

Two-dimensional Hooke's tensors – isotropic decomposition, effective symmetry criteria

A. BLINOWSKI, J. OSTROWSKA – MACIEJEWSKA
and J. RYCHLEWSKI (WARSZAWA)

ANY FOURTH RANK plane tensor \mathbf{H} obeying the "Hooke's" symmetries ($H_{ijkl} = H_{jikl} = H_{klij}$) can be split into three parts, behaving differently under the two-dimensional space rotation and belonging to the three different, mutually orthogonal, two-dimensional subspaces remaining stable under the rotation. Such representation leads to a convenient set of functionally independent invariants, vanishing of some of these invariants demarcating the transitions of the tensor to the higher symmetry class. A non-trivial effective condition of orthotropy has been obtained. Some problems concerning the necessary and complete set of measurements of the elastic properties are also encountered.

1. Introduction

LARGE VARIETY of engineering problems of structural mechanics concerning the applications of natural or man-made anisotropic composite materials can be effectively analyzed with the use of the plane stress and/or strain state concepts. Thus the convenient description of the plane elasticity and limit criteria is not only of theoretical, but also of practical interest. In some recent papers [8, 9], it was shown that some problems, which, due to their discouraging complexity, look rather boring and demanding time-consuming analysis in general (three-dimensional) case (cf. [5]), can be, with moderate efforts, effectively solved in the plane case.

In the present paper the authors will demonstrate an effective description of the properties of Hooke's tensor making easier both the better comprehension of the matter and the practical applications of the results. Almost all the considerations can be applied without change to elastic stiffness and/or compliance tensors as well as to the quadratic limit condition tensor. The results, together with the earlier obtained results presented in [8, 9] exhaust most of the practical aspects of the description of anisotropy of the plane, linearly elastic and quadratic limit properties⁽¹⁾.

2. Hooke's tensors

Our subject are plane tensors of the fourth rank $\mathbf{H} \in \mathcal{T}_4$, having the following internal symmetries:

$$(2.1) \quad H_{ijkl} = H_{jikl} = H_{ijlk} = H_{klij}.$$

⁽¹⁾ Some interesting but purely theoretical problems, like the polynomial integrity basis, remain out of the sphere of our interest in the present paper.

The most important tensors of this kind are the *stiffness tensors* and the *compliance tensors* of the theory of plane elasticity, thus in [3] it was proposed to call them *Hooke's tensors*. Among the applications of Hooke's tensors one can mention their role as linear operators $\alpha \rightarrow \mathbf{H} \cdot \alpha$, bilinear forms $(\alpha, \beta) \rightarrow \alpha \cdot \mathbf{H} \cdot \beta$ or as quadratic functionals $\alpha \rightarrow \alpha \cdot \mathbf{H} \cdot \alpha$, e.g. functionals of energy or the limit stress intensity [1]. A Hooke's tensor \mathbf{H} can play a role of the stiffness or compliance tensor only if $\alpha \cdot \mathbf{H} \cdot \alpha \geq 0$ for every α .

In the present section we shall present the important decompositions of the Hooke's tensors, useful for the analysis of the symmetries and the invariance. It would be convenient to begin with recalling the notions and the notation for the second rank tensors.

2.1. Second rank plane tensors

All the *isotropic orthogonal decompositions* of the plane second rank tensors are included in the following formula:

$$(2.2) \quad \mathcal{T}_2 = \mathcal{S} + \mathcal{A} = \mathcal{P} + \mathcal{D} + \mathcal{A}, \quad 2^2 = 3 + 1 = 1 + 2 + 1,$$

where \mathcal{S} is the three-dimensional space of plane *symmetric* tensors $\alpha^T = \alpha$, \mathcal{A} is the one-dimensional space of *skew-symmetric* tensors $\alpha^T = -\alpha$, \mathcal{P} is one-dimensional space of *isotropic* tensors $u\mathbf{1}$ and \mathcal{D} is the plane of the two-dimensional deviators: $\alpha^T = \alpha$, $\text{tr } \alpha = 0$. These decompositions are orthogonal, $\mathcal{S} \perp \mathcal{A}$, $\mathcal{P} \perp \mathcal{D}$. To these decompositions correspond the following *orthogonal decompositions of unity* \mathcal{I} of $\mathbb{E} \otimes \mathbb{E}$ (see (A.5))

$$(2.3) \quad \mathcal{I} = \mathcal{I}_S + \mathcal{I}_A = \mathcal{I}_P + \mathcal{I}_D + \mathcal{I}_A,$$

where

$$(\mathcal{I}_S)_{ijkl} = \frac{1}{2} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

$$(\mathcal{I}_P)_{ijkl} = \frac{1}{2} \delta_{ij}\delta_{kl}.$$

The unity \mathcal{I}_S of the space \mathcal{S} (see (A.5)) acting on the second rank tensors, $\alpha \rightarrow \mathcal{I}_S \cdot \alpha$, performs an orthogonal projection of the space \mathcal{T}_2 onto the \mathcal{S} space, hence $\mathcal{I}_S \cdot \alpha = \alpha$ iff $\alpha \in \mathcal{S}$. The other unities \mathcal{I}_D , \mathcal{I}_P , \mathcal{I}_A act in a similar way.

In the forthcoming considerations the one-dimensional space \mathcal{A} and its unity \mathcal{I}_A will remain out of the scope of our interest.

Taking an arbitrary Cartesian basis $\omega_1, \omega_2, \omega_3$ in \mathcal{S} and an arbitrary Cartesian basis τ_1, τ_2 in \mathcal{D} , one can write

$$(2.4) \quad \begin{aligned} \mathcal{I}_S &= \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + \omega_3 \otimes \omega_3, \\ \mathcal{I}_D &= \tau_1 \otimes \tau_1 + \tau_2 \otimes \tau_2, \\ \mathcal{I}_P &= \frac{1}{2} \mathbf{1} \otimes \mathbf{1}. \end{aligned}$$

For every tensor $\omega \in S$ takes place the well-known *spectral decomposition*

$$(2.5) \quad \omega = \omega_1 \mathbf{w}_1 \otimes \mathbf{w}_1 + \omega_2 \mathbf{w}_2 \otimes \mathbf{w}_2,$$

where $\mathbf{w}_1, \mathbf{w}_2$ is the Cartesian basis in the *physical plane* $\omega \mathbf{w}_1 = \omega_1 \mathbf{w}_1, \omega \mathbf{w}_2 = \omega_2 \mathbf{w}_1$. Thus every deviator has the following canonical form:

$$(2.6) \quad \tau = \mathbf{d} \otimes \mathbf{d} - \mathbf{d}^\perp \otimes \mathbf{d}^\perp = 2\mathbf{d} \otimes \mathbf{d} - |\mathbf{d}|^2 \mathbf{1},$$

where $\mathbf{d} \cdot \mathbf{d}^\perp = 0, |\mathbf{d}| = |\mathbf{d}^\perp|$. It can be also represented as $\tau = \mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}, \mathbf{n} \cdot \mathbf{m} = 0$. The interpretation of the deviators as stresses is shown in Fig. 1; thus we shall further call them *pure shears*.

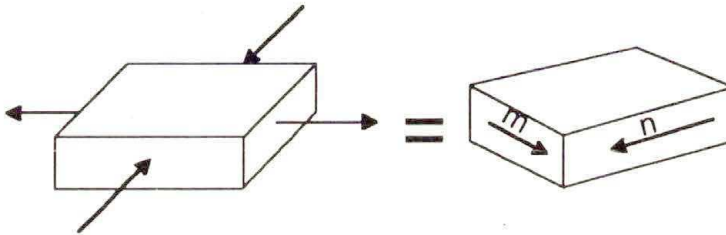


FIG. 1.

The following expression, uniquely representing a tensor $\omega \in S$ as the orthogonal sum of the isotropic tensor and the pure shear:

$$(2.7) \quad \omega = \pi \mathbf{1} + \tau = \pi \mathbf{1} + \mathbf{d} \otimes \mathbf{d} - \mathbf{d}^\perp \otimes \mathbf{d}^\perp$$

we shall call *the isotropic decomposition of a second rank tensor*.

The *rotations* \mathbf{R} (and the *mirror reflection* \mathbf{M}) of the physical plane act in \mathcal{T}_2 according to the rule: $\alpha \rightarrow \mathbf{R} * \alpha$, where

$$(2.8) \quad \mathbf{R} * \alpha \equiv \mathbf{R} \alpha \mathbf{R}^T.$$

One-dimensional subspaces \mathcal{P}, \mathcal{A} are the axes of every rotation $\mathbf{R}*$. In the plane of deviators \mathcal{D} , a rotation $\mathbf{R}(\varphi)$ of the physical plane by the angle φ acts as a rotation $\mathbf{R}(\varphi)*$ by the double angle 2φ (Fig. 2). Indeed, since

$$(2.9) \quad \mathbf{R}(\varphi)\mathbf{d} = \cos \varphi \mathbf{d} + \sin \varphi \mathbf{d}^\perp, \quad \mathbf{R}(\varphi)\mathbf{d}^\perp = -\sin \varphi \mathbf{d} + \cos \varphi \mathbf{d}^\perp$$

thus

$$(2.10) \quad \tau \cdot (\mathbf{R} * \tau) = |\tau|^2 \cos 2\varphi.$$

The action of the mirror reflections is similar.

If a Cartesian basis $(\mathbf{n}_1, \mathbf{n}_2)$ in the physical plane is chosen, then the vectors \mathbf{x} are represented by the pairs of numbers (x_1, x_2) and the tensors by the

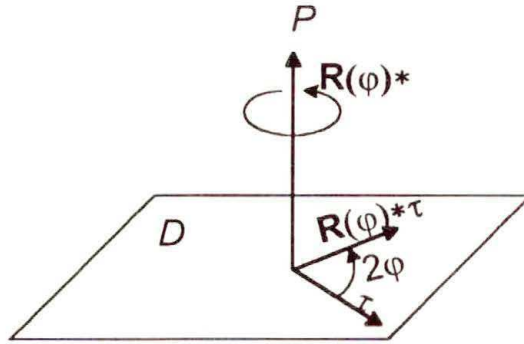


FIG. 2.

number-valued matrices 2×2 (α_{ij}). Then

$$(2.11) \quad \mathbf{1} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{R}(\varphi) \sim \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix},$$

$$\mathbf{R} * \boldsymbol{\alpha} \sim \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix}.$$

2.2. Decompositions of Hooke's tensors

During the last decade the spectral decomposition of the elasticity tensors (which can be traced back to Lord Kelvin and has been recalled by J. RYCHLEWSKI in early eighties [2]), is becoming almost canonical and even finds its way to textbooks [10]. In the plane case, such a decomposition of the two-dimensional Hooke's tensor has the form

$$(2.12) \quad \mathbf{H} = \chi_{\text{I}} \boldsymbol{\omega}_{\text{I}} \otimes \boldsymbol{\omega}_{\text{I}} + \chi_{\text{II}} \boldsymbol{\omega}_{\text{II}} \otimes \boldsymbol{\omega}_{\text{II}} + \chi_{\text{III}} \boldsymbol{\omega}_{\text{III}} \otimes \boldsymbol{\omega}_{\text{III}},$$

where the tensors $\boldsymbol{\omega}_K$ ($K = \text{I}, \text{II}, \text{III}$), called *the proper states* – the eigenelements of the symmetric linear operator $\boldsymbol{\alpha} \rightarrow \mathbf{H} \cdot \boldsymbol{\alpha}$ constitute an orthonormal basis

$$(2.13) \quad \boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL},$$

χ_K being the corresponding eigenvalues⁽²⁾.

It is not difficult to observe that, if one of the proper states is a pure shear, then the other two should be mutually coaxial. Indeed, if, say, $\boldsymbol{\omega}_{\text{III}}$ is a deviator,

⁽²⁾ For the case of the elastic stiffness tensor \mathbf{S} J. RYCHLEWSKI proposed [2] to call these eigenvalues denoted by λ_K *the Kelvin moduli*, their reciprocals $1/\lambda_K$ are the eigenvalues of the elastic compliance tensor \mathbf{C} , which has the same *elastic proper states* as \mathbf{S} , while the independent parameters, defining the elastic proper states κ_I , he proposed to call *the stiffness distributors*.

then there exists such a basis in the two-dimensional space that

$$(2.14) \quad \omega_{III} \sim \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

Thus, due to the orthogonality condition, the other two proper states should have in the same basis the following diagonal representations:

$$(2.15) \quad \omega_I \sim \begin{bmatrix} \cos \alpha & 0 \\ 0 & \sin \alpha \end{bmatrix}, \quad \omega_{II} \sim \begin{bmatrix} -\sin \alpha & 0 \\ 0 & \cos \alpha \end{bmatrix}$$

(we recall here that α has nothing in common with any rotation, it is merely a convenient parameter).

Obviously, such a set of proper states describes the orthotropic material – the reflections with respect to any of the two common proper axes of ω_I and ω_{II} merely changes the sign of ω_{III} leaving \mathbf{C} unchanged. Moreover – vanishing of the trace of at least one proper state is the *necessary condition* of orthotropy. If the pure shear $\boldsymbol{\tau}$ along the orthotropy axis were not a proper state, then it would give rise to non-vanishing diagonal terms in $\mathbf{H} \cdot \boldsymbol{\tau}$ tensor. The reflection would change the sign of $\boldsymbol{\tau}$ while $\mathbf{H} \cdot \boldsymbol{\tau}$ would change according to a different rule (diagonal terms are insensitive to such a transformation) i.e., against the assumption, the reflection would not preserve the shape of \mathbf{H} .

According to (2.6), the rotation by $\pi/2$ interchanging the vectors \mathbf{d} , \mathbf{d}^\perp (the change of sign is insignificant) transforms arbitrary traceless tensor $\boldsymbol{\tau}$ into $-\boldsymbol{\tau}$, hence if in (2.15) $\alpha = \pi/4$, then we are dealing with the tetragonal symmetry (the symmetry of the square). Observe that in such a case the hydrostatic state, (proportional to the unit tensor) must be a proper elastic state.

At last, if the two Kelvin moduli, corresponding to the two pure shear proper states, are equal – one obtains the case of isotropic material. We shall prove in the forthcoming consideration, that no other symmetries of the plane Hooke's tensors are possible.

The spectral decomposition (2.12) is an exact counterpart of the spectral decomposition (2.5). Let us find a counterpart of the isotropic decomposition (2.7).

The rotations of the physical plane \mathbf{R} act on the fourth-order tensors according to the rule $\mathbf{A} \rightarrow \mathbf{R} * \mathbf{A}$ (see (A.2)). It is evident that every Hooke's tensor \mathbf{H} , being rotated preserves its "Hookean nature", any linear combination of Hooke's tensors produces again a Hooke's tensor. Thus the set of all Hooke's tensors is the *tensorial space* (see (A.2)) $\mathcal{H} \subset \mathcal{T}_4$. For further considerations only this space will be of our interest; it is evident that $\dim \mathcal{H} = 6$. We have to find an isotropic decomposition of the space \mathcal{H} .

The earlier introduced unities $\mathcal{I}_S, \mathcal{I}_P, \mathcal{I}_D$ are Hooke's tensors. Moreover, every isotropic Hooke's tensor is a linear combination of the two arbitrarily chosen

tensors out of this threesome. In such a way we obtain the tensorial plane $\mathcal{J} \subset \mathcal{H}$ consisting of the *isotropic Hooke's tensors*. The pair $\mathcal{I}_P, \mathcal{I}_D$ is an orthonormal basis in \mathcal{J} , thus every isotropic Hooke's tensor has a unique orthogonal decomposition

$$(2.16) \quad \mathcal{I} = \lambda_P \mathcal{I}_P + \lambda_D \mathcal{I}_D, \quad \mathcal{I}_P \cdot \mathcal{I}_D = 0.$$

The orthogonal complement of this plane \mathcal{J}^\perp is a four-dimensional tensorial space. Its possible isotropic decomposition can be only of the following form: $\mathcal{J}^\perp = \mathcal{A} + \mathcal{B}$, $4 = 2 + 2$. Indeed, all one-dimensional tensorial subspaces in \mathcal{H} belong to \mathcal{J} . The conditions of orthogonality of the tensor \mathbf{H} to \mathcal{J} , $\mathcal{I}_P \cdot \mathbf{H} = 0$, $\mathcal{I}_D \cdot \mathbf{H} = 0$ are of the following form

$$(2.17) \quad \mathbf{1} \cdot \mathbf{H} \cdot \mathbf{1} = H_{iijj} = 0, \quad \text{Tr } \mathbf{H} \equiv H_{ijij} = 0.$$

These conditions meet e.g. all the tensors from the set \mathcal{A} of the following form:

$$(2.18) \quad \mathbf{1} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \mathbf{1}, \quad \boldsymbol{\tau} \in \mathcal{D}.$$

Since for every rotation \mathbf{R}^* the tensor $\mathbf{R}^*(\mathbf{1} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \mathbf{1}) = \mathbf{1} \otimes (\mathbf{R}^* \boldsymbol{\tau}) + (\mathbf{R}^* \boldsymbol{\tau}) \otimes \mathbf{1}$ remains in \mathcal{A} and every linear combination of the tensors from \mathcal{A} belongs to \mathcal{A} , thus \mathcal{A} is one of the two tensorial planes in \mathcal{J}^\perp , $\dim \mathcal{A} = \dim \mathcal{D} = 2$.

The last component of the isotropic decomposition of the space \mathcal{H} is the orthogonal complement \mathcal{B} of the space \mathcal{A} in \mathcal{J}^\perp . Let us find the general form of the tensors $\mathbf{D} \in \mathcal{B}$. From the orthogonality condition $\mathbf{D} \perp \mathcal{A}$ we have $(\mathbf{1} \cdot \mathbf{D}) \cdot \boldsymbol{\tau} = 0$ for every $\boldsymbol{\tau} \in \mathcal{D}$. Combining this with the condition $\mathbf{1} \cdot \mathbf{D} \cdot \mathbf{1} = 0$ one can see that $(\mathbf{1} \cdot \mathbf{D}) \cdot \boldsymbol{\alpha} = 0$ for every $\boldsymbol{\alpha} \in \mathcal{S}$, therefore $\mathbf{1} \cdot \mathbf{D} = \mathbf{0} \in \mathcal{S}$. Making use of the spectral decomposition

$$(2.19) \quad \mathbf{D} = \lambda_I \boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I + \lambda_{II} \boldsymbol{\omega}_{II} \otimes \boldsymbol{\omega}_{II} + \lambda_{III} \boldsymbol{\omega}_{III} \otimes \boldsymbol{\omega}_{III}$$

from the conditions $\mathbf{1} \cdot \mathbf{D} = \mathbf{0}$, $\text{Tr } \mathbf{D} = 0$, one obtains readily

$$(2.20) \quad \begin{aligned} \lambda_I \text{tr } \boldsymbol{\omega}_I &= \lambda_{II} \text{tr } \boldsymbol{\omega}_{II} = \lambda_{III} \text{tr } \boldsymbol{\omega}_{III} = 0, \\ \lambda_I + \lambda_{II} + \lambda_{III} &= 0. \end{aligned}$$

The only solution, other than $\mathbf{D} = \mathbf{0}$, is the following one

$$(2.21) \quad \lambda_{III} = -\lambda_I, \quad \lambda_{II} = 0, \quad \text{tr } \boldsymbol{\omega}_I = \text{tr } \boldsymbol{\omega}_{III} = 0.$$

Thus every tensor $\mathbf{D} \in \mathcal{B}$ can be uniquely expressed in the following form

$$(2.22) \quad \mathbf{D} = \boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}^\perp \otimes \boldsymbol{\tau}^\perp = 2\boldsymbol{\tau} \otimes \boldsymbol{\tau} - |\boldsymbol{\tau}|^2 \mathcal{I}_D,$$

where $\boldsymbol{\tau} \cdot \boldsymbol{\tau}^\perp = 0$, $|\boldsymbol{\tau}| = |\boldsymbol{\tau}^\perp|$. It is not difficult to check that the tensor \mathbf{D} is totally symmetric and traceless, i.e.

$$(2.23) \quad D_{\sigma(i)\sigma(j)\sigma(k)\sigma(l)} = D_{ijkl}, \quad D_{iikl} = 0,$$

where σ is an arbitrary permutation. Thus the plane \mathcal{B} consists of the *plane fourth rank deviators* [3] ⁽³⁾. Thus, concluding:

Isotropic complete decomposition of the space of plane Hooke's tensors has the following form

$$(2.24) \quad \mathcal{H} = \mathcal{J} + \mathcal{A} + \mathcal{B}, \quad 6 = (1 + 1) + 2 + 2,$$

where \mathcal{J} is the plane of space of isotropic Hooke's tensors, \mathcal{A} is the plane of the tensors: $\mathbf{1} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \mathbf{1}$, $\boldsymbol{\tau}$ denoting pure shear, \mathcal{B} is the plane of the fourth-rank plane deviators. In other words: the isotropic decomposition of every plane Hooke's tensor has the following form:

$$(2.25) \quad \mathbf{H} = \lambda_{\mathcal{P}} \mathcal{I}_{\mathcal{P}} + \lambda_{\mathcal{D}} \mathcal{I}_{\mathcal{D}} + (\mathbf{1} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \mathbf{1}) + \mathbf{D},$$

the four components defined above being mutually orthogonal; invariants $\lambda_{\mathcal{P}}$ and $\lambda_{\mathcal{D}}$ and the deviators $\boldsymbol{\tau}$, \mathbf{D} are the linear isotropic functions of Hooke's tensor.⁽⁴⁾

The rotation $\mathbf{R}(\varphi)^*$ is a rotation of the six-dimensional space \mathcal{H} around the fixed plane \mathcal{J} . It is evident that the tensorial plane \mathcal{A} rotates by the double angle 2φ . The deviatoric plane \mathcal{B} rotates by the quadruple angle 4φ , because, according to the formulae

$$(2.26) \quad \begin{aligned} \mathbf{R}(\varphi)^* \boldsymbol{\tau} &= \cos 2\varphi \boldsymbol{\tau} + \sin 2\varphi \boldsymbol{\tau}^{\perp}, \\ \mathbf{R}(\varphi)^* \boldsymbol{\tau}^{\perp} &= -\sin 2\varphi \boldsymbol{\tau} + \cos 2\varphi \boldsymbol{\tau}^{\perp}, \end{aligned}$$

taking $\mathbf{D} = \boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}^{\perp} \otimes \boldsymbol{\tau}^{\perp}$, one obtains

$$(2.27) \quad \mathbf{D} \cdot [\mathbf{R}(\varphi)^* \mathbf{D}] = |\mathbf{D}|^2 \cos 4\varphi.$$

3. Hooke's tensors as the second rank tensors

The intriguing similarity between the canonical forms of the pure shears $\boldsymbol{\tau}$ (2.6) and the Hooke's deviators \mathbf{D} (2.22) can be noticed. The explanation of this fact is simple and leading to the well known techniques of handling stiffness tensors, commonly used in the engineering applications.

The space \mathcal{T}_4 can be considered, if it is convenient, as any tensorial product $\mathcal{T}_i \otimes \mathcal{T}_j$, $i + j = 4$. The representation $\mathcal{T}_4 = \mathcal{T}_2 \otimes \mathcal{T}_2$ turns out to be especially useful; it means that the tensors of the *fourth rank* are considered as the "second" rank tensors from the sequence $\otimes^p \mathcal{T}_2$, $p = 1, 2, \dots$. This is particularly useful in the case of the Hooke's tensors.

⁽³⁾ The last expression will be called the *canonical form* of the deviator \mathbf{D} ; (in [3] the canonical form of the plane deviator of arbitrary rank has been shown).

⁽⁴⁾
$$2\lambda_{\mathcal{P}} = \mathbf{1} \cdot \mathbf{H} \cdot \mathbf{1}, \quad 2\lambda_{\mathcal{D}} = \text{Tr} \mathbf{H} - \lambda_{\mathcal{P}}, \quad 2\boldsymbol{\tau} = \mathbf{H} \cdot \mathbf{1} - \lambda_{\mathcal{P}} \mathbf{1},$$

$$\mathbf{D} = \mathbf{H} + \lambda_{\mathcal{P}} \mathcal{I}_{\mathcal{P}} - \lambda_{\mathcal{D}} \mathcal{I}_{\mathcal{D}} - \frac{1}{2} [(\mathbf{H} \cdot \mathbf{1}) \otimes \mathbf{1} + \mathbf{1} \otimes (\mathbf{H} \cdot \mathbf{1})].$$

The space of the plane symmetric second-rank tensors is the symmetrised tensorial square of the physical plane E ,

$$(3.1) \quad S = \text{sym } E \otimes E, \quad \dim S = 3.$$

Quite similarly, it is convenient to consider the space of the Hooke's tensors as the symmetrised tensorial square of the S space,

$$(3.2) \quad \mathcal{H} = \text{sym } S \otimes S, \quad \dim \mathcal{H} = 6.$$

In other words: Hooke's tensors can be considered as the symmetric "second rank" tensors, generated by the tensors $\alpha \in S$, exactly in the same manner as the tensors $\alpha \in S$ are generated by the vectors $\mathbf{x} \in E$. Such a viewpoint is correct and useful, under the following important condition, however: the orthogonal group $\mathcal{O}(S)$ of the transformations of the Euclidean three-dimensional space S contains such rotation and mirror reflections, which are not generated by the rotations and the reflections of the physical plane E , for example, the rotation transforming the isotropic tensor $\mathbf{1} \in \mathcal{P}$ into the pure shear $\boldsymbol{\tau} \in \mathcal{D}$, $|\boldsymbol{\tau}| = |\mathbf{1}| = \sqrt{3}$. Such rotations and reflections remain out of the scope of our interests.

If $\{\boldsymbol{\nu}_K\}$, $K = \text{I, II, III}$ is a Cartesian basis in S , then $\{\boldsymbol{\nu}_K \otimes \boldsymbol{\nu}_L\}$ is a Cartesian basis in $S \otimes S$, thus we can write

$$(3.3) \quad \mathbf{H} = H_{KL}(\boldsymbol{\nu}_K \otimes \boldsymbol{\nu}_L).$$

Symmetrising dyads $\boldsymbol{\nu}_K \otimes \boldsymbol{\nu}_L$ one obtains a basis in the space of the Hooke's tensors.

The usefulness of the description of the Hooke's tensors as the "second rank" tensors can be demonstrated using the three following examples:

1. Taking in the last relation the proper states $\boldsymbol{\omega}_K$ of the tensor \mathbf{H} as the base elements $\boldsymbol{\nu}_K$, one obtains the spectral decomposition of the tensor \mathbf{H} (2.12).

2. According to the new view on \mathcal{H} , we shall express the rotations $\alpha \rightarrow \mathbf{R} * \alpha$ and $\mathbf{H} \rightarrow \mathbf{R} * \mathbf{H}$ in the following form

$$(3.4) \quad \mathbf{R} * \boldsymbol{\alpha} = \mathcal{R} \cdot \boldsymbol{\alpha}, \quad \mathbf{R} * \mathbf{H} = \mathcal{R} \circ \mathbf{H} \circ \mathcal{R}^T,$$

where $\mathcal{R} \in S \otimes S$, $\mathcal{R}^T \circ \mathcal{R} = \mathcal{I}_S$.

Since $\boldsymbol{\alpha} \rightarrow \mathbf{R} * \boldsymbol{\alpha}$ is the rotation of the three-dimensional space S around the unit base vector $\mathbf{1}/\sqrt{2}$ by the double angle (Fig. 3), therefore⁽⁵⁾

$$(3.5) \quad \mathcal{R} = \mathcal{I}_P + \cos 2\varphi \mathcal{I}_D + \sin 2\varphi \mathbf{E}_D,$$

⁽⁵⁾ This is a generalization of the rotation in the three-dimensional Euclidean vector space around the unit vector \mathbf{n} by the angle φ , $\mathbf{R} = \mathbf{n} \otimes \mathbf{n} + \cos \varphi (\mathbf{1} - \mathbf{n} \otimes \mathbf{n}) + \sin \varphi \mathbf{E}$, where $\mathbf{E} \equiv \mathbf{n}_1 \wedge \mathbf{n}_2 \equiv \mathbf{n}_1 \otimes \mathbf{n}_2 - \mathbf{n}_2 \otimes \mathbf{n}_1$ and $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}\}$ is an orthonormal basis.

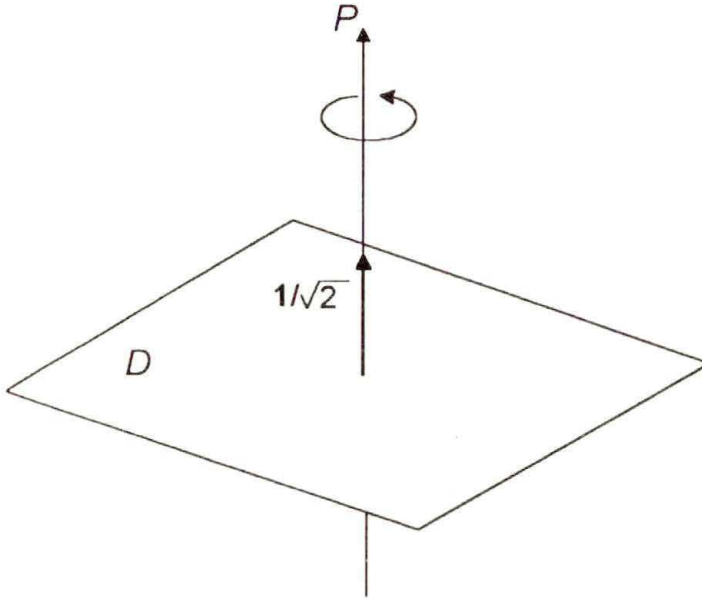


FIG. 3.

where \mathbf{E}_D is the tensor of orientation of the deviatoric plane \mathcal{D} , i.e.

$$(3.6) \quad \mathbf{E}_D \equiv \boldsymbol{\alpha} \otimes \boldsymbol{\beta} - \boldsymbol{\beta} \otimes \boldsymbol{\alpha},$$

where $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ is an arbitrary basis in \mathcal{D} left-oriented in the orientation of \mathbf{E}_D .

3. Let us adopt in \mathcal{S} an orthogonal basis, generated by the isotropic state $\mathbf{1} \in \mathcal{P}$ and the pure shears $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{D}$, $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0$, $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = \sqrt{2}$. Symmetrised and normalised tensor products of these tensors generate the following Cartesian basis in \mathcal{H} :

$$(3.7) \quad \begin{aligned} \mathbf{B}_1 &= \mathcal{I}_P, & \mathbf{B}_2 &= \frac{1}{\sqrt{2}} \mathcal{I}_D, \\ \mathbf{B}_3 &= \frac{1}{2\sqrt{2}} (\mathbf{1} \otimes \boldsymbol{\alpha} + \boldsymbol{\alpha} \otimes \mathbf{1}), & \mathbf{B}_4 &= \frac{1}{2\sqrt{2}} (\mathbf{1} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \mathbf{1}), \\ \mathbf{B}_5 &= \frac{1}{2\sqrt{2}} (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} - \boldsymbol{\beta} \otimes \boldsymbol{\beta}), & \mathbf{B}_6 &= \frac{1}{2\sqrt{2}} (\boldsymbol{\alpha} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\alpha}). \end{aligned}$$

Clearly the pairs $(\mathbf{B}_1, \mathbf{B}_2)$, $(\mathbf{B}_3, \mathbf{B}_4)$, $(\mathbf{B}_5, \mathbf{B}_6)$ are the bases in the corresponding tensorial spaces \mathcal{J} , \mathcal{A} , \mathcal{B} . Hence the matrix of rotation \mathbf{R}^* has in the basis

$\mathbf{B}_1, \dots, \mathbf{B}_6$ the following form:

$$(3.8) \quad \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cos 2\varphi & -\sin 2\varphi & & \\ & & \sin 2\varphi & \cos 2\varphi & & \\ & & & & \cos 4\varphi & -\sin 4\varphi \\ & & & & \sin 4\varphi & \cos 4\varphi \end{bmatrix}.$$

The decomposition

$$(3.9) \quad \mathbf{H} = H_1\mathbf{B}_1 + \dots + H_6\mathbf{B}_6$$

can be reduced by denoting

$$(3.10) \quad \boldsymbol{\tau} = \frac{1}{2\sqrt{2}}(H_3\boldsymbol{\alpha} + H_4\boldsymbol{\beta}), \quad \mathbf{D} = H_5\mathbf{B}_5 + H_6\mathbf{B}_6$$

to the isotropic decomposition (2.25).

The last example leads to some interesting relations, which will turn out to be useful in the forthcoming considerations. Introducing the following notation:

$$(3.11) \quad \begin{aligned} R_1 &\equiv \sqrt{H_3^2 + H_4^2}, & R_2 &\equiv \sqrt{H_5^2 + H_6^2}, \\ \cos \beta &\equiv \frac{H_3}{R_1}, & \sin \beta &\equiv \frac{H_4}{R_1}, \\ \cos \gamma &\equiv \frac{H_5}{R_2}, & \sin \gamma &\equiv \frac{H_6}{R_2}, \end{aligned}$$

one can write

$$(3.12) \quad \mathbf{H} = H_1\mathbf{B}_1 + H_2\mathbf{B}_2 + R_1(\cos \beta\mathbf{B}_3 + \sin \beta\mathbf{B}_4) + R_2(\cos \gamma\mathbf{B}_5 + \sin \gamma\mathbf{B}_6).$$

The angles β and γ are not merely the handy parameters, they change under the rotation of the physical plane. Using the representation (3.8), one can write

$$(3.13) \quad \mathbf{R} * \mathbf{H} = H_1\mathbf{B}_1 + H_2\mathbf{B}_2 + R_1[\cos(\beta + 2\varphi)\mathbf{B}_3 + \sin(\beta + 2\varphi)\mathbf{B}_4] \\ + R_2[\cos(\gamma + 4\varphi)\mathbf{B}_5 + \sin(\gamma + 4\varphi)\mathbf{B}_6].$$

This relation clearly discloses the geometric interpretation of the angles β and γ (see e.g. Fig. 4).

Let us establish the way of choice of bases in \mathcal{S} and in $\mathcal{S} \otimes \mathcal{S}$. Let $\{\mathbf{n}_1, \mathbf{n}_2\}$ be a Cartesian basis in the physical plane E ; we shall adopt the following Cartesian basis in \mathcal{S} ⁽⁶⁾

$$(3.14) \quad \boldsymbol{\nu}_I = \mathbf{n}_1 \otimes \mathbf{n}_1, \quad \boldsymbol{\nu}_{II} = \mathbf{n}_2 \otimes \mathbf{n}_2, \quad \boldsymbol{\nu}_{III} = \frac{1}{\sqrt{2}}(\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_1).$$

Note that $\boldsymbol{\nu}_I + \boldsymbol{\nu}_{II} = \mathbf{1} \in \mathcal{P}$, $\boldsymbol{\nu}_I - \boldsymbol{\nu}_{II} \in \mathcal{D}$, $\boldsymbol{\nu}_{III} \in \mathcal{D}$.

⁽⁶⁾ The coefficient $1/\sqrt{2}$ in the expression for $\boldsymbol{\nu}_{III}$ is essential. Taking instead the symmetric part $(1/2)(\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_1)$, we would not obtain the Cartesian basis, compare [12].

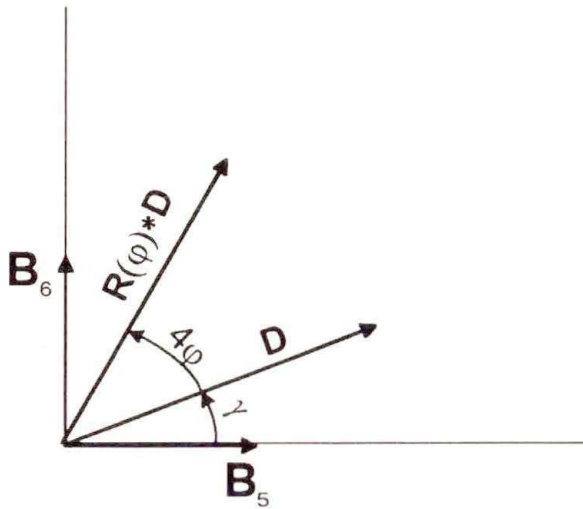


FIG. 4.

In $S \otimes S$ we shall take the Cartesian basis $\nu_K \otimes \nu_L, K, L = I, II, III$. The expressions for the tensors $\alpha \in S$ and $A \in S \otimes S$,

$$(3.15) \quad \alpha = \alpha_I \nu_I, \quad A = A_{IJ}(\nu_I \otimes \nu_J)$$

in the fixed basis $\{n_1, n_2\}$ are determined by the mutually unique relations:

$$(3.16) \quad \alpha \sim \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \\ \sqrt{2}\alpha_{12} \end{bmatrix},$$

$$(3.17) \quad A \sim \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{1111} & A_{1122} & \sqrt{2}A_{1112} \\ A_{2211} & A_{2222} & \sqrt{2}A_{2212} \\ \sqrt{2}A_{1211} & \sqrt{2}A_{1222} & 2A_{1212} \end{bmatrix},$$

$$(3.18) \quad \alpha_K \beta_K = \alpha_{ij} \beta_{ij} = \alpha \cdot \beta,$$

$$(3.19) \quad A_{IJ} B_{IJ} = A_{ijkl} B_{ijkl} = A \cdot B;$$

moreover:

$$(3.20) \quad \varepsilon_I = C_{IJ} \alpha_J \Leftrightarrow \varepsilon_{ij} = C_{ijkl} \alpha_{kl} \Leftrightarrow \beta = C \cdot \alpha,$$

$$(3.21) \quad A_{IJ} = B_{IK} C_{KJ} \Leftrightarrow A_{ijkl} = B_{ijpq} C_{pqkl} \Leftrightarrow A = B \circ C.$$

The representations of some important tensors have the following form:

$$(3.22) \quad \begin{aligned} \mathcal{I} &\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & \mathcal{I}_P &\sim \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \mathcal{I}_D &\sim \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, & \mathbf{E}_D &\sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \end{aligned}$$

while, for the base tensors \mathbf{B}_K , one obtains

$$(3.23) \quad \begin{aligned} \mathbf{B}_1 &\sim \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \mathbf{B}_2 &\sim \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \\ \mathbf{B}_3 &\sim \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \mathbf{B}_4 &\sim \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \\ \mathbf{B}_5 &\sim \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, & \mathbf{B}_6 &\sim \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}. \end{aligned}$$

Under such a choice, the components H_I of the Hooke's tensor \mathbf{H} in the base \mathbf{B}_K and the components of the "second rank" representation of the same tensor H_{ij} are related by the following equalities:

$$(3.24) \quad \begin{aligned} H_1 &= \frac{H_{11} + H_{22} + 2H_{12}}{2}, & H_2 &= \frac{H_{11} + H_{22} - 2H_{12} + 2H_{33}}{2\sqrt{2}}, \\ H_3 &= \frac{H_{11} - H_{22}}{\sqrt{2}}, & H_4 &= (H_{13} + H_{23}), \\ H_5 &= \frac{H_{11} + H_{22} - 2H_{12} - 2H_{33}}{2\sqrt{2}}, & H_6 &= (H_{13} - H_{23}). \end{aligned}$$

At last, the representation of the rotation tensor \mathcal{R} given by (3.4), (3.5) has the following form:

$$(3.25) \quad \mathcal{R}_{KL} \sim \frac{1}{2} \begin{bmatrix} 1 + \cos 2\varphi & 1 - \cos 2\varphi & -\sqrt{2}\sin 2\varphi \\ 1 - \cos 2\varphi & 1 + \cos 2\varphi & \sqrt{2}\sin 2\varphi \\ \sqrt{2}\sin 2\varphi & -\sqrt{2}\sin 2\varphi & 2\cos 2\varphi \end{bmatrix}.$$

4. Invariants and symmetries – effective formulae

4.1. Symmetries

The problems of the symmetries of plane Hooke's tensor has already been discussed in the previous section in terms of proper elastic states and Kelvin moduli. In the case of the “second rank” representations, the matter is also not difficult if only an axis of the presumed symmetry is known. In such a case, taking one base vector, say \mathbf{n}_1 along this axis, one can determine the convenient “second rank” representation H_{KL} of the Hooke's tensor (⁷). Inspecting the shape of this representation and recalling some considerations from the Subsec.2.2, particularly the expressions (2.15), (2.16) and the two subsequent paragraphs of text, one can easily tell, what kind of symmetry we really observe, depending on the shape of the representation of the Hooke's tensor, namely:

full symmetry (isotropy),

$$H_{KL} \sim \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & (a-b) \end{bmatrix},$$

symmetry of the square, (tetragonal),

$$H_{KL} \sim \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix},$$

symmetry of rectangle (orthotropy)

$$H_{KL} \sim \begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & d \end{bmatrix}.$$

The problem arises if we find H_{13} and/or H_{23} different from zero: it is difficult to say, in this case, if there is no symmetry at all or, maybe, we have chosen a wrong axis. We must check up in this case if there exists such a rotation by the angle φ which annihilates the terms containing \mathbf{B}_4 and \mathbf{B}_6 in the expression (3.13). To this end the following two relations must hold true:

$$(4.1) \quad \begin{aligned} \sin(\beta + 2\varphi) &\equiv \sin \beta \cos 2\varphi + \cos \beta \sin 2\varphi = 0, \\ \sin(\gamma + 4\varphi) &\equiv \sin \gamma \cos 4\varphi + \cos \gamma \sin 4\varphi = 0, \end{aligned}$$

or

$$(4.2) \quad \tan 2\varphi = -\tan \beta, \quad \tan 4\varphi = -\tan \gamma.$$

(⁷) The corresponding measurement rules will be discussed in the last subsection of this section.

These two equations can be fulfilled simultaneously if the following relation holds true:

$$(4.3) \quad \tan \gamma = \frac{2 \tan \beta}{1 - \tan^2 \beta} \equiv \tan 2\beta.$$

Using (3.11) one can rewrite this condition in terms of the representation of the Hooke's tensor in the basis $\{\mathbf{B}_K\}$ obtaining the following effective symmetry criterion for the Hooke's tensor \mathbf{H} :

Hooke's plane tensor \mathbf{H} obeys at least orthotropic symmetry if and only if the components of its representation in the basis $\{\mathbf{B}_K\}$ fulfil the following relation:

$$(4.4) \quad J_5 \equiv (H_3^2 - H_4^2)H_6 - 2H_3H_4H_5 = 0.$$

We shall prove in the next subsection that J_5 is invariant under rotation (but not under reflections). The condition (4.4) is trivially fulfilled, if R_1 and/or R_2 vanish. Looking at the relation (3.13) one can readily observe that:

$R_1 = 0$ yields symmetry of the square, while simultaneous vanishing of R_1 and R_2 give rise to the isotropy of the Hooke's tensor.

As it has already been shown, the presence of the plane of symmetry bears orthotropy. We shall prove now that

The only possible non-trivial (i.e. different from the total isotropy) rotational symmetry of the plane Hooke's tensor is the invariance under the rotation by $\pi/2$ – the tetragonal one.

Indeed, in virtue of the uniqueness of the tensor decomposition in given orthonormal basis and the functional independence of $\sin(\cdot)$ and $\cos(\cdot)$, to preserve the plane Hooke's tensor under the two-dimensional rotation by the angle $2\pi/n$ one has to fulfil the following two conditions:

$$(4.5) \quad \beta + \frac{4\pi}{n} = \beta + 2\pi m \quad \text{or} \quad R_1 = 0,$$

$$(4.6) \quad \gamma + \frac{8\pi}{n} = \gamma + 2\pi k \quad \text{or} \quad R_2 = 0,$$

where n , m and k are arbitrary integers. The only (non-trivial) solution of (4.5) and (4.6) is: $R_1 = 0$, $n = 4$, $k = 1$, what proves our assertion⁽⁸⁾.

⁽⁸⁾ One may ask, why by cutting off a slice perpendicularly to the axis of the trigonal symmetry of the three-dimensional body we are gaining additional rotational symmetry? A closer inspection of the case shows that the trigonal symmetry of the three-dimensional body is connected with shearing in the planes orthogonal to the axis of the trigonal symmetry. This shearing stiffness is immaterial in the case of a plane state.

4.2. Invariants

Looking at the relation (3.13) one can tell at once that the following four quantities:

$$\begin{aligned}
 J_1 \equiv H_1 &= \frac{1}{2}(H_{11}+H_{22} + 2H_{12}), \\
 J_2 \equiv H_2 &= \frac{1}{2\sqrt{2}}(H_{11} + H_{22} - 2H_{12} + 2H_{33}), \\
 J_3 \equiv R_1^2 &= H_3^2 + H_4^2 = \frac{1}{2}(H_{11} - H_{22})^2 + (H_{13} + H_{23})^2, \\
 J_4 \equiv R_2^2 &= H_5^2 + H_6^2 = \frac{1}{8}(H_{11} + H_{22} - 2H_{12} - 2H_{13})^2 + (H_{13} - H_{23})^2
 \end{aligned}
 \tag{4.7}$$

are invariants of the *proper orthogonal group* (the group of rotations). The plane Hooke's tensor, however, has in general six independent components, while the proper plane orthogonal group is one-parametric, thus one can expect five functionally independent invariants.

Let us denote:

$$\widehat{\beta} \equiv \beta + 2\varphi, \quad \widehat{\gamma} \equiv \gamma + 4\varphi.
 \tag{4.8}$$

Certainly the quantity

$$\psi = \widehat{\gamma} - 2\widehat{\beta} = \gamma - 2\beta
 \tag{4.9}$$

is invariant with respect to the proper orthogonal group, and, moreover, it is (modulo 2π) uniquely determined by the components of Hooke's tensor in an arbitrary basis. On the other side, if the values of the previous four invariants as well as ψ are known, then the relation (3.13) determines the Hooke's tensor to within the accuracy of an arbitrary rotation. Thus these five invariants constitute a *complete functionally independent set of invariants with respect to the proper orthogonal group* (*complete irreducible hemitropic function basis*).

Tracing the derivation of the orthotropy condition (4.4) one can observe that its left-hand term can be expressed by ψ ⁽⁹⁾ and the condition (4.4) can be rewritten as follows:

$$J_5 = R_1^2 R_2 \sin \psi = 0.
 \tag{4.10}$$

Any reflection tensor in two-dimensional space can be represented as the superposition of the axes exchange (reflection) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and some rotation, the axes exchange merely changes the sign of the terms containing \mathbf{B}_3 and \mathbf{B}_6 changing β into $\pi - \beta$ and γ into $-\gamma$, i.e. ψ (taken modulo 2π) changes its sign; thus only $\cos \psi$ but not ψ itself or $\sin \psi$ is invariant with respect to the *complete orthogonal group* (i.e. containing both the rotation and the mirror reflections), while the previous

⁽⁹⁾ The square, or absolute value of this term can be considered as an invariant measure of deviation from the orthotropy.

four rotationally-invariant terms are the invariants of the complete orthogonal group as well.

These considerations lead to the following important conclusion:

No rotation in the plane of the stress (strain) of the plane elastic state is able to change the sign of ψ , thus the class of the materials of the lowest symmetry can be subdivided into two classes of "left" and "right" materials, depending on the sign of ψ .

The "left" materials can be changed into "right" ones by the off-plane turning them upside-down. This means that the sheets of such a material have two distinct sides, which should be specially marked in order to make the information on the elastic properties meaningful.

For completeness we shall express the obtained invariants in terms of the four-index representation of the Hooke's tensors.

Using relation (3.17) to express the first two invariants (4.7) by the components H_{ijkl} it is not difficult to observe that the following two identities hold true:

$$(4.11) \quad H_1 = \frac{1}{2}(H_{1111} + H_{1122} + H_{2211} + H_{2222}) = \frac{1}{2}H_{iijj} = \lambda_P,$$

$$(4.12) \quad H_2 = \frac{1}{2\sqrt{2}}(H_{1111} - H_{1122} - H_{2211} + H_{2222} + 2H_{1212} + 2H_{2121}) \\ = \frac{1}{2\sqrt{2}}(2H_{ijij} - H_{iijj}) = \sqrt{2}\lambda_D.$$

The expressions for the remaining invariants are not straightforward. Observe that they depend only on the traceless part \mathbf{H}' of the Hooke's tensor \mathbf{H} (3.12):

$$(4.13) \quad \mathbf{H}' = R_1(\cos \beta \mathbf{B}_3 + \sin \beta \mathbf{B}_4) + R_2(\cos \gamma \mathbf{B}_5 + \sin \gamma \mathbf{B}_6),$$

or

$$(4.14) \quad H'_{ijkl} = H_{ijkl} - \frac{1}{6}[H_{pprr}(2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) + H_{pprr}(2\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl})], \\ (H'_{iill} = 0, \quad H'_{ilil} = 0).$$

Substituting relations (3.7), (3.11) and (3.23) into (4.13) one obtains the following representation of the plane second rank tensor $\mathbf{H}' \cdot \mathbf{1}$:

$$(4.15) \quad [H'_{ijkk}] = \frac{R_1}{\sqrt{2}} \begin{bmatrix} \cos \beta & \sin \beta \\ \sin \beta & -\cos \beta \end{bmatrix}.$$

Thus

$$(4.16) \quad H'_{ijkk}H'_{ijll} = R_1^2.$$

Due to the orthonormality of the base tensors $\{\mathbf{B}_K\}$, directly from (4.13) it follows that

$$(4.17) \quad \mathbf{H}' \cdot \mathbf{H}' \equiv H'_{ijkl} H'_{ijkl} = R_1^2 + R_2^2,$$

hence:

$$(4.18) \quad R_2^2 = H'_{ijkl} H'_{ijkl} - H'_{ijpp} H'_{ijqq}.$$

The most time-consuming is the derivation of the last relation – the one describing the “shape of deviatoric part”, i.e. expressing the functions of ψ in terms of polynomial invariants of the Hooke's tensor. Omitting the tedious calculations⁽¹⁰⁾ we present the following result:

$$(4.19) \quad H'_{ijkl} H'_{klmn} H'_{mniij} = \frac{3}{2\sqrt{2}} R_1^2 R_2 \cos \psi.$$

It is not difficult (however it can be fairly boring) to show that our set of invariants: H_1 , H_2 , R_1^2 , R_2^2 and $\text{Tr}(\mathbf{H}'^3)$ is equivalent to the set of invariants obtained by ZHENG [9], who proved that they constitute the complete irreducible isotropic function basis.

One cannot expect to find an expression of such a kind for $\sin \psi$. There is a simple reason for this: all the polynomial scalar expressions obtained by the contraction are *invariant with respect to complete orthogonal group* while $\sin \psi$, as we have already shown, is *the hemitropic function of the plane Hooke's tensor*.

The last question, concerning the invariants of the Hooke's tensors, which has to be discussed are the conditions of positive definiteness

$$(4.20) \quad \boldsymbol{\alpha} \cdot \mathbf{H} \cdot \boldsymbol{\alpha} \geq 0$$

for every $\boldsymbol{\alpha} \in \mathcal{S}$, which are required for most applications of the Hooke's tensors. In the case of the spectral decomposition of the stiffness (compliance) tensor the problem reduces to the trivial conditions of non-negativeness of the three Kelvin moduli, which are equivalent to the conditions:

$$(4.21) \quad \begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &\geq 0, \\ \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 &\geq 0, \\ \lambda_1 \lambda_2 \lambda_3 &\geq 0. \end{aligned}$$

Recalling that in the basis of proper states the representation of the Hooke's tensor is diagonal, and taking into account that all three expressions (4.21) are invariant with respect to *any orthogonal transformation in $\mathcal{S} \otimes \mathcal{S}$* (including those,

⁽¹⁰⁾ The following interesting relations can make this boring procedure slightly simpler: $\mathbf{B}_3^2 = \frac{1}{4}[\mathbf{B}_1 + \sqrt{2}(\mathbf{B}_2 + \mathbf{B}_5)]$, $\mathbf{B}_4^2 = \frac{1}{2}[\mathbf{B}_1 + \sqrt{2}(\mathbf{B}_2 - \mathbf{B}_5)]$, $\mathbf{B}_5^2 = \mathbf{B}_2$, $\mathbf{B}_6^2 = \mathbf{B}_2$, $\text{sym}(\mathbf{B}_3 \mathbf{B}_4) = \frac{1}{2\sqrt{2}} \mathbf{B}_6$, $\text{sym}(\mathbf{B}_3 \mathbf{B}_5) = \frac{1}{2\sqrt{2}} \mathbf{B}_3$, $\text{sym}(\mathbf{B}_3 \mathbf{B}_6) = \frac{1}{2\sqrt{2}} \mathbf{B}_4$, $\text{sym}(\mathbf{B}_4 \mathbf{B}_5) = -\frac{1}{2\sqrt{2}} \mathbf{B}_4$, $\text{sym}(\mathbf{B}_4 \mathbf{B}_6) = \frac{1}{2\sqrt{2}} \mathbf{B}_3$, $\text{sym}(\mathbf{B}_5 \mathbf{B}_6) = 0$.

which do not correspond to any rotation of the physical plane), after some rearrangements one can express these conditions by the polynomial invariants of \mathbf{H} as the fourth-rank tensor

$$(4.22) \quad \begin{aligned} \text{Tr } \mathbf{H} &\geq 0, \\ (\text{Tr } \mathbf{H})^2 - \text{Tr } \mathbf{H}^2 &\geq 0, \\ \frac{1}{3} \text{Tr } \mathbf{H}^3 - \frac{1}{2} \text{Tr } \mathbf{H} \text{Tr } \mathbf{H}^2 + \frac{1}{6} (\text{Tr } \mathbf{H})^3 &\geq 0. \end{aligned}$$

The same relations can be expressed in the language of the invariants generated by the isotropic decomposition as follows:

$$(4.23) \quad \begin{aligned} H_1 + \sqrt{2}H_2 &\geq 0, \\ H_2^2 + 2\sqrt{2}H_1H_2 - R_1^2 - R_2^2 &\geq 0, \\ \sqrt{2}H_1(H_2^2 - R_2^2) - R_1^2(H_2 - R_2 \cos \psi) &\geq 0. \end{aligned}$$

It is not difficult to notice that no restriction on the sign of ψ has been imposed by the “thermodynamic” condition of the positive definiteness of the Hooke’s tensor. Thus both “left” and “right” materials are thermodynamically admissible.

4.3. The rules of the measurements

The procedures of measurements of the elastic properties in the case of materials supplied in the form of sheets and foils very seldom include direct measurements of the shear moduli⁽¹⁾; not only the standard, but even more sophisticated laboratory equipment is usually rather inappropriate for such measurements. Usually the Young moduli and Poisson ratio in the chosen directions are measured and then, if needed, the other elastic constants are calculated.

Let us denote the direction of uniaxial tension by x_1 and let \mathbf{C} denote the elastic compliance tensor, then the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\epsilon}$ have the following representations:

$$(4.24) \quad \boldsymbol{\sigma} \sim \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{\epsilon} \sim \begin{bmatrix} \sigma C_{11} \\ \sigma C_{21} \\ \sigma C_{31} \end{bmatrix}.$$

Consequently, by the definitions of the Young modulus E and the Poisson ratio ν , one can write:

$$(4.25) \quad \frac{1}{E} = C_{11}, \quad \frac{\nu}{E} = -C_{21}.$$

In general we have to determine six unknown elastic constants; to this end we should take at least three specimens oriented at three different angles φ_i ($i = 1, 2, 3$) with respect to some fixed material basis. Performing measurements we

⁽¹⁾ We shall leave aside in this paper the acoustic measurement techniques.

would obtain then six quantities: E_i, ν_i ($i = 1, 2, 3$). Making use of the Eq. (3.13), and substituting expressions (3.23) for the base tensors \mathbf{B}_K , one can write relations (4.25) in the following form:

$$(4.26) \quad \frac{1}{E_i} = \frac{1}{2}C_1 + \frac{1}{2\sqrt{2}}C_2 + \frac{1}{\sqrt{2}}R_1 \cos(\beta + 2\varphi_i) + \frac{1}{2\sqrt{2}}R_2 \cos(\gamma + 4\varphi_i),$$

$$(4.27) \quad -\frac{\nu_i}{E_i} = \frac{1}{2}C_1 - \frac{1}{2\sqrt{2}}C_2 - \frac{1}{\sqrt{2}}R_2 \cos(\gamma + 4\varphi_i).$$

Using (3.11) one can rewrite Eq. (4.26) and (4.27) in the following form:

$$(4.28) \quad \frac{1}{2}C_1 + \frac{1}{2\sqrt{2}}C_2 + \frac{\cos 2\varphi_i}{\sqrt{2}}C_3 - \frac{\sin 2\varphi_i}{\sqrt{2}}C_4 + \frac{\cos 4\varphi_i}{2\sqrt{2}}C_5 - \frac{\sin 4\varphi_i}{2\sqrt{2}}C_6 = \frac{1}{E_i},$$

$$(4.29) \quad \frac{1}{2}C_1 - \frac{1}{2\sqrt{2}}C_2 - \frac{\cos 4\varphi_i}{2\sqrt{2}}C_5 + \frac{\sin 4\varphi_i}{2\sqrt{2}}C_6 = -\frac{\nu_i}{E_i}.$$

Taking $i = 1, 2, 3$ we obtain the system of the three pairs of equations for six unknown constants C_K . The determinant Δ of this system can be expressed as follows:

$$(4.30) \quad \Delta = 2\sqrt{2} \sin^2(\varphi_1 - \varphi_2) \sin^2(\varphi_2 - \varphi_3) \sin^2(\varphi_3 - \varphi_1) \times \cos(\varphi_1 - \varphi_2) \cos(\varphi_2 - \varphi_3) \cos(\varphi_3 - \varphi_1).$$

Hence the following rule of the measurements should be observed:

For the determination of the plane Hooke's tensor for the material of no (or unknown) symmetry, using the uniaxial tension tests one should take at least three specimens whose axes are neither parallel nor orthogonal to each other.⁽¹²⁾

It is not difficult to show that if the axes of orthotropy are known, only two specimens are necessary (the one along an orthotropy axis and the other under the angle of $\pi/4$ being particularly convenient). In the case of the isotropy recognized in advance, only one specimen is necessary.

Appendix

A.1. Plane tensors

Two-dimensional Euclidean plane E consisting of the elements $\mathbf{x}, \mathbf{y}, \dots$ with the scalar product $\mathbf{x} \cdot \mathbf{y}$ we shall call the *physical plane* (it can be e.g. the plane tangent

⁽¹²⁾ This result is not quite unexpected: it is not difficult to observe (compare (4.27)) that for the orthogonal directions $\nu_i/E_i \equiv \nu_j/E_j$.

to the median surface of the shell at an arbitrary point). The plane E generates the *plane Euclidean tensors* as the elements of the tensorial powers $\mathcal{T}_p \equiv \otimes^p E$, $p = 1, 2, \dots$. Every tensor $\mathbf{A} \in \mathcal{T}_p$ is a finite linear combination of the simple tensors $\mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_p$.

A.2. The rotations and mirror reflections of the tensors

Every rotation $\mathbf{x} \rightarrow \mathbf{R}\mathbf{x}$ of the physical plane, $\mathbf{R} \in E \otimes E$, rotates all the tensors, $\mathbf{A} \rightarrow \mathbf{R} * \mathbf{A}$. The operators \mathbf{R}^* are linear and defined on the simple tensors as follows: $\mathbf{R}^*(\mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_p) = \mathbf{R}\mathbf{x}_1 \otimes \dots \otimes \mathbf{R}\mathbf{x}_p$. Similarly act the mirror reflections $\mathbf{x} \rightarrow \mathbf{M}\mathbf{x}$, $\mathbf{M} \in E \otimes E$.

A.3. Tensorial spaces

Every linear subspace $\mathcal{U} \subset \mathcal{T}_p$ invariant under the rotations and the mirror reflections of the physical plane, $\mathbf{R}*\mathcal{U} = \mathcal{U}$, $\mathbf{M}*\mathcal{U} = \mathcal{U}$, we call the *tensorial space*. The representation of the tensorial space \mathcal{U} (as well as the whole space \mathcal{T}_p) in the form of direct sum of the tensorial spaces $\mathcal{U} = \mathcal{U}_1 + \dots + \mathcal{U}_k$ we call the *isotropic decomposition* of this space. The linear operators mapping \mathcal{U} onto itself, particularly the rotations and the reflections, can be considered as the tensors from $\mathcal{U} \otimes \mathcal{U}$.

A.4. The tensors of the second and fourth rank

In the present paper we use the *second* rank tensors denoted (except for \mathbf{I} , \mathbf{R} , \mathbf{M}) as $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, \dots and the *fourth* rank tensors denoted as \mathbf{A} , \mathbf{B} , \dots . The tensorial operations which we use can be expressed in the well-known language of the Cartesian representations as follows:

$$(A.1) \quad \begin{aligned} \mathbf{x} \cdot \mathbf{y} &\leftrightarrow x_i y_i, & \boldsymbol{\alpha} \mathbf{x} &\leftrightarrow \alpha_{ij} x_j, \\ \text{tr } \boldsymbol{\alpha} &= \alpha_{ii}, & \boldsymbol{\alpha} \otimes \boldsymbol{\beta} &\leftrightarrow \alpha_{ij} \beta_{pq}, \\ \mathbf{A} \cdot \boldsymbol{\alpha} &\leftrightarrow A_{ijpq} \alpha_{pq}, & \text{Tr } \mathbf{A} &= A_{pqpq}, \\ \mathbf{A} \cdot \mathbf{B} &= A_{pqrs} B_{pqrs}, & \mathbf{A} \circ \mathbf{B} &\leftrightarrow A_{ijpq} B_{pqkl}. \end{aligned}$$

The following relations hold true

$$(A.2) \quad \begin{aligned} (\mathbf{x} \otimes \mathbf{y})\mathbf{z} &= (\mathbf{y} \cdot \mathbf{z})\mathbf{x}, & (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \cdot \boldsymbol{\tau} &= (\boldsymbol{\beta} \cdot \boldsymbol{\tau})\boldsymbol{\alpha}, \\ (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \circ (\boldsymbol{\tau} \otimes \boldsymbol{\nu}) &= (\boldsymbol{\beta} \cdot \boldsymbol{\tau})\boldsymbol{\alpha} \otimes \boldsymbol{\nu}, \\ R * (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) &= (R * \boldsymbol{\alpha}) \otimes (R * \boldsymbol{\beta}). \end{aligned}$$

A.5. The tensorial unities

The tensorial *unity* of the plane E we shall denote by \mathbf{I} , while the *unity* of the space $E \otimes E$ – by \mathcal{I} , thus $\mathbf{I}\mathbf{x} = \mathbf{x}$, $\mathcal{I}\boldsymbol{\alpha} = \boldsymbol{\alpha}$ for all $\mathbf{x} \in E$, $\boldsymbol{\alpha} \in (E \otimes E)$. In a similar

way one can introduce the tensorial *unity* $\mathcal{I}_{\mathcal{U}} \in \mathcal{U} \otimes \mathcal{U}$ into every tensorial space \mathcal{U} . In the language of Cartesian representations:

$$(A.3) \quad (\mathbf{1})_{ij} = \delta_{ij}, \quad (\mathcal{I})_{ijkl} = \delta_{ik}\delta_{jl}.$$

A.6. Euclidean tensor spaces

In every tensor space \mathcal{T}_p a scalar product $\mathbf{A} \cdot \mathbf{B}$, defined for the simple tensors: $(\mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_p) \cdot (\mathbf{y}_1 \otimes \dots \otimes \mathbf{y}_p) = (\mathbf{x}_1 \mathbf{y}_1) \dots (\mathbf{x}_p \mathbf{y}_p)$ can be introduced, yielding 2^p -dimensional Euclidean space. Every orthonormal basis in \mathcal{T}_p we shall call *Cartesian*. Only such rotations of the Euclidean spaces \mathcal{T}_p remain in the scope of our interest, which are generated by rotations of the physical space, as described in A.2.

Acknowledgement

Most of the results reported in the present paper were obtained in the framework of the research supported by the grant of the Committee of the Scientific Research (Poland) No. 3 P404 027 05.

References

1. J. RYCHLEWSKI, *Elastic energy decomposition and the limit criteria* [in Russian], *Uspekhy Mekh. – Advances in Mech.*, 7, 3, 51–80, 1984.
2. J. RYCHLEWSKI, *On the Hooke's law*, *PMM*, 48, 420–435 (in Russian, see English translation: *Prikl. Matem. Mekhan.*, 48, 303–314, 1984).
3. J. RYCHLEWSKI, *Tensors* [in Polish], PWN, Warszawa 1995.
4. J. RYCHLEWSKI, *Unconventional approach to linear elasticity*, *Arch. Mech.*, 47, 2, pp. 149–171, 1995.
5. A. BLINOWSKI and J. OSTROWSKA-MACIEJEWSKA, *On the elastic orthotropy*, *Arch. Mech.*, 48, 1, 1996.
6. A. BLINOWSKI, *Mechanics of the elastic and plastic bodies* [in Polish, a textbook], Politechnika Białostocka, Białystok 1989.
7. S. SUTCLIFFE, *Spectral decomposition of the elasticity tensor*, *J. Appl. Mech, Trans. of ASME*, 59, pp. 762–774, December 1992.
8. Q.-S. ZHENG, *Two-dimensional tensor function representation for all kinds of material symmetry*, *Proc. of the Royal Society of London, Series A*, 443, 1917, pp. 127–138, October 1993.
9. Q.-S. ZHENG, *A note on representation for isotropic functions of 4th-order tensors in 2-dimensional space*, *Z. Angew. Math. Mech.*, 74, 8, pp. 357–359, 1994.
10. J. OSTROWSKA-MACIEJEWSKA, *Mechanics of deformable bodies* [in Polish], PWN, Warszawa 1994.
11. M.M. MEHRABADI, S.C. COWIN, *Eigentensors of linear anisotropic elastic materials*, *Q. J. Mech. Appl. Math.*, 43, pp. 16–44, 1990.
12. J.F. NYE, *Physical properties of crystals*, Oxford Univ. Clarendon Press, London and New York 1957.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received July 4, 1995.