}{\partial H} dH = p\, dq^* - d^p \psi_e(p,H) \tag{58}$$

on account of Eq. (23.2).

It concerns all those materials (rate-dependent or rate-independent) that behave elastically in a vicinity of every natural state, and for which it is possible to construct a strain energy function ϕ_e. *From Eq.* (22.1) *it follows that the invariant incremental plastic work* dW^p *when taken with reverse sign plays a role of the potential for* $d^p p$.

By combining Eq. (58) with the total differential of (15) one arrives at the following invariant decomposition of incremental total work:

$$dW = p\,dq = dW^e + dW^p, \quad dW^e = d\phi_e, \quad dW^p = -d^p\phi_e(q,H). \tag{59.1–3}$$

Accordingly the invariant increment of elastic work is equal to the total differential of $\phi_e = \Delta\phi - \phi_s$.

From Eq. (58) it follows that incremental irreversible mechanical work dW^p differs from the work of p expended in the incremental permanent strain, i.e., $dW^p \neq p\,dq^*$ independently of the range of plastic deformations. It also does not coincide with the work, say $(dW)_p$, of p done on the plastic part of an increment of total strain (cf. (54)). The connections between $(dW)_p$ and dW^p are

$$dW^p = (dW)_p - d^p\psi_e + p\frac{\partial(d^p\psi_e)}{\partial p} \tag{60}$$

on account of Eqs. (22.2) and (58).

In the thermodynamics of elastic-plastic materials, the plastic part of the increment of free energy function when taken with reverse sign represents the incremental energy dissipation dD, i.e., $dD = -d^p\phi$, provided that the total strain q is regarded as state variable. Combination of the plastic part of Eq. (14) with (58) shows that it is equal to the difference between the invariant incremental plastic work and the increment of the stored free energy

$$dD = dW^p - d\phi_s(H). \tag{61}$$

This relationship is valid for all elastic-plastic materials, including those whose elastic properties depend on prior plastic straining.

Remark. As an elementary example consider elastic-plastic damaging materials. Suppose that for some choice of reference configuration and strain measure the elastic strain energy takes the form

$$\varrho^R\phi_e(q,H) = \frac{1}{2}[q - q^*(H)]L(H)[q - q^*(H)], \quad \bar{p} \equiv \varrho^R p = L(H)[q - q^*(H)], \tag{62}$$

what implies the following form of the elastic complementary energy:

$$\varrho^R\psi_e = -\frac{1}{2}M(H)\bar{p}^2 \Leftrightarrow \psi_e = -\frac{1}{2}\varrho^R M p^2, \quad q = q^* + M(H)\bar{p} \tag{63}$$

on account of Eqs. (16) and (17). Here $M = L^{-1}$, and the set of internal variables H contains the damage parameter that describes the gradual degradation of elastic properties of a material (cf. the illustration shown in Fig. 4).

Application of the formula (63) to Eqs. (22.2) and (58) gives

$$\begin{aligned} d^p q &= dq^* + \bar{p}dM(H), \\ \varrho^R dW^p &= \bar{p}dq^*(H) + \frac{\bar{p}^2}{2}dM(H). \end{aligned} \tag{64}$$

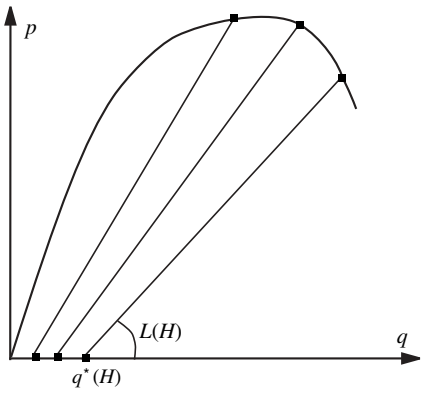

Fig. 4. Degradation of elastic properties

Hence the product $pd^pq = (dW)_p$ overestimates the actual invariant incremental plastic work dW^p,

$$\varrho^R dW^p = \varrho^R (dW)_p - \frac{\bar{p}^2}{2} dM(H), \tag{65}$$

since as a rule $dM(H) > 0$ in the course of damage.

2.4 Elastic-plastic materials with intrinsic master elasticity

From Eq. (60) it follows that $(dW)_p$ coincides with the invariant incremental plastic work dW^p whenever $d^p\psi_e$ is a homogeneous function of degree one with respect to p:

$$p\frac{\partial}{\partial p}(d^p\psi_e) = d^p\psi_e.$$

However, no such function satisfies the conditions (17.2). Therefore, coincidence may happen only if ψ_e is independent of H or equivalently if ϕ_e depends on H only through dependence of $q - q^*(H)$ on H, i.e., $\phi_e = \phi_e[q - q^*(H)]$. This should be read in proper context. Neither $(dW)_p$ nor ψ_e are invariant under the change of strain measure or/and reference configuration. Therefore, for a given material $(dW)_p$ may be equal to dW^p for one choice of q and be different from dW^p for the other choice.

Let us distinguish the class of materials for which there exists such q, say $\bar{q}$, that ϕ_e when expressed in terms of $\bar{q}$ has the special form

$$\phi_e = \phi_e[\bar{q} - \bar{q}^*(H)]. \tag{66}$$

Since ϕ_e depends on H only through dependence of $\bar{q} - \bar{q}^*(H)$ on H, the elastic behavior of the material is described by a single "master" constitutive law, provided it is expressed in terms of quantities work-conjugate to $\bar{q}$. The material can be said to exhibit *master elasticity* that is independent of the prior loading history. For such a material the elastic complementary energy expressed in terms of $\bar{p}$ conjugate to $\bar{q}$ is independent of H:

$$\psi_e = \psi_e(\bar{p}) \tag{67}$$

(cf. Eqs. (16) and (17)).

The following corollaries are immediate consequences of Eqs. (22.2) and (58):

i) For every elastic-plastic material with master elasticity there exists the conjugate couple $(\bar{p}, \bar{q})$ for which

$$d^p\bar{q} = d\bar{q}^*(H), \quad dW^p = \bar{p}d^p\bar{q} = \bar{p}d\bar{q}^*(H). \tag{68.1,2}$$

ii) Conversely, if for a given material there exists the conjugate couple $(\bar{p}, \bar{q})$ such that either Eq. (68.1) or (68.2) holds, then the behavior of the material is characterized by intrinsic master elasticity, so that ϕ_e and ψ_e when expressed in terms of $(\bar{p}, \bar{q})$ have the special form (66) and (67).

The decision whether the actual material can be categorized as exhibiting master elasticity must rest upon careful theoretical analysis of experimental data. For example, the unloading curves in simple tension when traced in the nominal stress-engineering strain plane may have different slopes for different prestrains, whereas when they are drawn in the Kirchhoff true stress-logarithmic strain plane they may be formed from a single "master curve". In the latter case the logarithmic strain is the example of $\bar{q}$.

2.5 Special case: updated Lagrangian formalism

We shall now briefly discuss the situation when at the generic instant t of a process of deformation the reference configuration of the bar is assumed to coincide with its actual configuration (updated Lagrangian description). From Eqs. (1)–(3) it readily follows that in the limit $l^R \to l(t), \varrho^R \to \varrho(t), A^R \to A(t), U \to 1$ $(f(U) \to 0, f'(U) \to 1)$ we have

$$q \to 0 \quad \text{and} \quad dq \to dl(t)/l(t) \tag{69}$$

for all measures. That is, when the current configuration is regarded as common reference configuration, all strain measures vanish, whereas their increments coincide (to the first order) and are equal to the increment of the logarithmic strain $d\varepsilon_L = dl(t)/l(t)$. In the uniaxial case $d\varepsilon_L = D_{11}dt$, where D_{11} is the axial component of the usual Eulerian strain-rate. Thus, in 1D updated Lagrangian description all strain rates dq/dt coincide with D_{11}. By calculating the limits in Eq. (5) it is found that

$$p(t) = \tau(t) \quad \text{and} \quad \bar{p}(t) = \varrho p = \sigma. \tag{70}$$

In the considered situation all specific and standard work-conjugate stresses are equal to the Kirchhoff's and Cauchy's stress, respectively. Moreover, the limiting values of Eq. (7) are

$$\begin{aligned} &\varrho d\tau = d\sigma - \sigma(d\varrho/\varrho), \quad dp = d\tau - 2m\tau d\varepsilon_L, \\ &d\bar{p} = \varrho dp = d\sigma - 2m\sigma d\varepsilon_L - \sigma(d\varrho/\varrho), \quad 2m \equiv \left.\frac{d^2f}{dz^2}\right|_{z=1} + 1. \end{aligned} \tag{71.1–4}$$

Equation (71.2), when written in the rate form, illustrates the 1D axial component of Hill's family of objective stress-rates $\overset{(m)}{\sigma}_{11}dt = d\bar{p}$

$$\overset{(m)}{\sigma}_{11} = \dot{\sigma} - 2m\sigma D_{11} - \sigma(\dot{\varrho}/\varrho), \quad \dot{\sigma} \equiv \frac{d\sigma(t)}{dt}. \tag{72}$$

The parameter m represents the selected strain measure ($m = 0$ for the logarithmic measure). Let $\mathcal{L}(m)$ be the limit of instantaneous tangent moduli L. In the elastic domain we have $\overset{(m)}{\sigma}_{11} = \mathcal{L}(m)D_{11}$. Since D_{11} is the common limit of rate of all strains we also have $\overset{(m_1)}{\sigma}_{11} = \mathcal{L}(m_1)D_{11}$ for another choice of strain measure represented by the parameter m_1.

Elimination of D_{11} between these two equations (with account of Eqs. (72)) gives the transformation rule for the tangent elastic modulus and its inverse $\mathcal{M} = \mathcal{L}^{-1}$,

$$\mathcal{L}(m_1) = \mathcal{L}(m) - 2(m_1 - m)\sigma, \quad \mathcal{M}(m_1) = \mathcal{M}(m)[1 - 2(m_1 - m)\mathcal{M}(m)\sigma]^{-1}. \tag{73}$$

This can be derived directly from Eq. (45) since the limit values of Q and Γ occurring in Eqs. (28) and (46) are

$$Q = 1, \quad \Gamma = 2(m_1 - m). \tag{74}$$

Hence, the updated Lagrange form of the transformation rule (42) is

$$d^p p_1 = d^p p \equiv \overset{0}{\sigma}{}^p dt. \tag{75}$$

Similarly as strain-rates, the plastic parts of all work-conjugate stress-rates are independent (to the first order) of the selected strain measure, provided that the current configuration is taken as the common reference one. The updated Lagrangian rate-form of the transformation rule (49) is

$$\overset{(m_1)}{\sigma}{}_{11} = \overset{(m)}{\sigma}{}_{11} - 2(m_1 - m)\sigma D_{11}. \tag{76}$$

It can be also obtained directly from Eq. (72).

The additive partitioning of the strain increment (19.2) can now be presented as

$$\overset{(m)}{\sigma} = \mathcal{L}(m)D_{11} + \overset{0}{\sigma}{}^p, \quad D_{11} = \mathcal{M}(m)\overset{(m)}{\sigma} + D^p_{11}(m), \tag{77}$$

where $D^p_{11}(m)dt$ are the limits of $d^p q$. Both $D^e_{11}(m) = D_{11} - D^p_{11}(m)$ and $D^p_{11}(m)$ depend on the choice of strain measure characterized by the cooefficient m (cf. Eq. (71.2)), i.e., the additive partitioning of D_{11} is not an invariant concept. The transformation rules for D^p_{11} and $(dW)_p$ are the updated counterparts of Eqs. (48) and (54)

$$\begin{aligned} D^p_{11}(m_1) &= [1 - 2\sigma(m_1 - m)\mathcal{M}(m)]^{-1} D^p_{11}(m), \\ (dW)^1_p &= [1 - 2\sigma(m_1 - m)\mathcal{M}(m)]^{-1}(dW)_p. \end{aligned} \tag{78}$$

As expected, the relative difference

$$\frac{D^p_{11}(m)}{D^p_{11}(m_1)} - 1 = \frac{2\sigma(m_1 - m)}{\mathcal{L}(m) - 2\sigma(m_1 - m)} \approx 2(m_1 - m)\frac{\sigma}{\mathcal{L}(m)} \tag{79}$$

is a small quantity for most metallic solids.

Finally, let $\mathcal{L}^{e-p}(m)$ be the tangent elastic-plastic modulus defined by the first-order updated Lagrangian rate version of (50),

$$\overset{(m)}{\sigma} = \mathcal{L}^{e-p}(m)D_{11}. \tag{80}$$

The transformation rules for $\mathcal{L}^{e-p}$ and $\mathcal{L}$ are the same (cf. Eq. (73)). Hence, if the intrinsic eigenstate associated with the m_1-strain measure is defined by $\mathcal{L}^{e-p}(m_1) = 0$, it will manifest itself at states where

$$\mathcal{L}^{e-p}(m) = 2(m_1 - m)\sigma \quad \text{or} \quad \overset{(m)}{\sigma} = 2(m_1 - m)\sigma D_{11}. \tag{81}$$

In particular, for $m_1 = 1/2$ Eq. (81) becomes the engineering necking criterion (a state where the applied force attains its maximum)

$$\overset{(m)}{\sigma} = (1 - 2m)\sigma D_{11} \Leftrightarrow \dot{\sigma} - \sigma(\dot{\varrho}/\varrho) = \sigma D_{11}. \tag{82.1,2}$$

Equations (81) and (82) are updated Lagrangian versions of Eqs. (52) and (8), respectively. To get Eq. (82.2) use was made of Eq. (72).

3 Eckart-Mandel theoretical framework

i) There is a slightly different framework (cf., e.g., [1] and [11]) for the structure of elastic-plastic constitutive laws where the changes in the current shape of a material element are described treating the variable conceptual stress-free configuration as the reference one. The primary kinematical quantities are elastic strain q_e and incremental permanent strain ε^*. In the uniaxial stress state they are defined as (cf. Fig. 1)

$$q_e = f(U_e) \quad \text{and} \quad d\varepsilon^* = \frac{dl^*}{l^*} \quad \text{and} \quad U_e = \frac{l}{l^*}\,, \tag{83}$$

respectively. The scale function f has the same mathematical properties as that occurring in Eq. (1). In what follows, f will be assumed to be defined by Eq. (2) so that

$$q_e = \frac{U_e^{2n_1} - 1}{2n_1}, \tag{84}$$

where now the arbitrary scale exponent is denoted by n_1. The increments of elastic strain dq_e and permanent strain $d\varepsilon^*$ together determine the current change in the length of a bar. However, when $n_1 \neq 0$, their sum is not equal to an increment of any total strain.

In view of the relationship $dl = l^* dU_e + l d\varepsilon^*$ the incremental total work can now be expressed in the form

$$dW = \tau d\varepsilon^* + p_e dq_e, \tag{85}$$

where

$$p_e = \frac{Pl}{m_0} U_e^{-2n_1} = (1 + 2n_1 q_e)^{-1} \tau \tag{86}$$

is the elastic specific stress work-conjugate to q_e in the course of elastic unloading processes ($d\varepsilon^* = 0$). Note that p_e is not equal to the stress p work-conjugate with the total strain (cf. (6)), unless $n_1 = n = 0$.

Within the Eckart–Mandel (E-M) theoretical framework the specific free energy ϕ is assumed to be a function of q_e and H,

$$\phi(q_e, H) - \phi_0 = \phi_e(q_e, H) + \phi_s(H), \tag{87}$$

so that

$$p_e = \frac{\partial \phi_e(q_e, H)}{\partial q_e}, \quad \frac{\partial \phi_e}{\partial q_e} = \phi_e = 0, \quad \text{for} \quad q_e = 0. \tag{88.1,2}$$

Here ϕ_e and ϕ_s are the elastic strain energy and the part of free energy that is independent of q_e, respectively. They have the same physical meaning as the quantities defined in Sect. 2.1.b. However, the function $\phi_e(q_e, H)$ is different from the function $\phi_e(q, H)$. The dual potential has also different physical meaning. It is defined by

$$\psi_e^{(1)}(p_e, H) = \phi_e - p_e q_e, \quad \psi_e^{(1)}(0, H) = \left.\frac{\partial \psi_e^{(1)}}{\partial q_e}\right|_{q_e = 0} = 0 \tag{89.1,2}$$

such that the inverse relation to Eq. (88) becomes

$$q_e = -\frac{\partial \psi_e^{(1)}(p_e,H)}{\partial p_e}. \tag{90}$$

For materials with *elastic properties insensitive to prior plastic straining* the functions ϕ_e and $\psi_e^{(1)}$ are independent of H. At the end of this Section it is shown that in a uniaxial situation every such material exhibits master elasticity properties (cf. Sect. 2.4). However, *not every material possessing master elasticity has elastic properties insensitive to prior plastic straining*.

ii) Introduce the notation

$$d^h\mathcal{A}(q_e,H) = \frac{\partial \mathcal{A}(q_e,H)}{\partial H}dH \quad \text{and} \quad d^h\mathcal{A}(p_e,H) = \frac{\partial \mathcal{A}(p_e,H)}{\partial H}dH$$

for the H-differential of any constitutive function $\mathcal{A}(q_e,H)$ or $\mathcal{A}(p_e,H)$. By calculating H-differentials of (89.1), (90) and (88.1) one gets the following identities:

$$\begin{aligned} &d^h\psi_e^{(1)}(p_e,H) = d^h\phi_e(q_e,H), \\ &d^hq_e = -\frac{\partial}{\partial p_e}\Big[d^h\psi_e^{(1)}(p_e,H)\Big] \end{aligned} \tag{91.1,2}$$

and

$$d^hp_e = \frac{\partial d^h\psi_e^{(1)}(q_e,H)}{\partial q_e} = -\frac{L^*}{\varrho_0}d^hq_e(p_e,H) \tag{92}$$

that hold for any dual couple (q_e,p_e) connected by Eq. (88.1). Here L^* is the elastic tangent modulus defined as

$$L^*(q_e,H) \equiv \varrho_0\frac{\partial^2\phi_e(q_e,H)}{\partial q_e^2} \to M^* = L^{*-1} = -\varrho_0\frac{\partial^2\psi_e^{(1)}(p_e,H)}{\partial p_e^2} \tag{93}$$

and ϱ_0 is the density in the thermodynamical reference state. The incremental elastic response of a material is described by two equivalent equations

$$dq_e = \varrho_0M^*dp_e + d^hq_e(p_e,H) \Leftrightarrow dp_e = \frac{L^*}{\varrho_0}dq_e + d^hp_e(q_e,H). \tag{94}$$

The description of the total incremental deformation behavior consists of Eq. (94) and appropriate equations for $d\varepsilon^*$ and d^hq_e (or d^hp_e). The latter requires knowledge of the evolution equation for H.

In analogy to (59.2) the total differential of ϕ_e is indentified with the invariant incremental elastic work

$$dW^e = d\phi_e = p_edq_e + d^h\phi_e(q_e,H), \tag{95}$$

whereas the invariant incremental plastic work dW^p is (cf. Eqs. (85) and (91.1))

$$dW^p = dW - dW^e = \tau d\varepsilon^* - d^h\phi_e(q_e,H) = \tau d\varepsilon^* - d^h\psi_e^{(1)}(p_e,H). \tag{96}$$

It is seen that in the course of plastic flow ($dH \neq 0$) the incremental elastic work is not equal to p_edq_e unless the elastic properties of a material are not influenced by prior plastic straining.

As may be seen, there is a certain mathematical analogy between the equations presented here and those discussed in Sect. 2.1. *The basic difference is that now the specification of the free energy function does not require knowledge of the detailed dependence of any plastic strain on H.* Instead, the additional constitutive equation for the increment of the permanent strain $d\varepsilon^*$ must be established. From the thermodynamics point of view this fact follows from different choices

of state variables and thermodynamical forces. In fact, the incremental energy dissipation dD is $dD = -d^p\phi(q,H)$, when a total strain is taken as state variable, and

$$dD = dW^p - d\phi_s(H) = \tau d\varepsilon^* - d^h\phi_e(q_e,H) - d\phi_s(H) \tag{97}$$

when elastic strain is treated as external state parameter. Thus, now $d\varepsilon^*$ and dH appear as rates of irreversible flows.

Within the E-M theoretical framework it is meaningful to investigate relativity and invariance of constitutive objects under different choices of elastic strain measure. The transformation formulas will be identical to that presented in Sect. 2.2, provided q is replaced by q_e and l^R/l_1^R is set to be unity. In particular, ϕ_e, ϕ_s, $p_e dq_e$, $d^p\phi_e$, dW^p, $d^h q_e dp_e$, $d^h p_e dq_e$ are the examples of basic invariants, under change of elastic strain measure.

iii) There is a definite connection between the Hill-Rice (H-R) framework and the E-M framework. This is briefly discussed beneath.

When the description of material properties within the E-M theoretical framework is completed then the rate equations linking the increments of a total strain q and its work-conjugate stress p can be derived with use of the appropriate kinematical relationships only. Under uniaxial stress state this task is relatively simple.

Choose any reference configuration of a bar and denote by U its stretch $U = l/l^R$. It can be multiplicatively decomposed as $U = U_e U_p(H)$ where $U_p = l^*(H)/l^R$, so that $\varepsilon^*(H) = \ln U_p = \ln(l^*/l^R)$. Let $q = (U^{2n}-1)/2n = (U_p^{2n}U_e^{2n}-1)/2n$ be any total strain measure of the family (2). Eliminating U_e between this equation and Eq. (84) leads to the definite relationship between q_e, q and $U_p(H) = \exp[\varepsilon^*(H)]$:

$$q_e = q_e(q,H) = \frac{1}{2n_1}\Big[(1+2nq)^{n_1/n}U_p^{-2n_1}(H) - 1\Big], \tag{98.1}$$

$$q = q(q_e,H) = \frac{1}{2n}\Big[(1+2n_1q_e)^{n/n_1}U_p^{2n}(H) - 1\Big]. \tag{98.2}$$

This can be viewed as basic transformation formulas for the transition from the H-R description to the E-M description. The important connection following from Eq. (98) is the identity

$$\tau d\varepsilon^* = -p_e d^p q_e(q,H), \quad d^p q_e(q,H) \equiv \frac{\partial q_e(q,H)}{\partial H}dH, \tag{99}$$

that can be verified using Eqs. (86) and (98.1). If one inserts (98.1) into $\phi_e(q_e,H)$ then the result is the function $\phi_e(q,H)$ occurring in Eq. (14) (it is invariant under the change of the reference configuration and the total strain measure)

$$\phi_e[q_e(q,H),H] = \phi_e(q,H). \tag{100}$$

By calculating the partial derivative of this equation with respect to q one finds the relation between the stresses p_e and p (cf. Eqs. (15) and (88)):

$$\begin{aligned} &p = Q^* p_e, \quad p_e = \overset{-1}{Q^*} p, \\ &\overset{-1}{Q^*} = \frac{\partial q(q_e,H)}{\partial q_e} = \frac{1+2nq}{1+2n_1q_e}, \quad Q^* = \frac{\partial q_e}{\partial q} = \frac{1+2n_1q_e}{1+2nq}. \end{aligned} \tag{101}$$

From Eq. (100) it follows also that the definitions (58) and (96) of the incremental plastic work are equivalent: $dW^p = -d^p\phi_e(q,H) = -p_e d^p q_e - d^h\phi_e(q_e,H) = \tau d\varepsilon^* - d^h\phi_e(q_e,H)$, on

account of (99). Likewise the additive decompositions of dW presented in Eqs. (59.1) and (95)–(96) are equivalent.

There are a number of identities following from Eq. (100). Use of them enables to show full equivalence between the incremental relation (94) and (19). This matter, however, will not be pursued further here.

iv) Substitute $n = 0$ into Eq. (98) (logarithmic total strain measure is adopted in the H-R description) to find

$$q_e = \frac{1}{2n_1}\{\exp[2n_1(q - \varepsilon^*)] - 1\}. \tag{102}$$

If $\phi_e = \phi_e(q_e)$ then $\phi_e(q,H) = \phi_e(q - \varepsilon^*)$, where q and ε^* are total and permanent logarithmic strains, respectively. Thus, the function $\phi_e(q,H)$ has the property (66), and the material with elastic properties insensitive to prior plastic straining belongs to the class of materials exhibiting master elasticity (cf. Sect. 2.4). The converse of this statement is not valid. To see it substitute, for example, $n = 1/2$ into Eq. (98.2) (engineering total strain) and find

$$q - q^*(H) = (U_e - 1)U_p(H) = [(1 + 2n_1 q_e)^{\frac{1}{2n_1}} - 1]U_p(H), \tag{103}$$

where $q^*(H) = U_p(H) - 1$. Hence $\phi_e(q - q^*)$ does not imply $\phi_e = \phi_e(q_e)$ because U_p depends on H.

4 Concluding remarks

The instructive presentation of the subtle aspects of invariances (under the change of strain measure and/or reference configuration) in solid mechanics is not an easy task even in the case of simple tension (extension). Presuming that the incremental free energy of a specimen is the basic invariant, we have derived Hill's transformation formula for the number of fundamental quantities in a manner different from that of Hill. The following quantities are shown to be invariant in the sense that their transformation rules are linear and homogeneous:

- The incremental strain dq and the work-conjugate stress p
- The plastic increment of work conjugate stress $d^p p$
- The incremental irreversible mechanical work $dW^p = -d^p\phi_e$ (identified here as an invariant incremental plastic work) which plays the role of a potential for $d^p p$–these properties seem to have been not demonstrated so far in the literature.

The other quantities (such as tangent elastic modulus L, total increment of work-conjugate stress dp, plastic increment of total strain $d^p q$, work of work-conjugate stress done on $d^p q$, i.e., $(dW)_p = pd^p q$) were shown to have more complex transformation rules. Their knowledge helps to judge whether the mathematically distinct constitutive descriptions employing different strain measures concern the same physical material. Moreover, they help to distinguish the universal concepts (e.g., notion of incremental strain cycle) from others that are not invariant (e.g., notion of incremental stress cycle, linearity, positive semi-definiteness of tangent moduli).

We found some difficulties to give the rigorous mathematical description of the statement "elastic properties are not influenced by prior plastic deformation" in the situation when total strain is regarded as state parameter (H-R formalism). To this end in this paper we decided to distinguish the class of materials with "intrinsic master plasticity" (Sect. 2.4).

This problem does not occur when the E-M theoretical framework is applied, since then ϕ_e does not depend on H. It has been shown that if (in the uniaxial situation) the total logarithmic strain is used and $\phi_e = \phi_e[q - q^*(H)]$ then the materials with intrinsic master plasticity have the elastic properties insensitive to prior plastic straining.

In uniaxial case the basic interrelations between the two theoretical frameworks are expressed in terms of the connections (98). In that sense it is shown here that they are equivalent. However, use of the mobile Lagrangian description (E-M theoretical framework) brings (on phenomenological ground) an extra information upon the structure of the incremental energy dissipation.

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