

# Brief Note

## Three-particle motion under gravity in Stokes flow: an equilibrium for spheres in contrast to “an end-of-world” for point particles

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*In Memory of Professor Henryk Zorski*

EVOLUTION OF THREE IDENTICAL solid spheres falling under gravity in a low-Reynolds number flow is investigated for a symmetric initial configuration. Three spheres aligned horizontally at equal distances evolve towards an equilibrium relative configuration while the point particles collapse onto a single point in a finite time.

### 1. Introduction

THE STRUCTURE AND VELOCITY FIELD of a sedimenting non-Brownian suspension have been recently shown to differ significantly from the equilibrium state [1]. Explanation of this behavior is of the great interest. Evolution of the suspension is governed by many-body hydrodynamic interactions between its individual particles. Therefore it is important to understand the evolution of exemplary systems of only several solid particles moving under gravity in a quiescent viscous infinite fluid. Dynamics of the simplest interesting cluster consisting of three identical point particles was analyzed numerically, starting from positions aligned horizontally [2]. In general, for unequal distances from the middle particle to the other ones, the system exhibits a chaotic behavior [2]. However, it has not yet been investigated what happens for the equal distances, neither for point-particles nor for solid spheres.

Therefore in this paper, we present evolution of a symmetric initial configuration of three solid spheres, with two equal interparticle distances, and we illustrate how do hydrodynamic interactions between three identical spheres differ from those predicted by the point-particle approximation [3].

## 2. The problem

We consider time-dependent symmetric configurations of the sphere centers located in a vertical plane and illustrated in Fig. 1.

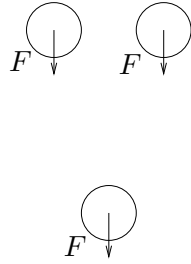


FIG. 1. Symmetric configurations of the spheres in a vertical plane.

The fluid velocity  $\mathbf{v}$  and pressure  $p$  satisfy the Stokes equations [4],

$$(2.1) \quad \eta \nabla^2 \mathbf{v} - \nabla p = \mathbf{0}, \quad \nabla \cdot \mathbf{v} = 0,$$

with the fluid viscosity  $\eta$ . Stick boundary conditions are assumed at the surfaces of the spheres. Positions of the sphere centers  $\mathbf{r}_i(t)$  satisfy the following system of ordinary differential equations, called the Stokesian dynamics [5],

$$(2.2) \quad \dot{\mathbf{r}}_i(t) = \left[ \sum_{k=1}^3 \boldsymbol{\mu}_{ik} \right] \cdot \mathbf{F}, \quad i = 1, 2, 3.$$

The mobility matrices  $\boldsymbol{\mu}_{ik}$  are evaluated from the multipole expansion with the multipole order  $L = 4$ . The algorithm from Ref. [6] and its accurate numerical FORTRAN implementation described in Ref. [7] are used. In the point-particle approximation [4],

$$(2.3) \quad \boldsymbol{\mu}_{ii} = \frac{1}{3\pi\eta d} \mathbf{I}, \quad \text{and} \quad \boldsymbol{\mu}_{ik} = \frac{1}{8\pi\eta r_{ik}} (\mathbf{I} + \hat{\mathbf{r}}_{ik} \hat{\mathbf{r}}_{ik}), \quad \text{for } i \neq k.$$

From the symmetry of Stokes equations it follows that the configuration, shown in Fig. 1, stays symmetric in the course of time. Using the frame of reference and the notation indicated in Fig. 2, we obtain from Eqs. (2.2) a simple system of equations for the symmetric relative motion,

$$(2.4) \quad \dot{x} = v_x(x, z),$$

$$(2.5) \quad \dot{z} = v_z(x, z),$$

with the distances  $x$ ,  $z$  normalized by the sphere diameter  $d$ , the velocities  $v_x$ ,  $v_z$  by the Stokes velocity  $U_s = F/(3\pi\eta d)$ , and time by  $d/U_s$ . For point-particles, the dynamics (2.4)–(2.5) is scale-invariant. For spheres, it depends on  $d$ .

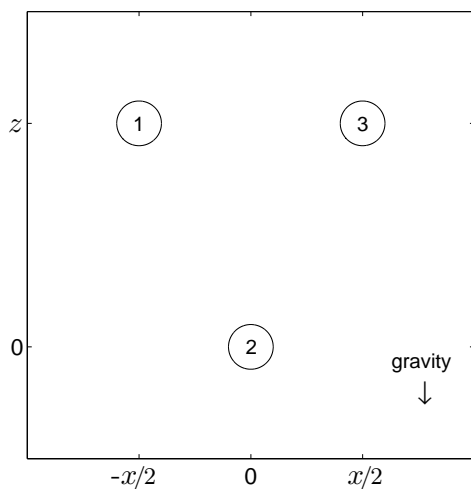


FIG. 2. Frame of reference and notation.

The dynamics is integrated by the adaptive fourth-order Runge–Kutta method.

### 3. Results

First, we evaluate evolution of spheres aligned horizontally and we compare it with the point-particle approximation. After a time, the relative configuration is shown in Fig. 3. The point particles have collapsed onto a single point, with infinite velocities. Obviously, spheres have evolved differently than point-particles. They seem to have stopped at a “kissing” configuration of spheres 1 and 3, with the sphere 2 (tactfully) separated.

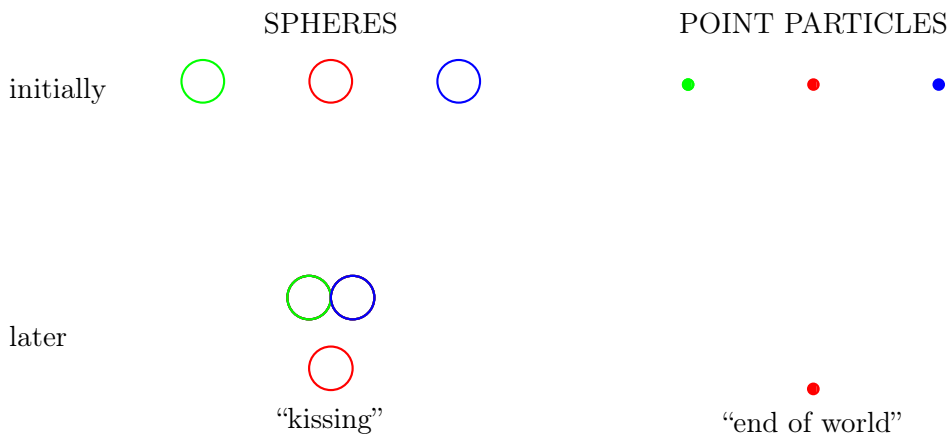

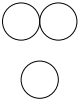

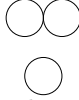


FIG. 3. Evolution of particles aligned horizontally.

It is of interest to explore what is the equilibrium configuration: do the spheres

tend to  or  ?

To answer this question, the relative motion has been investigated when spheres 1 and 3 touch each other [3]. The relative vertical velocity  $v_z(1, z)$ , evaluated numerically, is plotted in Fig. 4. This function has two zeros, which correspond to two equilibria:

 at  $z = \sqrt{3}/2$  and  at  $z = 1.579$  [3]. The

last value has been calculated for  $L = 4$ ; it is recovered with 2% accuracy even at  $L = 1$ . Moreover, the corresponding equilibrium also exists if two vertical walls are added at  $x = \pm H$  (or at  $y = \pm D$ ). In the presence of the walls, for  $L = 4$ , the equilibrium interparticle distance differs from  $z = 1.579$  by less than 2% if  $H > 10.0$  (or  $D > 8.6$ ).

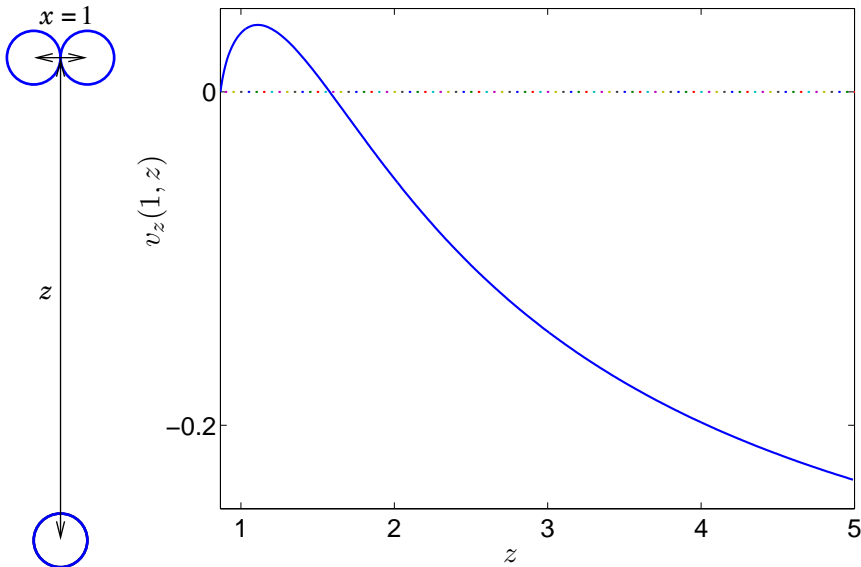



FIG. 4. Relative vertical velocity  $v_z(1, z)$  versus vertical separation  $z$ .

The phase portrait of the relative dynamics of the spheres (2.4)–(2.5), evaluated in Ref. [3], is depicted in Fig. 5. All the trajectories are symmetric under reflection in the plane  $z = 0$ , superposed with the time reversal. The equilibrium  is a stable node, and its reflected counterpart is unstable [3]. In Fig. 6, a typical sphere trajectory is compared with the point-particle approximation. Initially the particles are aligned horizontally. Sizes of the spheres and the stable equilibrium are explicitly shown.

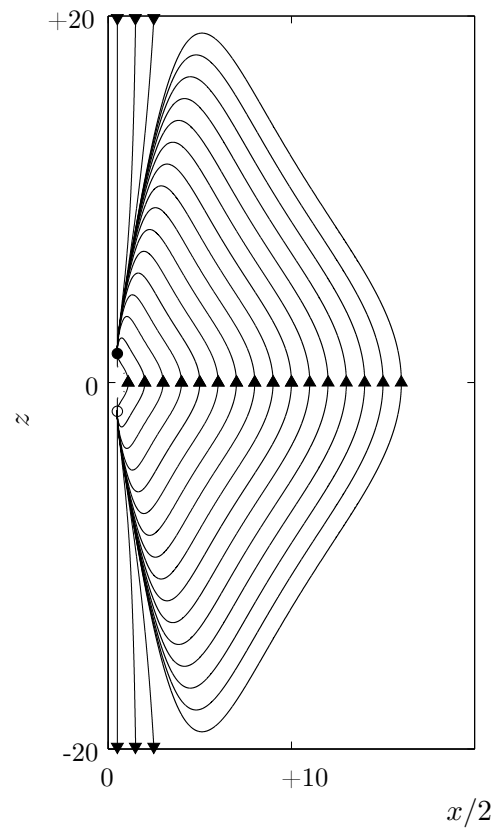


FIG. 5. Phase portrait of the relative dynamics (2.4)–(2.5). Stable and unstable nodes are marked as  $\bullet$  and  $\circ$ , respectively.

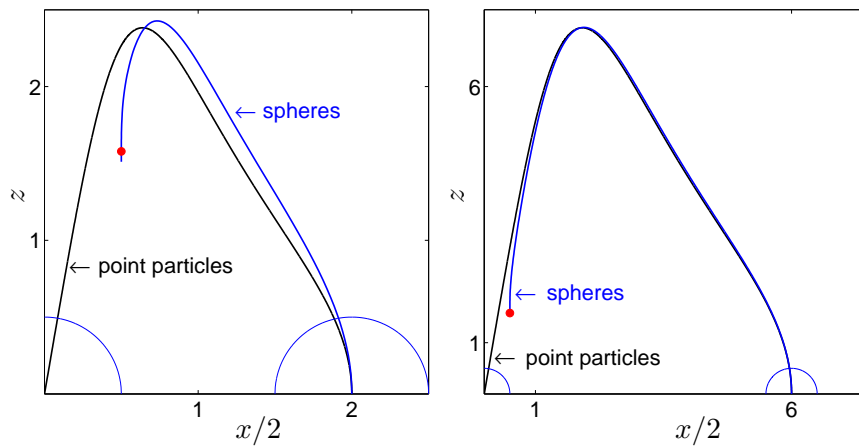


FIG. 6. Typical relative trajectories of close and distant spheres are compared with the corresponding point-particle approximation. Initially centers are at the same configuration, aligned horizontally.

## 4. Conclusions

We have found an equilibrium configuration of three identical spheres sedimenting steadily under gravity in a low-Reynolds-number fluid flow [3]. At the equilibrium, the two upper spheres touch each other and are well-separated from the lower one. The equilibrium is stable against the symmetric perturbations, and it is reached after infinite time [3]. We have shown that point-particles with the same initial condition evolve differently – they experience “the end-of-world”, collapsing onto a single point in a finite time.

## References

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