

## **THEORETICAL RESULTS AND NUMERICAL STUDY ON THE NONLINEAR REFLECTION AND TRANSMISSION OF PLANE SOUND WAVES**

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*The comparison between theoretical and numerical solutions of the reflection/transmission problem for the acoustic plane wave normally incident on the discontinuity surface between two nonlinear lossy media was presented. Numerical calculations made under the assumption that the two media have the same impedance, allow to single out the effect of nonlinearities in the description of the reflection and transmission phenomena, so they agreed with theoretical predictions. It was shown that theoretically obtained and numerically calculated results mutually confirmed themselves*

### INTRODUCTION

The reflection and transmission of acoustic disturbances is a fundamental problem of the propagation theory in inhomogeneous media. Even in the linear propagation case beyond the known classical solutions the problems which we meet here are difficult and the corresponding literature is wide [1]. On the contrary in the case of nonlinear description of this problem the acoustic literature is very scant. Few theoretical and experimental papers presented in Refs. [2-4] are worth mentioning. Though the titles of this papers suggest undertake of this problem but next their essence was eliminated by wrong suppositions or as a small effect. They stimulated the author to perform this study. The author has not found in the literature the solution of the fundamental nonlinear problem; how to determine the reflected and transmitted disturbances as a function of the incident disturbance in the shape of a plane wave incident normally on the plane boundary between two media. In other words to determine nonlinear quantities corresponding to reflection and transmission coefficients known from the linear propagation theory. Their shape results from the propagation equation and from the general continuity conditions of the fields on the boundary. The proper number

of equations and unknown variables cause that they can not be a subject of definition like in [2] (giving in wrong results) or can not be constructed in the way of intuitive generalization of ideas (for instance impedances) known from linear theory. The phenomenon, which will be considered in this paper is a part – often fundamental- of many technologies. For example, in medical ultrasonic diagnostic methods important information is obtained due to the detection of reflections from boundaries between different tissues. Often equilibrium parameters (impedances) of tissues differ from each other by a very small amount. It arises a question - which may be important not only for medical diagnostics- how the nonlinear effects impact on general picture of reflection and transmission phenomenon? The nonlinearity parameter  $B/A$  for different soft tissues can vary from 5.8 (cardiac muscle) to 11 (fatty tissue)[6]. Can the media which differ by this parameter only (or generally by third order material parameters) be differentiated from each other due to different reflection? What is the qualitative and quantitative effect of this phenomenon? The aim of this paper is to find a response to the aforementioned question in its simplest one-dimensional geometrical configuration without taking into consideration any transverse disturbances, which may occur in relation to the beam axis.

Derivations of the reflection-transmission (R-T) operators for the plane wave incident normally on the boundary between nonlinear media was presented in detail in the papers [6],[7]. They were shortly repeated in chapter I and II. In chapter IV we present numerical solutions describing passage of the pulse plane wave through the boundary between different nonlinear (or linear-nonlinear) media. The main objective of this paper is the comparison of the theoretical predictions with the numerical calculations results. The selections and applications in the numerical calculations of the characteristics relations (resulting from theory) between material parameters permits on reciprocal verification theory and numerical algorithm (positive for both if results are conformable).

1. ASUMPTIONS AND PROBLEM FORMULATION

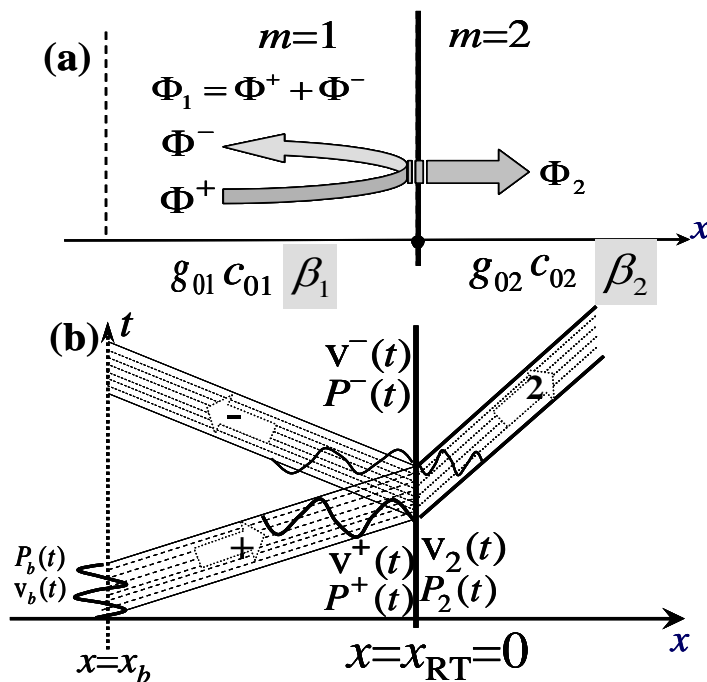


Fig.1. Diagram of the fields decomposition and material parameters distribution in the first and the second media -(a). Time-space diagram of the incident reflected and transmitted pulses.

Two adjacent media with a plane interface between them are considered. Parameters and material function characteristic for the first (left) medium and the second medium will be denoted by  $m=1,2$  respectively (Fig.1). The dynamics of the system is one-dimensional. In analysis a normalized system of variables and parameters will be used. The normalized Cartesian coordinates in space  $x$  and time  $t$  and the normalized equilibrium density and sound speed are given respectively by the relations

$$x \equiv x' / k_0, \quad t \equiv t' \Omega_0, \quad g_{0m} \equiv \rho_{0m} / \rho_0, \quad c_{0m} \equiv c_{m0} / c_0, \quad \Omega_0 = k_0 c_0. \quad (1)$$

Where,  $\rho_{0m}, c_{m0}$  are the equilibrium density and speed of sound of the  $m$ -Th medium;  $\rho_0, c_0$  are density and speed of sound of the reference medium (of course it can be  $\rho_0 = \rho_{01}, c_0 = c_{10}$ ). In the case of the isotropic solid medium third order parameters  $\beta_m$  (normalized by  $\rho_0 c_0^2$ ) describes material and geometrical nonlinearities of the medium. For fluids  $\beta_m \equiv (\gamma_m + 1) / 2$ , where  $\gamma$ -exponent of the adiabat or  $\beta_m \equiv (1 + (B/2A)_m)$ ,  $(B/A)_m$  is the nonlinearity parameter. After normalization the formulas analyzed in this paper contain Mach number  $q \equiv P_b / \rho_0 c_0^2$  ( $= v_b / c_0$ ) where  $P_b, v_b$  characteristics Pressure or velocity ( $P_b \equiv \max_t |P_b(t)|$  see Fig.1).  $x_{RT} (= 0)$  denotes the coordinate of the boundary plane (Fig.1).

The following representations of the fields in media (nonlinear and lossy) will be used.

$$\Phi_1(x, t) = \Phi^+(t - \frac{x}{c_{01}}, x) + \Phi^-(t + \frac{x}{c_{01}}, -x), \quad x \leq x_{RT}, \quad m=1 \quad (2)$$

$$\Phi_2(x, t) = \Phi_2(t - \frac{x}{c_{02}}, x), \quad x \geq x_{RT}, \quad m=2, \quad (3)$$

where  $\Phi_1 = \Phi^+ + \Phi^-$  denotes the composition of the incident  $\Phi^+$  and reflected  $\Phi^-$  waves.

From the definition  $\mathbf{v} \equiv \nabla \Phi$  it means that the velocity field is linear in respect to the potential. Therefore, the following decompositions of the velocity, corresponding to (2), (3), are valid

$$v_1(x, t) = v^+(t - \frac{x}{c_{01}}, x) + v^-(t + \frac{x}{c_{01}}, -x), \quad (4)$$

$$v_2(x, t) = v_2(t - \frac{x}{c_{02}}, x). \quad (5)$$

The relation between acoustical pressure and potential follows from the definition  $\mathbf{v} \equiv \nabla \Phi$  and from the general equations of motion of the continuous medium, and is not linear,  $P_1 = P_1[\Phi_1] = P_1[\Phi^+ + \Phi^-] \neq P_1[\Phi^+] + P_1[\Phi^-] = P^+ + P^-$ , nevertheless  $P_m \cong -g_{0m} \partial_t$  is a very good approximation. We would like to stress that the allowance of the neglecting terms in the  $P_m$  operator no affect on the obtained results [6,7]. Then similarly, the following decompositions of the acoustical pressure are obtained

$$P_1(x, t) = P^+(t - \frac{x}{c_{01}}, x) + P^-(t + \frac{x}{c_{01}}, -x), \quad (6)$$

$$P_2(x, t) = P_2(t - \frac{x}{c_{02}}, x). \quad (7)$$

Firstly, we would like to determine the relations between  $v^+, v^-, v_2$  and  $P^+, P^-, P_2$  on the interface at  $x = x_{RT} = 0$ . Then we will research the boundary conditions for  $v^-, v_2$  and  $P^-, P_2$  on the plane between two different nonlinear and lossy media. This problem will be solved if we find the functions  $\mathbf{R}, \mathbf{T}$  (operators) such that,

$$v^- = \mathbf{R}'_v[\{m\}; v^+] = \mathbf{R}_v[\{m\}; v^+] \circ v^+ \quad (8)$$

$$v_2 = \mathbf{T}'_v[\{m\}; v^+] = \mathbf{T}_v[\{m\}; v^+] \circ v^+ \quad (9)$$

where  $\mathbf{R}_v[\cdot; \cdot]$ ,  $\mathbf{T}_v[\cdot; \cdot]$  are the reflection and transmission operators; in general case  $\circ$  - denote operation characteristics for the operator. Generally in almost all considered cases  $\circ \equiv \otimes$  is convolution (in time or Fourier frequency domain), but sometimes in special cases  $\circ \equiv \cdot$  is the ordinary multiplication;  $\{m\}$  denotes a set of the material parameters which characterizes the media. In linear case we have a ‘‘classical’’ problem of reflection and transmission of the plane wave. For no absorbing media  $v^- = \hat{\mathbf{R}}_v[\{m\}] \circ v^+ = R_v[\{m\}] \cdot v^+$ ,

$$\hat{v}_2 = \hat{\mathbf{T}}_v[\{m\}] \circ v^+ = \hat{T}_v[\{m\}] \cdot \hat{v}^+. R, T \text{ are reflection and transmission coefficients.}$$

As a base of our theoretical description we assume equation (24) referred in [7]. It describes finite amplitude potential disturbances in lossy media. For the  $m$ -th homogenous phase of the medium it takes the form

$$\partial_{tt}\Phi_m - c_m^2 \partial_{xx}\Phi_m + 2\mathcal{A}_m \partial_t \Phi_m + q_m \partial_t (\partial_t \Phi_m)^2 = 0 + o^2, \quad q_m \equiv q \beta_m / c_{0m}^2 \quad m=1,2 \quad (10)$$

Generally  $\mathcal{A}_m$  is the convolution type operator of absorption  $\mathcal{A}_m \Phi \equiv A_m(t) \otimes \Phi(x, t)$ . For classical absorption  $\mathcal{A}_m = -\alpha_2^m \partial_{xx} \square - \alpha_2^m \partial_{tt} + o^2$ ,  $\alpha_2^m$  is the dimensionless hybrid viscosity.  $o^l$  is a small quantity of the order  $l$  respect  $q$  or  $\alpha = \|\mathcal{A}\|$ ,  $o^l = o(q + \alpha)^l$ .

Secondly, we would like compare theoretical with numerical calculations results. In numerical calculations we will use unitary description of the nonlinear disturbances in the heterogeneous stationary media. It heave the form

$$\partial_{tt}P - c^2 g \partial_x \frac{1}{g} \partial_x P + 2\mathcal{A} \partial_t P - \bar{q} \partial_{tt}(P)^2 = 0 + o^2, \quad \bar{q} \equiv q \beta / g c^2, \quad (11)$$

In our case  $g(x) \equiv g_{0m}$ ;  $c(x) \equiv c_{0m}$ ;  $\bar{q}(x) \equiv q \beta_m / g_{0m} c_{0m}^2$ ;  $P(x, t) \equiv P_m(x, t) = -g_{0m} \partial_t \Phi_m$  for  $x \in m$ -th phase of the medium. Then Eq.(11) reduces to Eqs.(10) for  $x \in (m$ -th phase of the medium).

Functions

$$\tau^+(t, x) = t - \frac{x}{c_{01}}, \quad \tau^-(t, x) = t + \frac{x}{c_{01}}, \quad x \leq 0 \quad (12)$$

$$\tau_2(t, x) = t - \frac{x}{c_{02}}, \quad x \geq 0 \quad (13)$$

are a set of characteristics of the d'Alambertian operator  $\square_m \equiv c_{0m}^2 \partial_{xx} - \partial_{tt}$ , then for linear ( $q \equiv 0$ ) and lossless media ( $\mathcal{A}_m \equiv 0$ ),  $\Phi_1, \Phi_2$  depends only on the characteristics and  $\square_2 \Phi_2(\tau_2) \equiv 0$ ,  $\square_1 \Phi_1 \equiv 0 \equiv \square_1(\Phi^+(\tau^+) + \Phi^-(\tau^-))$ . In the characteristics coordinates  $(\tau_2^{+,-}; x)$   $\partial_x \Phi^{+,-} = 0 = \partial_x \Phi_2$ .

In the solutions of the equations with absorbing and nonlinear terms  $\mathcal{A}_m \neq 0, q \neq 0$  (of the range  $o^1$ ) dependence on the additional coordinate (in our case  $x$ ) should occurs in the space scale  $\lambda_{\alpha, q} \equiv \min(1/\alpha, 1/q)$  ( $1/q$ -normalized shock distance;  $\alpha = \|\mathcal{A}\|$ ). In the coordinates containing characteristics (Eq.(12),(13))  $\partial_x(\Phi_1(\square_1, x), \Phi_2(\square_1, x)) = o^1$  however  $\partial_{xx}(\Phi_1; \Phi_2) = o^2$  (see.[6,7]). Then after substitution Eqs.(2,3) in too Eqs.(10), and using coordinates with characteristic (retarded time) we obtain

$$\partial_x(\Phi^+ - \Phi^-) = -\frac{1}{c_{01}} \bar{\mathcal{A}}_1 (\partial_t \Phi^+ + \partial_t \Phi^-) - \frac{q_1}{2c_{01}} (\partial_t \Phi^+ + \partial_t \Phi^-)^2 + o^2; \quad \Phi^{+,-} = \Phi^{+,-}(\tau^{+,-}, x), \quad (14)$$

$$\partial_x \Phi_2 = -\frac{1}{c_{02}} \bar{\mathcal{A}}_2 \partial_t \Phi_2 - \frac{q_2}{2c_{02}} (\partial_t \Phi_2)^2 + o^2; \quad \Phi_2 = \Phi_2(\tau_2, x) \quad (15)$$

$$\partial_{\tau^+} \Phi^{+,-} = \partial_{\tau^-} \Phi^{+,-} = \partial_t \Phi^{+,-}, \quad \partial_{\tau_2} \Phi_2 = \partial_t \Phi_2$$

where  $\bar{\mathcal{A}}_m \equiv \int \mathcal{A}_m \bar{\mathcal{A}}_m \partial_t \Phi = \bar{A}_m(t) \otimes (\partial_t \Phi)$ , for a classically absorbing medium  $\bar{\mathcal{A}}_m = -\alpha_2^m \partial_t$ .

We see that equation (15) is the Burger's equation for  $P_2 = -g_{02} \partial_t \Phi_2$ . Equation (28) may be interpreted as the nonlinear coupled by means of the term  $(q_1/z_{01}) \partial_t (P^+ P^-)$  of two Burger's equations for  $P^+$  and  $P^-$ .

For assumed shapes  $\Phi^+, \Phi^-, \Phi_2$ , acoustical pressures and velocities take the form,

$$P^+ = -g_{01} \partial_t \Phi^+ = -g_{01} \partial_{\tau^+} \Phi^+ \quad (16)$$

$$P^- = -g_{01} \partial_t \Phi^- = -g_{01} \partial_{\tau^-} \Phi^- \quad (17)$$

$$P_2 = -g_{02} \partial_t \Phi_2 = -g_{02} \partial_{\tau_2} \Phi_2 \quad (18)$$

$$v^+ = \partial_x \Phi^+ = -\frac{1}{c_{01}} \partial_t \Phi^+ + \partial_x \Phi^+ = -\frac{1}{c_{01}} \partial_{\tau^+} \Phi^+ + \partial_x \Phi^+ = \frac{P^+}{z_{01}} + \partial_x \Phi^+, \quad (19)$$

$$v^- = \partial_x \Phi^- = \frac{1}{c_{01}} \partial_t \Phi^- + \partial_x \Phi^- = \frac{1}{c_{01}} \partial_{\tau^-} \Phi^- + \partial_x \Phi^- = -\frac{P^-}{z_{01}} + \partial_x \Phi^- \quad (20)$$

$$v_2 = \partial_x \Phi_2 = -\frac{1}{c_{02}} \partial_t \Phi_2 + \partial_x \Phi_2 = -\frac{1}{c_{02}} \partial_{\tau_2} \Phi_2 + \partial_x \Phi_2 = \frac{P_2}{z_{02}} + \partial_x \Phi_2 \quad (21)$$

where:  $z_{0m} \equiv g_{0m} c_{m0}$ , are equilibrium impedances. From (19), (20), (21) we obtain

$$P^+ = z_{01} (v^+ - \partial_x \Phi^+) \quad (22)$$

$$P^- = -z_{01} (v^- - \partial_x \Phi^-) \quad (23)$$

$$P_2 = z_{02} (v_2 - \partial_x \Phi_2) \quad (24)$$

The functions  $\partial_x \Phi^+, \partial_x \Phi^-, \partial_x \Phi_2$  show substantial differences between impedance relations for an ideal linear medium, and for lossy or nonlinear one.

## 2. CONTINUITY CONDITIONS. REFLECTION-TRANSMISSION OPERATORS

In this section all functions and relations on the interface are considered and analyzed. On this surface at  $x = x_{RT} = 0$  all functions and relations depend only on time  $t$ .

We assume continuity conditions in the conventional form

$$P_1 \equiv P^+ + P^- = P_2, \quad (25)$$

$$v_1 \equiv v^+ + v^- = v_2. \quad (26)$$

Apply (22), (23), (24) to reduce  $P^+, P^-, P_2$  from (25) we obtain

$$z_{01} ((v^+ - v^-) - \partial_x (\Phi^+ - \Phi^-)) = z_{02} (v_2 - \partial_x \Phi_2) \quad (27)$$

$$v^+ + v^- = v_2 \quad (28)$$

The needed terms  $\partial_x (\Phi^+ - \Phi^-), \partial_x \Phi_2$  were obtained from (14), (15) for  $x \rightarrow x_{RT}, x = x_{RT}$ . Using (16), (17), (18) and (22), (23), (24)

$$(\partial_x \Phi^+ - \partial_x \Phi^-) = \bar{\mathcal{A}}_1 (v^+ - v^-) - \frac{q_1 c_{01}}{2} (v^+ - v^-)^2 + o^2 \quad (29)$$

$$\partial_x \Phi_2 = \bar{\mathcal{A}}_2 v_2 - \frac{q_2 c_{02}}{2} v_2^2 + o^2 \quad (30)$$

Substituting (29), (30) into (27) and using (28) we obtain

$$\left[ z_1 + z_2 + w \cdot v^+ \right] v^- = \left[ z_1 - z_2 + \frac{1}{2} u \cdot v^+ \right] v^+ + \frac{1}{2} u \cdot (v^-)^2, \quad (31)$$

where:  $z_1, z_2$  are the operators of the linear impedance  $z_m \equiv z_{0m}(1 - \bar{\mathcal{A}}_m)$ ,  $m=1,2$ .  $w \equiv z_{01}c_{01}q_1 + z_{02}c_{02}q_2$ ,  $u \equiv z_{01}c_{01}q_1 - z_{02}c_{02}q_2$ . Due to absorption Eq.(31) may be an integral-differential nonlinear equation in time domain, or nonlinear convolution equation in the Fourier frequency domain. We obtain the simplest case with regard to the absorption when assuming classical absorption  $\bar{\mathcal{A}}_m = -\alpha_2^m \partial_t$  for both media (Riccati equation).

The abstract (operational) solution of equation (31) can be presented as follows

$$v^- = R'_v \left[ \{m\}; v^+ \right] \equiv \frac{1}{2w} \left[ 1 - \sqrt{1 - 4wR_0 v^+} \right] = R_v \left[ \{m\}; v^+ \right] \circ v^+ \quad (32)$$

$$R_v \left[ \{m\}; v^+ \right] \equiv \frac{1}{2w \circ R_0 v^+} \left[ 1 - \sqrt{1 - 4w \circ R_0 v^+} \right] R_0, \quad (33)$$

$$T_v \left[ \{m\}; v^+ \right] = 1 + R_v \left[ \{m\}; v^+ \right], \quad (34)$$

$$R_0 \equiv \frac{z_1 - z_2 + \frac{1}{2} u v^+}{z_1 + z_2 + w v^+}, \quad w \equiv \frac{\frac{1}{2} u \cdot}{z_1 + z_2 + w v^+}, \quad (35)$$

The factorizations of  $w$  and  $R_0$  operators was present in [6,7]. If absorption is absent  $z_m = z_{0m}$  (or negligibility)  $w[\cdot]$  and  $R_0[\cdot]$  factorize themselves in ordinary functions, and the operational solutions (32),(33),(34),(35) give the factorized (algebraic) solution,  $\circ = \cdot$ . Expanding the above formulas in respect to  $q$  and keeping the terms of and  $o^1$  only, we obtain

$$v^-(t) = \left( \frac{z_{01} - z_{02}}{z_{01} + z_{02}} + \frac{2z_{01}^2 z_{02}^2 q}{(z_{01} + z_{02})^3} \left( \frac{\beta_1}{z_{01} c_{01}} - \frac{\beta_2}{z_{02} c_{02}} \right) v^+(t) \right) v^+(t), \quad (36)$$

$$v_2(t) = \left( \frac{2z_{01}}{z_{01} + z_{02}} + \frac{2z_{01}^2 z_{02}^2 q}{(z_{01} + z_{02})^3} \left( \frac{\beta_1}{z_{01} c_{01}} - \frac{\beta_2}{z_{02} c_{02}} \right) v^+(t) \right) v^+(t), \quad (37)$$

We rewrite the above formulas in the form

$$v^- = R_v v^+ = (R_v + r_v v^+) v^+ \quad (38)$$

$$R_v \equiv \frac{z_{01} - z_{02}}{z_{01} + z_{02}}, \quad r_v \equiv \frac{q 2 z_{01}^2 z_{02}^2}{(z_{01} + z_{02})^3} \left( \frac{\beta_1}{z_{01} c_{01}} - \frac{\beta_2}{z_{02} c_{02}} \right), \quad (39)$$

$$v_2 = T_v v^+ = (T_v + r_v v^+) v^+, \quad T_v = 1 + R_v, \quad T_v = 1 + R_v = \frac{2z_{01}}{z_{01} + z_{02}} \quad (40)$$

In similar way, apply Eqs.(22), (23), (24) to reduce  $v^+, v^-, v_2$  from (26) we obtain reflection  $R_p$  and transmission  $T_p = 1 + R_p$  operators for treasure  $P^+$ . However the procedure is more complicated see [6,7].

$$P^-(t) = \left( \frac{z_{02} - z_{01}}{z_{01} + z_{02}} + \frac{2z_{01} z_{02}^2 q}{(z_{01} + z_{02})^3} \left( \frac{\beta_2}{z_{02} c_{02}} - \frac{\beta_1}{z_{01} c_{01}} \right) P^+(t) \right) P^+(t), \quad (41)$$

$$P_2(t) = \left( \frac{2z_{02}}{z_{01} + z_{02}} + \frac{2z_{01}z_{02}^2q}{(z_{01} + z_{02})^3} \left( \frac{\beta_2}{z_{02}c_{02}} - \frac{\beta_1}{z_{01}c_{01}} \right) \right) P^+(t), \quad (42)$$

$$P^- = R_p P^+ = (R_p + r_p P^+) P^+, \quad P_2 = T_p P^+ = (T_p + r_p P^+) P^+, \quad T_p = 1 + R_p \quad (43)$$

$$R_p = \frac{z_{02} - z_{01}}{z_{01} + z_{02}} = -R_v, \quad r_p \equiv \frac{2z_{01}z_{02}^2q}{(z_{01} + z_{02})^3} \left( \frac{\beta_2}{z_{02}c_{02}} - \frac{\beta_1}{z_{01}c_{01}} \right) = -r_v \frac{1}{z_{01}} + O^2. \quad (44)$$

### 3. DISCUSSION

The properties of nonlinear part of the boundary conditions for reflected-transmitted waves are described by  $r_p$  - the nonlinear reflection-transmission coefficient. Let us notice that the lack of differences of nonlinear parameters in both media ( $\beta_1 = \beta_2$ ) is not a sufficient condition for vanishing of the nonlinear component of the reflection-  $R_p$  and transmission-  $T_p$  operators. This condition have the form

$$r_p = 0 \quad \rightarrow \quad \frac{\beta_2}{z_{02}c_{02}} = \frac{\beta_1}{z_{01}c_{01}}. \quad (45)$$

However, in this case  $R_p = R_p$  where,

$$R_p = \frac{z_{02} - z_{01}}{z_{02} + z_{01}} = \frac{\beta_2 c_{01} - \beta_1 c_{02}}{\beta_1 c_{02} + \beta_2 c_{01}}. \quad (46)$$

It means in the case  $\beta_1 \neq \beta_2$  that, in spite of a linear dependence between the reflected and incident disturbances on the interface  $x=x_{RT}$ , the phenomenon of the interaction with the interface preserves its nonlinear character as before.

If one from interconnected media is linear (either  $\beta_1=0$  or  $\beta_2=0$ ), nonlinear reflection ( $r_p \neq 0$ ) also occurs. Let medium  $m=1$  be linear and no dissipative. Pulses propagating in this medium preserves their shapes given bay boundary conditions (boundary pulse time shape). For incident waves this is  $P_b(t)$  see Fig.1.b. The solution of Eq.(10)  $m=1$  for incident waves have the form

$$P^+(t, x) = P_b \left( t - \frac{x - x_b}{c_{01}} \right) \quad x_b \leq x \leq x_{RT} = 0. \quad (47)$$

Moving left (reflected) solution of Eq.(10) have the form  $P^-(t, x) = P^-(t + \frac{x}{c_{01}})$ . Where  $P^-(t) \equiv P^-(t, x=0)$  is the boundary condition. Accordingly Eqs.(41),(43) the boundary condition for reflected in  $x=x_{RT}=0$  wave have the form

$$P^-(t) = R_p P^+(t) = (R_p + r_p P^+(t)) P^+(t) = \left( R_p + r_p P_b \left( t - \frac{-x_b}{c_{01}} \right) \right) P_b \left( t - \frac{-x_b}{c_{01}} \right). \quad (48)$$

The solution for reflected wave takes the shape,

$$P^-(t, x) = P^-(t + \frac{x}{c_{01}}) = R_p P_b \left( t + \frac{x + x_b}{c_{01}} \right) + r_p P_b \left( t + \frac{x + x_b}{c_{01}} \right)^2 \quad x < 0. \quad (49)$$

In the case of nonlinear or absorbing media the shapes of reflecting (transmitting) and propagating pulses still changes. Nevertheless this changes are small on distances  $|\delta x| \ll \lambda_{\alpha, q}$ . Then propagating near the surface ( $x=x_{RT}=0$ ) the reflecting-transmitting short pulses almost preserves the shape given by boundary conditions (Reflection-transmission operators).

$$P^-(t, x) = P^-(t + \frac{x}{c_{01}}, -x) \cong R_p P^+(t + \frac{x}{c_{01}}) + r_p P^+(t + \frac{x}{c_{01}})^2, \quad |x - x_{RT}| \sim (\text{pulse length}) \ll \lambda_{\alpha, q} \quad (50)$$

Because the Mach number  $q$  is a small quantity even for relatively strong acoustical disturbances ( $q=0.001$  for water and  $P_b=2.25\text{MPa}$ ) then effects of the nonlinear reflection may be observed separately if  $z_{02} = z_{01}, R_p = 0$  ( $z_{02} \cong z_{01}, |R_p| < |r_p|$ ). In this cases

$$P^-(t, x) = P^-(t + \frac{x}{c_{01}}) = r_p \cdot P^+(t + \frac{x}{c_{01}})^2, \quad r_p = \frac{q}{4} \left( \frac{\beta_2}{c_{02}} - \frac{\beta_1}{c_{01}} \right), \text{ because } z_{01} = 1 \quad (51)$$

If  $\beta_1 \neq 0$  sign  $=$  should be replaced by  $\cong$  in Eq.(51). In next section we compare results of the numerical solution of Eq.(11) with theoretical one. Especially expressed by the formula (51).

#### 4. NUMERICAL CALCULATIONS

The numerical solutions of Eq.(11) are the response of the solving kernel on material parameters and on changing this parameters. The solving kernel is a general numerical image of the nonlinear and non homogeneous (respect coefficients) operator (11). Particularly not contain any theoretically received on the basis of Eq.(10) information about continuity conditions and the shapes of the (R-T) operators.

Incident pulses  $P^+(t, x)$  was excited by boundary pulse  $P^+(t, x=x_b) \equiv P_b(t)$ -see Fig.1. Where  $P_b(t) = P_{0b} \cdot \text{Env}(t-t_{en}) \cdot \sin(2\pi\nu_c \cdot (t-t_c))$ ,  $t_c=0$ ,  $t_{en}=2/\nu_c$ , is four cycles length sine wave with triangle envelope  $\text{Env}()$ , carrier frequency  $\nu_c=5$  MHz ( $\lambda_c=0.3$  mm in water) and maximal amplitude  $P_{0b}=1.125$  MPa,.

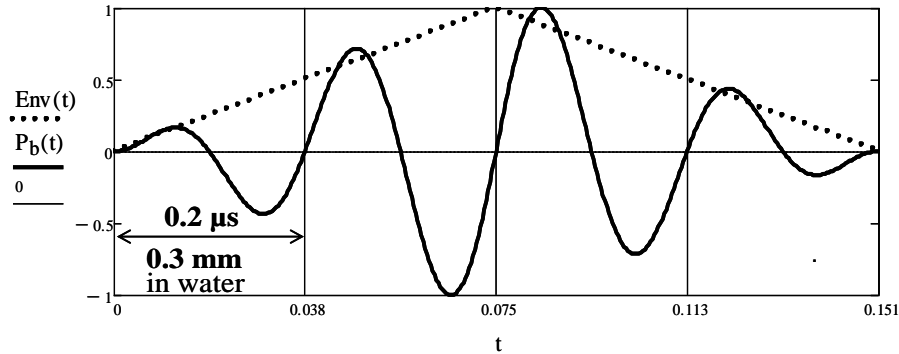


Fig. 2. Boundary pulse time shape for excitation of the incident  $P^+(t, x)$  pulses.

The numerical calculations were performed for medium consist three layer of 27mm, 3mm and 21mm thickness respectively (total thickness was 51[mm]=170  $\lambda_c$  ). The equilibrium material parameters of the first layer (like for water) was used for normalization, then  $g_{01}=1, c_{01}=1$ . We introduced third layer which copy properties of the first one. This is additional test for numerical algorithm and the way for showing effects of the sign changes of the differences between equivalent materials parameters ( $z, \beta, c$ ). In normalized system the Mach number  $q \equiv P_{0b} / \rho_0 c_0^2 = P_{0b} / \rho_{01} c_{01}^2 = 0.0005$  and  $\max_t |P_b(t)| = 1$ . Normalized carrier frequency and wavelength was equal  $\nu=170$  and  $\lambda = 2\pi/170$ . Because (pulse length)=  $4\lambda = 8\pi/170 \ll \lambda_{\alpha, q} = 1/q\nu = 1/170$  then conditions permits on applying of the formulas (50,51)



for interpretations of the space shapes of numerical solutions as preserving time shapes of the boundary conditions Eqs.(41,42) for reflecting-transmitting pulses are satisfying. Amplitude level of the incident pulses in  $x=x_{RT}=0$  were  $\max_t |P^+(t)| = 1$ .

A. Linear propagation  $\beta_m = 0$ .

As we see on Figs.3,4. Calculated (R-T) pulses heaves amplitudes and phases (depending on signs of differences) in accordance with theory. Additionally because  $c_{02} = 5/3 > c_{01} = c_{03}$  we observe that wave length in  $m=2$  is higher then in  $m=1,3$ .

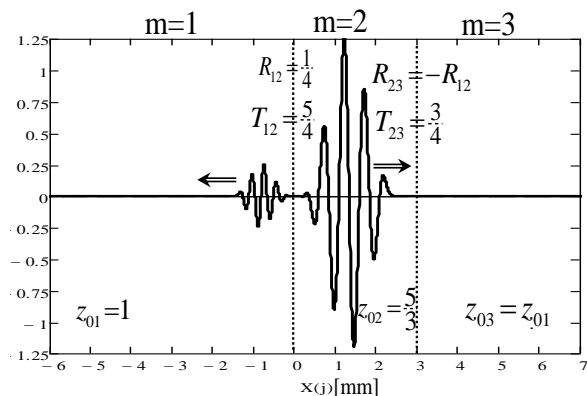


Fig. 3. The reflected and transmitted pulses after near first (R-T) plane in  $x_{RT}=0$ . pulse.

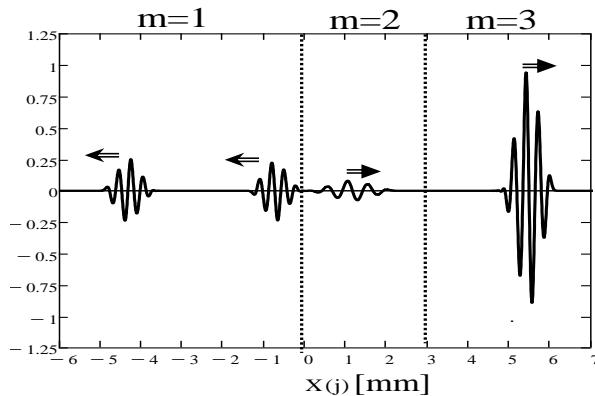


Fig. 4. R-T pulses. Second and next reflections passing of the second medium by incident

B. Nonlinear propagation

In this case the (R-T) operators are the same like in A. However  $\beta_1 = \beta_2 = \beta_3 = 3$  and  $r_{p12} \neq 0$  but  $R_{12}, T_{12} \square r_{12} = q(9/16)$  and effect of nonlinear reflection is masking by linear one. The same effects like in linear propagation case (R-T) are observed. Numerical results are in accordance with theoretically predicted.

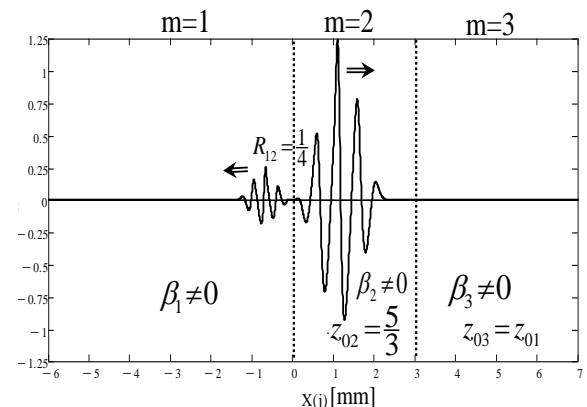


Fig. 5. The reflected and transmitted pulses after near first (R-T) plane in  $x_{RT}=0$ . pulse.

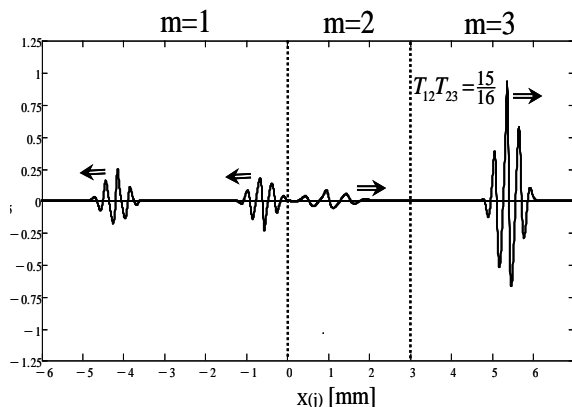


Fig. 6. R-T pulses. Second and next reflections passing of the second medium by incident

C. Nonlinear reflection. Linear-Nonlinear media.  $\beta_1=0, R_p=0; r_{p12} \neq 0$

In this case  $z_{01}=z_{02}, c_{01}=c_{02}=1, \beta_1=0, \beta_2=4, R_p=0, r_{p12}=(1/4)q\beta_2=q=0.0005$ . Then - accordingly theory- in reflect waves the effects of the nonlinear reflection should occur only. Reflected waves propagates linearly and should preserves the shape predicted by formula (51). It means should be proportional to "square" of incident wave shape. In the applied on Fig.7 scale the plots of the incident-transmitted pulses are out of the scale (vertical lines). For better visualization and comparison the plots  $0.0005 \times P_2(t-x) (\cong 0.0005 \times P^+(t-x))$  are showed -bold line. The numerically calculated shape and amplitude level of the reflecting pulse are in full accordance with theoretical predictions see. Fig. 7..

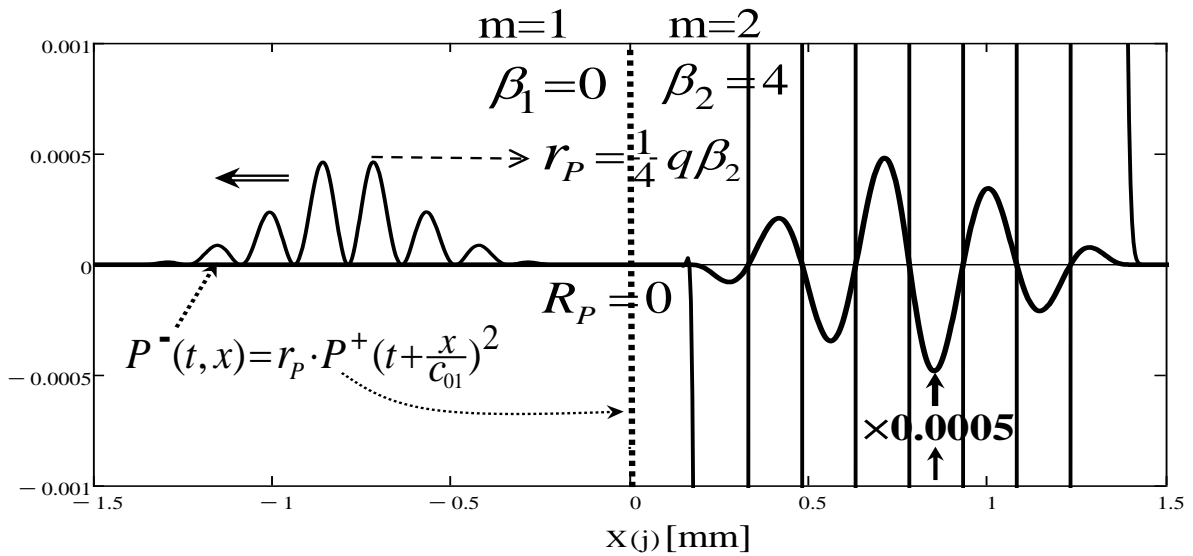


Fig. 7. Nonlinear reflection. Linear-nonlinear media.

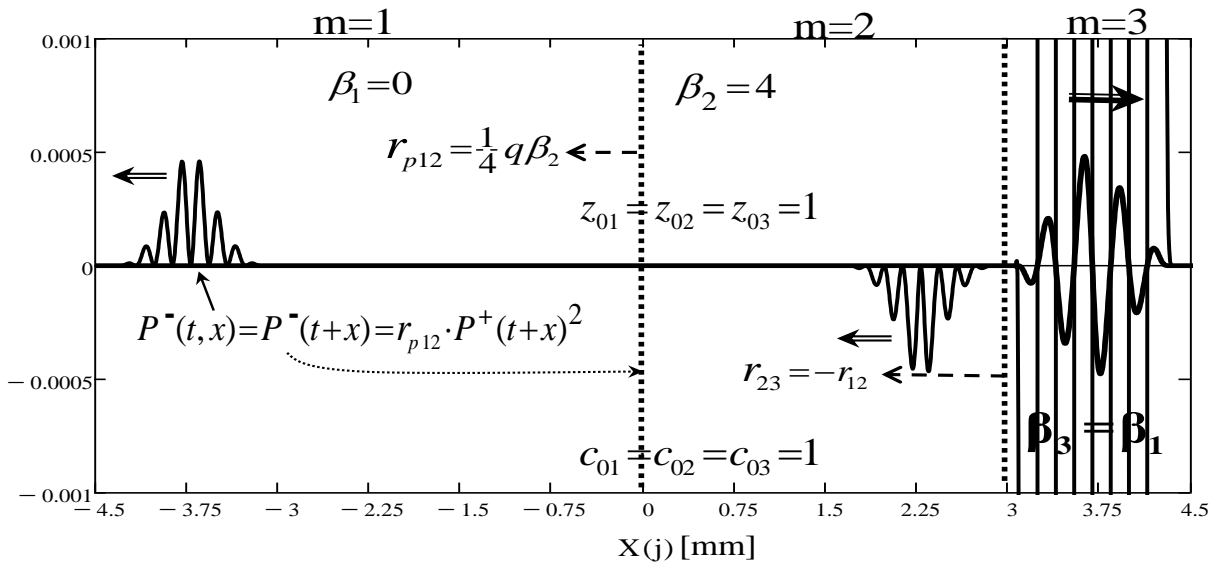


Fig. 8. Nonlinear reflection. Effect of sign changes of the  $r_p$ .  $r_{p23} = 0.25q(\beta_3 - \beta_2) = -0.25q\beta_2 = -r_{p12}$

D. Nonlinear reflection. Nonlinear-Nonlinear media.  $\beta_m \neq 0, R_p = 0; r_{p12} \neq 0$

In this case like above  $z_{01} = z_{02}, c_{01} = c_{02} = 1$ , but  $\beta_1 = 3, \beta_2 = 7, \beta_3 = 3, R_p = 0, r_{p12} = (1/4)q(\beta_2 - \beta_1) = q = 0.0005$ . Then -accordingly theory- in reflect waves the effects of the nonlinear reflection should occur only. However reflected waves propagates nonlinearly but near (R-T) planes should preserves predicted by formula (51) shape in very good approximation. It means (like in C) should be proportional to "squire" of incident wave shape. The numerically calculated shape and amplitude level of the reflecting pulse are in full accordance with theoretical predictions see. Fig.9.

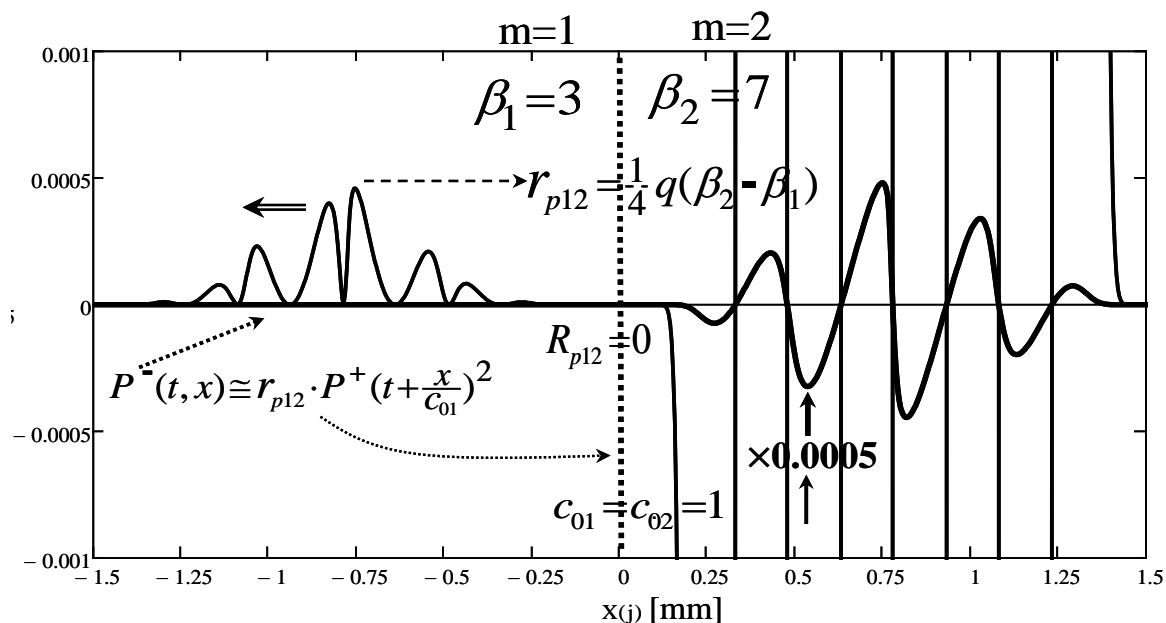


Fig. 9. Nonlinear reflection,  $R_p = 0, \beta_1 = 3, \beta_2 = 7, \beta_3 = 3$

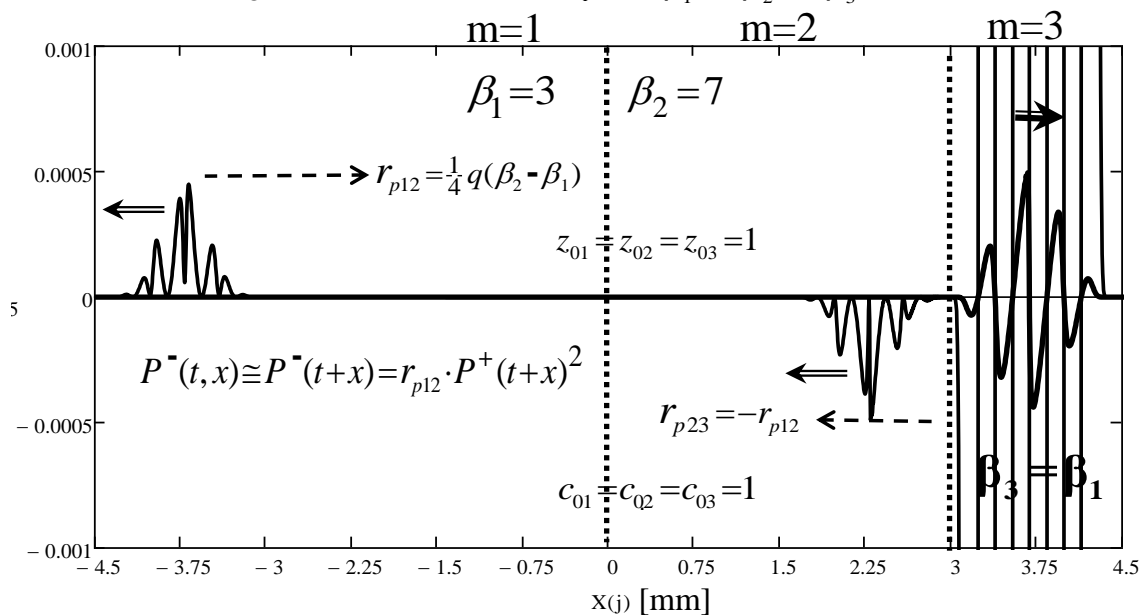


Fig.10. Nonlinear reflection. Effect of sign changes of the  $r_p$ .  $r_{p23} = 0.25q(\beta_3 - \beta_2) = -0.25q(\beta_2 - \beta_1) = -r_{p12}$

## 5. CONCLUSIONS

The numerically calculated and theoretically obtained results mutually confirmed themselves.

Much more widely discussion on the phenomenon of the nonlinear reflection and transmission was performed in [6,7]. Nevertheless we would like turn too attention on two features of the nonlinear reflection-transmission phenomenon: firstly nonlinear (R-T) occurring if only one from both (first respect incident wave!) media is nonlinear and secondly nonlinear (R-T) is nonlocal in Fourier frequency domain. The reflection and transmission of every Fourier component of the incident wave depends on the all remaining components as it follows from the properties of the auto correlation.

## ACKNOWLEDGEMENTS

This work has been supported by the Ministry of Science and Higher Education, Poland (Grant No. N N518 503339) and partially supported by the National Science Centre (grant no. 2011/03/B/ST7/03347).

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