A R C H I V E S O F M E T A L L U R G Y A N D M A T E R I A L S
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AN EXTENSION OF BURZYŃSKI HYPOTHESIS OF MATERIAL EFFORT ACCOUNTING FOR THE THIRD INVARIANT OF STRESS TENSOR

ROZSZERZENIE HIPOTEZY WYTĘŻENIA BURZYŃSKIEGO UWZGLĘDNIAJĄCE WPŁYW TRZECIEGO NIEZMIENNIKA TENSORA NAPRĘŻENIA

The aim of the paper is to propose an extension of the Burzyński hypothesis of material effort to account for the influence of the third invariant of stress tensor deviator. In the proposed formulation the contribution of the density of elastic energy of distortion in material effort is controlled by Lode angle. The resulted yield condition is analyzed and possible applications and comparison with the results known in the literature are discussed.
Keywords: Burzyński yield condition, strength differential effect, pressure sensitivity, Lode angle, third invariant of stress tensor deviator

Celem pracy jest propozycja rozszerzenia hipotezy wytężenia Burzyńskiego dla uwzględnienia wpływu trzeciego niezmiennika dewiatora tensora naprężenia. W proponowanym sformułowaniu udział gęstości sprężystej energii odkształcenia postaciowego jest kontrolowany przez funkcję zależną od kąta Lodego. Wyprowadzony warunek plastyczności porównano ze znanymi wynikami z literatury oraz przedyskutowano jego możliwe zastosowania.

1. Introduction

Experimental investigations related with mechanics of soils and rocks, the strength of concrete as well as failure processes and phase transformations in advanced metallic materials reveal that assumption about isotropic mechanical properties requires also an account for the third invariant of stress tensor deviator in the formulation of limit state criteria (failure criteria). The limit state criteria can determine the limit of elasticity, plasticity, phase transformations or rupture. The third invariant of stress tensor deviator corresponds to the Lode angle on the octahedral plane. The experimental observations show that the effect of Lode angle appears particularly visible, while shear processes are taking place. On the atomic level the shear process in a solid body is accompanied with the change of the configuration of atoms and valence electrons which are responsible for the bonds between the neighboring atoms. This leads to the change of the symmetry of groups of atoms and results in the change, mostly with downward tendency, of their energy of cohesion [3], [6]. Therefore an extent of the domination of shearing in deformed solid body has an essential influence on the energy-based measure of material effort. This leads to the conclusion that the contribution of shear modes of stress, quantified by means of the Lode angle, should be accounted for in the energetic criteria of failure. Due to the universality and multiscale character of energy, among many proposed hypotheses of material effort those with energy basis seem to be the most worthy considering from the physical point of view. The hypothesis of variable limit energy of volume change and distortion proposed originally by Burzyński [2] is the one which seems to be the most general of all – many other are just special cases of this one for certain values of its parameters [7], [13].

The aim of the paper is to propose an extension of the Burzyński hypothesis of material effort to account for the aforementioned influence of the Lode angle. The main idea of the proposed new formulation is that the contribution of the density of elastic energy of distortion in the measure of material effort is controlled by Lode angle. The resulted yield condition is analyzed from the point of view of possible applications and the comparison with the results presented in the literature is discussed [4], [5], [8], [9].

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2. The Burzyński energy-based hypothesis of material effort and its extension

2.1. The hypothesis of variable limit energy of volume change and distortion

Burzyński considered as a measure of material effort variable elastic energy density of distortion and a part of energy density of volumetric change controlled by a pressure dependent influence function. The limit condition takes in such a case the following form [2]:

$$\Phi_{\mathrm{f}} + \eta(p)\,\Phi_v = K \quad \text{where} \quad \eta(p) = \omega + \frac{\delta}{3p}, \tag{1}$$

where Φ_v is volumetric strain energy density, while Φ_f is distortional strain energy density. It is in general a three-parameter condition. The great convenience in application of it is due to the fact that after proper substitutions the set of parameters (K, ω, δ) can be replaced by (k_t, k_c, k_s) which are the limit values of stress under tension, compression and pure shear respectively Those three parameters can be measured experimentally. In case of the Burzyński condition (1) those three values give us full information about strength properties of isotropic solid. Sometimes it is useful to express the limit condition (1) in terms of the other two parameters proposed by Burzyński: the strength difference ratio $\kappa = \frac{k_c}{k_t} > 1$ and the so called "plasticity coefficient" which is a measure of mutual relations between all three limit quantities $\nu = \frac{k_c k_t}{2k_s^2} - 1$. Eq. (1) can be rewritten in the following form:

$$\frac{1+\nu}{3}\sigma_f^2 + 3(1-2\nu)\,p^2 + 3pk_t(\kappa - 1) = \kappa k_t^2, \tag{2}$$

where $p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ denotes hydrostatic stress and

$$\sigma_f = \sqrt{(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2}$$

is the stress invariant introduced by Burzyński [2]. For certain values of κ and ν the condition (2) is describing certain type of a limit surface in the space of principal stresses $(\sigma_1, \sigma_2, \sigma_3)$:

- $\nu < \frac{1}{2}$ – ellipsoidal surface with one of its axes parallel to p axis. For $\kappa \neq 1$ it is translated along p axis. In such a case the strength differential (SD) effect is involved. In the case of symmetric elastic range ($\kappa = 1$) limit surface is the same as in the case of Rychlewski's [10], [11] limit criterion for isotropic materials;
- $\nu = \frac{1}{2}$ – paraboloidal surface with its axis parallel to p axis and translated along it. It describes the SD effect. For $\kappa = 1$ the paraboloid of revolution becomes a cylinder – the criterion for pressure insensitive materials not exhibiting SD effect – this case corresponds to the Huber criterion [13];
- $\frac{1}{2} < \nu < \frac{3(k_c + k_t)}{8k_c k_t} - 1$ hyperboloidal limit surface (two sheets). Only one sheet has a physical meaning. If ν reaches its upper limit value the surface becomes a cone – the same as in the Drucker-Prager hypothesis [13], which was proposed independently 25 years later;

Note that the qualitative change of surface character is continuous however it changes very rapidly when ν is close to $\frac{1}{2}$. As it was already mentioned, the SD effect is involved by accounting for pressure influence on material effort – other authors (Raniecki and Mróz [9]) formulated limit conditions for pressure insensitive-materials describing the SD effect with use of the Lode angle dependence.

2.2. The proposed extension of the Burzyński hypothesis

The influence of the Lode angle on the Burzyński measure of material effort could be involved by the variable contribution of the energy density of distortion. The contribution is described by certain "influence function". The possible propositions of the form of such a function are given in the next subsection. Finally the extended material effort hypothesis of variable energy of volume change and distortion in the limit state reads:

$$\tilde{\eta}_f(\theta)\,\Phi_f + \tilde{\eta}_v(p)\,\Phi_v = K, \tag{3}$$

where the symbols $\tilde{\eta}_f$ and $\tilde{\eta}_v$ denote Lode angle influence function and pressure influence function, respectively. The condition (3) can be expressed in a following way:

$$\eta_f(\theta)\,q^2 + \eta_v(p)\,p^2 = K, \tag{4}$$

where

$$q = \sqrt{\frac{1}{3}\left[(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2\right]} \tag{5}$$

and $\theta = \frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right)$ is the Lode angle.

2.3. Propositions of influence functions η_v and η_f

2.3.1. Pressure influence function

In case of pressure influence function the choice of Burzyński's function seems to be well justified since hypothesis formulation given by Eq.(3) is a consistent extension of his own idea. Furthermore this function enables description of various relations between hydrostatic and deviatoric stresses – linear, paraboloidal, hyperboloidal and elliptical – and it was analyzed precisely by Burzyński in [2]. It is a two-parameter rational function which can be written in the form:

$$\eta_v(p) = \omega + \frac{\delta}{p} \tag{6}$$

2.3.2. The Lode angle influence functions

The Lode angle dependence, which is characteristic especially for brittle materials and soils, can be described:

- two-parameter power function (Raniecki and Mroz [9])
  $\eta_f(\theta) = [1 + \alpha\cos(3\theta)]^{\beta}$
- two-parameter exponential function (Raniecki and Mróz [9])
  $\eta_f(\theta) = 1 + \alpha\left[1 - e^{-\beta(1+\cos(3\theta))}\right]$
- one-parameter trigonometric function (Lexcellent [5])
  $\eta_f(\theta) = \cos\left[\frac{1}{3}\arccos\left[1 - \alpha(1 - \cos(3\theta))\right]\right]$
- two-parameter trigonometric function (Podgórski [8])
  $\eta_f(\theta) = \frac{1}{\cos\left(\frac{\pi}{6} - \beta\right)}\cos\left[\frac{1}{3}\arccos(\alpha\cos(3\theta)) - \beta\right]$

Valuable summary of propositions of Lode angle influence functions was made by Bardet in [1]. The earlier attempts to account for the Lode angle effect are discussed by Życzkowski [13].

2.3.3. Convexity condition

A general discussion on convexity condition was presented in [12] for arbitrary chosen form of influence functions such that $K - \eta_p(p) > 0$:

$$\left\{\begin{array}{l} \sqrt{3}\left[6\eta_f\frac{\partial^2\eta_f}{\partial\theta^2} - \left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4\left(\eta_f\right)^2 + \cdots \right.\\ \left. \cdots + \frac{\partial^2\eta_p}{\partial p^2}\left(\left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4\left(\eta_f\right)^2\right) + \frac{\eta_f}{\left(K - \eta_p\right)}\left(\frac{\partial\eta_p}{\partial p}\right)^2\left(\frac{\partial^2\eta_f}{\partial\theta^2} + 4\eta_f\right)\right] > 0 \quad , \\ 3\left[2\left(K - \eta_p\right)\frac{\partial^2\eta_p}{\partial p^2} + \left(\frac{\partial\eta_p}{\partial p}\right)^2\right]\left[2\eta_f\frac{\partial^2\eta_f}{\partial\theta^2} - \left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4\left(\eta_f\right)^2\right] > 0 \end{array}\right. \tag{7}$$

where: $\eta_p(p) = \eta_v(p) \cdot p^2$

The above conditions can be applied in material identification procedures basing on numerical fitting of the results of simulation using assumed model to the results obtained from experiment. It is in fact an optimization problem in which the given convexity condition can be considered as an inequality constraint on the resultant optimal solution.

3. Yield surface specification

3.1. Iyer and Lissenden experiments on Inconel 718

An attempt to find certain form of a yield condition basing on (4) and experimental data available in literature was made. Inconel 718 alloy was considered. Various tests including tension, compression and pure shear as well as composition of two of those loadings one followed by another were performed for Inconel 718 by Iyer and Lissenden [4]. Values of plasticity limit (considered as an offset limit being a stress causing 0,2% permanent plastic strain) and proportionality limit has been estimated basing on uniaxial and pure shear tests.

The proportionality limit estimation is especially important since it denotes Hooke's law validity range. However it seems that mathematical formalism of the considered yield condition could be well used also in case of plasticity limit estimation.

TABLE 1

	Plasticity limit	Proportionality limit
Tension:	k_t^{pl}= 779 MPa	k_t^H= 500 MPa
Compression:	k_c^{pl}= 878 MPa	k_c^H= 610 MPa
Pure shear:	k_s^{pl}= 473 MPa	k_s^{pl}= 323 MPa

One can observe that the considered material exhibit significant strength-differential effect – in case of plasticity limit $\kappa = \frac{k_c}{k_t} \approx 1.13$ and in case of proportionality limit $\kappa \approx 1.22$. Mutual relations between those limit stresses can be estimated by Burzyński's "plasticity coefficient" $\nu = \frac{k_c k_t}{2k_s^2} - 1$. For the plasticity limit $\nu \approx 0.53$, for proportionality limit $\nu \approx 0.46$. One can see that this value is close to 0.5, the value distinguished by Burzyński, for which yield condition becomes either a paraboloid of revolution or a cylinder. In fact yield condition used in [4] being simple combination of powers of stress tensor invariants, namely:

$$aI_1^2 + b \cdot J_2 + c \cdot \mathrm{sgn}\,(J_3) \cdot (J_3)^{2/3} - 1 = 0$$
$$a = 2.610^{-7}\frac{1}{\mathrm{MPa}^2} \quad b = 1.81510^{-5}\frac{1}{\mathrm{MPa}^2} \quad c = 2.210^{-6}\frac{1}{\mathrm{MPa}^2} \tag{8}$$

is represented in stress space by an ellipsoid with its axis parallel to $p$axis and with slightly deformed cross-section (see Fig. 2). This is due to the assumed elliptical relation between hydrostatic and deviatoric stresses Parameters a, b, c and seven other (viscoplastic) material parameters were found numerically by fitting simulated processes to those observed during experiments The first attempt was to determine values of parameters so they fit the shearing data well, however the obtained result were unrealistic, especially shear threshold stress was much too low. Finally parameters a, b, c of yield condition (8) were found in [4] by fitting numerical simulation to various experimental data, however unequal weights were used for different samples in such way so the correlation between model and experiment was optimal.

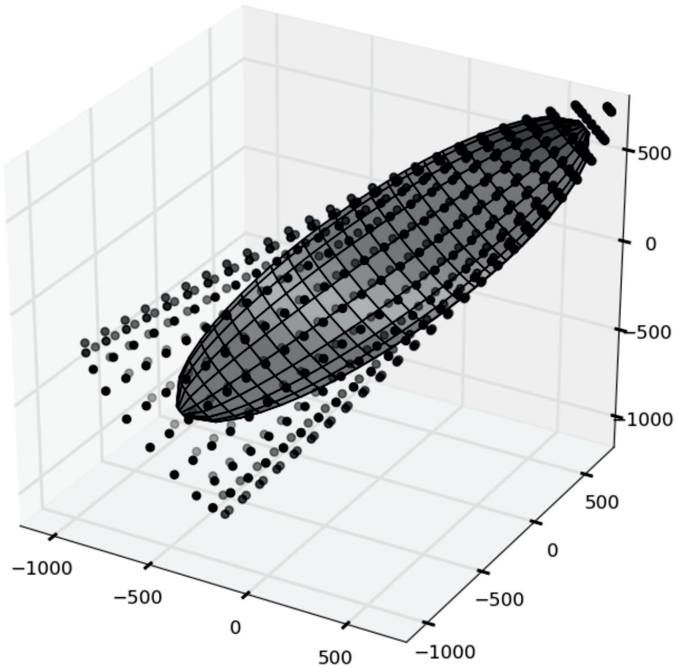

Fig. 1. Ellipsoidal yield surface used in [4] and the paraboloid yield surface according to the proposed criterion given by Eq. (9)

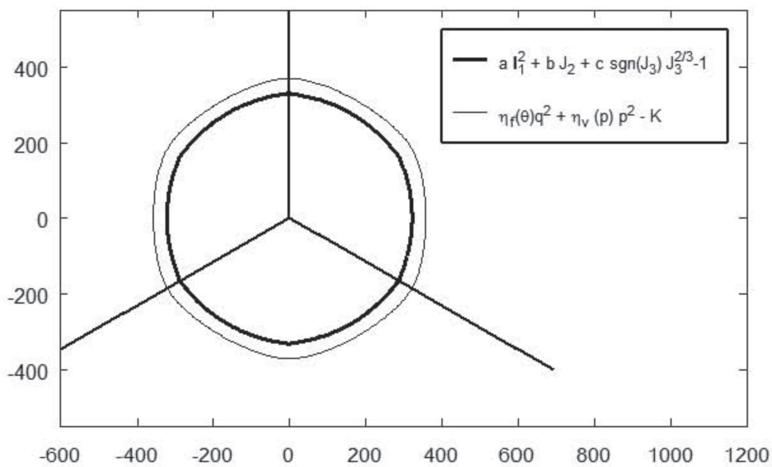

Fig. 2. The comparison of the cross-sections of the two yield surfaces, used in [4] and given by Eq. (9), at the deviatoric (octahedral) plane

3.2. Why a paraboloid yield surface?

Referring to the Burzyński hypothesis relatively low value of limit shear stress is characteristic for ellipsoidal yield surfaces ($\nu < 0.5$) so it seems that assuming different character of a yield surface (i.e. paraboloidal, what indicates calculated $\nu \approx 0.5$) might provide higher precision in model identification It is also common assumption (based on experimental results) that metals are either pressure-insensitive (cylindrical yield surface as in Huber-Mises hypothesis [13]) or that hydrostatic stress is a safe stress state (paraboloidal yield surface). Thus paraboloidal yield surface seems to be much more natural and reasonable choice than the ellipsoidal one, which is typical for instance in case of foams and cellular materials. Furthermore elliptic relation between deviatoric and hydrostatic stress is symmetric with respect to both of those quantities (sign insensitive) so only the Lode angle might be considered as the cause of SD effect. Assuming that the difference between values of limit stresses under tension and compression depends only on different modes of shearing (Lode angle) omitting the influence of pressure is unjustifiable. The only objective verification of a surface character would be performing experiments under high pressure.

3.3. Yield condition specification

Yield surface given by condition (4) was determined for Inconel 718 assuming that relation between hydrostatic and deviatoric stresses is paraboloidal. To obtain such relation Burzyński's function was chosen as pressure influence function. Analysis of the cross-section of yield surface given by Eq.(8) parallel to octahedral plane indicates that the Lode angle dependence has a hexagonal character. Among all presented Lode angle influence functions only the one by Podgórski enables description of such a dependence. However it had to be slightly modified since its maximums were rotated referring to the maximums obtained from Eq.(8). Finally yield condition of a following form was considered:

$$\frac{q^2}{\cos\left(\frac{\pi}{6}-\beta\right)}\cos\left[\frac{1}{3}\arccos\left[\alpha\cos\left(3\left(\theta-\frac{\pi}{2}\right)\right)\right]-\beta\right]+ \\ +\left(\omega+\frac{\delta}{p}\right)p^2-K=0 \tag{9}$$

Unknown parameters can be determined from a nonlinear system of equations – each equation is obtained from satisfying yield condition (9) in simple loading cases like uniaxial tension or compression or pure shear for which values of p, q and θ are known. For higher precision more complex stress states could be analyzed – i.e. shearing with tension, shearing with compression or (as Burzyński proposed) biaxial and triaxial uniform tension and compression.

Yet different method of yield condition specification was chosen. Since surface given by Eq.(8) was determined by optimal fitting it to various multiaxial tests, it seems that its character is quite well estimated especially for small values of hydrostatic stress - no tests under high pressure were performed and hydrostatic component of a stress tensor in each test was also not significantly high due to lack of stress concentration in the samples. That's why an attempt of fitting paraboloidal yield surface given by Eq.(4) to the surface (not only single points in stress space corresponding with basic strength tests) given by Eq.(8) was made. To achieve proper fitting Levenberg-Marquardt algorithm was implemented replacing yield surface used from [4] by a large set of points. For reasons presented below, the fitting took place only for positive values of hydrostatic stress. As it was mentioned in subsection 3.2, the aim of the authors was to determine a paraboloidal yield surface – different character of paraboloidal and ellipsoidal relation between q and p for negative values of pressure would cause too big error for any efficient use of Levenberg-Marquardt algorithm. Another reason is that results presented in [4], are much more reliable for positive values of pressure – the material was characterized in [4] basing on five tests (tension, compression and three shearing tests) yet each of them had a different influence on the final result due to use of inequal weights. The only source of hydrostatic component in those tests are uniaxial tests. Tension test was taken with the weight 40% while compression test with the weight 10%.

As a measure of an error of fitting a distance between both surfaces along q direction was considered. It appeared that the resultant values of unknown parameters of function given by Eq.(9) obtained from the performed computation depend strongly on their initial values used in iteration if the number of estimated parameters was greater than two. However the number of parameters could be reduced by assuming some of their values i.e.:

- $\omega = 0$ – due to fact that relation between p and q was assumed paraboloidal.
- $\beta = \dfrac{\pi}{6}$ – due to fact that Lode's angle dependence has a hexagonal character.
- $\alpha = 0.9$ – smoothing of corners of an octahedral cross-section depend on the value of this parameter (for $\alpha = 0$ – no Lode's angle dependence; $\alpha = 1, \beta = \dfrac{\pi}{6}$ – strictly hexagonal dependence – like i.e. in Coulomb-Tresca-Guest hypothesis). This value was estimated so that octahedral cross-sections of both surfaces correspond sufficiently well.

Parameters $\delta \approx 174,85$ MPa and $K \approx 127871$ MPa^2 were found using the mentioned Levenberg-Marquardt algorithm. Comparison of both surfaces given by Eq.(8) and Eq.(9) and their octahedral cross-sections' comparison are shown in Fig.(1) and Fig.(2). Surface given by Eq. (9) was presented in form of points only to make the figure of two intersecting surfaces clear.

4. Summary

A proposition of a yield criterion of clear physical (energetic) interpretation for isotropic and homogeneous materials involving pressure sensitivity with the resulting SD effect and the Lode angle dependence and was introduced. Many commonly used yield criteria are involved as special cases for certain form of influence functions and certain values of their parameters. It is an extension of the Burzyński yield condition accounting for influence of the third invariant of stress tensor deviator. General convexity condition for obtained yield surface for arbitrary chosen forms of influence functions was derived. Certain form of yield condition for Inconel 718 alloy was formulated referring to experiments by Iyer and Lissenden [4] by yield surface identification.

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