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# Incompressible limit for a magnetostrictive energy functional

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**Abstract.** The modern materials undergoing large elastic deformations and exhibiting strong magnetostrictive effect are modelled here by free energy functionals for nonlinear and non-local magnetoelastic behaviour. The aim of this work is to prove a new theorem which claims that a sequence of free energy functionals of slightly compressible magnetostrictive materials with a non-local elastic behaviour, converges to an energy functional of a nearly incompressible magnetostrictive material. This convergence is referred to as a Γ-convergence. The non-locality is limited to non-local elastic behaviour which is modelled by a term containing the second gradient of deformation in the energy functional.

**Key words:** gamma-convergence, incompressibility, magnetostrictive material, second gradient of deformation, existence of minimizers.

## 1. Introduction

Magnetostrictive properties of smart materials are stronger when the considered structural element is comparatively thin. The magnetostrictive materials are potentially important as actuator and sensor materials. However, tension brittleness limits their applications. When metals such as iron, nickel or cobalt are combined with a polymer in a composite, the polymer matrix improves elastic behaviour while magnetostrictive properties, due to metal particles, are still preserved.

Mathematical models of plates or shells are generally used for the elasticity description of thin structural elements. The models possess an internal elastic energy, which depends on higher gradients of elastic deformation, and are known as non-local models. Frequently, deformation under external loads (magnetic or mechanical) applied to elements of metals and their alloys is sufficiently small, hence the theory of small deformations may be used. However, if magnetostrictive elements are made of composites with a polymeric matrix, the deformation is no longer small. Thus, the theory of large elastic deformations should be employed. Such an example of composite is shown in [1] where a polyurethane elastomer mixed with polycrystalline powders of Terfenol-D has been studied.

With large deformations, rubber-like material models may be used as appropriate for the analysis of stresses in such composites. Let us notice that the polymer is usually a very poorly compressible material, and there is a need to introduce an incompressible nonlinear model of magnetostrictive materials undergoing external loads.

In what follows, we study relationships between a special store energy form of an incompressible magnetostrictive material element and near incompressible material energy models. The problem is solved by the penalty method introducing a near incompressibility. Such an approach seems to be fruitful from the numerical point of view when FEM is employed, see [2,3]. The penalty method introduces isochoric-volumetric decoupling of the magnetoelastic energy functional. The measure of the volume change is represented by Jacobian of the deformation. The volumetric strain energy term is proportional to the bulk modulus of the considered material and is also inversely proportional to a small parameter.

We follow the model of a magnetostrictive finite body with prescribed boundary conditions and a non-local elastic behaviour, deforming under the external magnetic field introduced in [4]. In the model, the energy density function has been decomposed into isochoric and volumetric parts. The theorem on existence of minimizers of the magnetostrictive energy functional has been proved; therefore, the so-called existence problem for magnetostrictive body has been solved. The form of magneto-elastic strain energy density studied in [4] allowed us to define the penalization term, and to formulate a boundary value problem for nearly incompressible magnetostrictive bodies. Recently, we have proved in [5] that there exists a sequence of nearly incompressible bodies for which solutions of existence problems converge to a solution of an incompressible magnetostrictive existence problem. The proof was based on the direct method of the calculus of variations. In the present paper, we apply a special convergence type of nearly incompressible magnetostrictive models (i.e. appropriate energy functionals) to an incompressible magnetostrictive model. The convergence of functionals is understood as the Γ-convergence. The gamma convergence method is so strong that we can easily recover the main result of [5].

The Γ-convergence is an abstract notion of the functional convergence introduced by De Giorgi [6, 7]. A detailed presentation of the Γ-convergence theory may be found in Attouch [8], Braides [9] and Dal Maso [10].

Certain necessary definitions and properties of the Γ-convergence will be presented in Sec. 2. The necessary notions and definitions of energy functionals and admissible function sets of deformation and magnetization are presented in Secs. 3 and 4, respectively. The main result of the paper

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is contained in Sec. 5, where the theorem on the existence of the Γ-limit for the sequence of nearly incompressible magnetostrictive energy functionals is proved. Additionally, in Sec. 6 we show that from the new obtained result we can easily recover the result as a consequence of the theorem the result of our previous paper [5], i.e. the existence of the incompressible magnetostrictive problem is resolved. In Sec. 7 we comment some analytical forms of the penalization term in the considered energy functional.

## 2. Preliminaries

The model of a magnetostrictive body with a non-local elastic behaviour, deformed under the external magnetic field is introduced in [4]. The energy functional for the model, is additionally defined and the theorem on the existence problem is proved. Following the results of [4] we proved in [5] that there exists a sequence of weakly compressible bodies for which solutions of existence problems converge to a solution of an incompressible magnetostrictive existence problem. The proof was based on the direct method of the calculus of variations. Now, our aim is to demonstrate that the sequence of slightly compressible magnetostrictive models, i.e. appropriate energy functionals, converges to the incompressible magnetostrictive model.

Let us remind the necessary definitions and properties of the Γ-convergence.

**2.1. Definition of Γ-convergence.**
**Definition 2.1.** Let $(X,\tau)$ be a metrizable topological space, and let $\{G_\varepsilon\}_{\varepsilon>0}$ be a sequence of functionals from $X$ into $\mathbb{R}$.

- (a) $\Gamma(\tau)$- $\liminf$, denoted by $G_i$, is defined on $X$ by
$$G_i(u)=\Gamma(\tau)\text{-}\liminf_{\varepsilon\to 0} G_\varepsilon(u)=\min_{\{u_\varepsilon \xrightarrow{\tau} u\}} \liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon).$$
- (b) $\Gamma(\tau)$- $\limsup$, denoted by $G_s$, is defined on $X$ by
$$G_s(u)=\Gamma(\tau)\text{-}\limsup_{\varepsilon\to 0} G_\varepsilon(u)=\min_{\{u_\varepsilon \xrightarrow{\tau} u\}} \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon).$$
- (c) A sequence $\{G_\varepsilon\}_{\varepsilon>0}$ is $\Gamma(\tau)$-convergent iff $G_i=G_s$; so we can write
$$G=\Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon.$$

**2.2. Properties of Γ-convergence.** Let $G_\varepsilon\colon (X,\tau)\to\overline{\mathbb{R}}$ be a sequence $\Gamma(\tau)$-convergent to $G=\Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon$. Then the following properties hold (Braides [9], Dal Maso [10]):

**Theorem 2.2.**

1. The functionals $G_i$ and $G_s$ are $\tau$-lower semicontinuous ($\tau$-l.s.c.).
2. $$G(u)=\Gamma(\tau)-\lim_{\varepsilon\to 0} G_\varepsilon(u) \Leftrightarrow$$
$$\begin{cases} \forall\,\{u_\varepsilon \xrightarrow{\tau} u\},\ G(u)\le \liminf\limits_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon), \\ \forall u\in X\ \exists\, u_\varepsilon \xrightarrow{\tau} u \ \text{ such that } \ G(u)\ge \limsup\limits_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon). \end{cases}$$
3. If $\Phi\colon X\to\mathbb{R}$ is a $\tau$-continuous functional, then
$$\Gamma(\tau)\text{-}\lim_{\epsilon\to 0}(G_\epsilon+\Phi)=\Gamma(\tau)\text{-}\lim_{\epsilon\to 0} G_\epsilon+\Phi=G+\Phi.$$
and $\Phi$ is called a perturbation functional.

The following theorem is a key step in the investigations:

**Theorem 2.3. (Fundamental Property of Γ-convergence).** Let $G=\Gamma(\tau)-\lim_{\varepsilon\to 0} G_\varepsilon$, and let us assume that there exists $\tau$-relatively compact set $X_0\subset X$ such that $\inf_{X_0} G_\varepsilon=\inf_X G_\varepsilon$ (for all $\varepsilon>0$). Then $\inf_X G=\lim_{\varepsilon\to 0}(\inf_X G_\varepsilon)$. Besides, if $\{u_\varepsilon\}_{\varepsilon>0}$ is such that $\lim_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon)=\lim_{\varepsilon\to 0}\inf_X G_\varepsilon$ then $\tau$-point of convergence of the sequence $\{u_\varepsilon\colon \varepsilon\to 0\}$ minimizes $G$ on $X$.

## 3. Energy functionals of slightly incompressible and incompressible deformable magnetics

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$, which is a reference configuration of the considered magnetoelastic body. We assume that the deformation $\boldsymbol{y}:\Omega\to\mathbb{R}^3$ is of the class of $W^{2,2}(\Omega,\mathbb{R}^3)$, and the magnetic field $\boldsymbol{m}:\Omega\to\mathbb{R}^3$ has a regularity of the class $W^{1,2}(\Omega,\mathbb{R}^3)$. Since the magnetization vector $\boldsymbol{m}$ is controlled and measured in actual configuration we assume, according to [4], that $\boldsymbol{m}=\boldsymbol{m}(\boldsymbol{y})\in W^{1,2}(\boldsymbol{y}(\Omega),\mathbb{R}^3)$, where $\boldsymbol{y}(\Omega)$ denotes the image of the domain $\Omega$ under the deformation $\boldsymbol{y}$
$$\boldsymbol{m}(\boldsymbol{y})\equiv(\boldsymbol{m}\circ\boldsymbol{y}):\Omega\to\mathbb{R}^3.$$

In the paper [4], Luskin and Rybka proposed, for the case of magnetostrictive crystals, the energy, where the term of a surface energy in the domain walls is described by an integrand containing a combination of second derivatives of the deformation $\boldsymbol{y}\in W^{2,2}(\Omega,\mathbb{R}^3)$.

The free energy functional, whose minimizers describe the macroscopic behaviour of a weakly compressible magnetostrictive body caused by the applied magnetic field $\boldsymbol{h}$ reads
$$\begin{aligned} J_\varepsilon(\boldsymbol{y},\boldsymbol{m})&=\int_\Omega \kappa\left|\nabla^2\boldsymbol{y}(\boldsymbol{x})\right|^2 d\boldsymbol{x} \\ &+\int_\Omega W_\varepsilon\left(\nabla\boldsymbol{y}(x),\boldsymbol{m}(\boldsymbol{y}(x))\right)d\boldsymbol{x} \\ &+\int_{\boldsymbol{y}(\Omega)}\alpha\left|\nabla_z\boldsymbol{m}(\boldsymbol{z})\right|^2 d\boldsymbol{z} \\ &-\int_{\boldsymbol{y}(\Omega)}\boldsymbol{h}(\boldsymbol{z})\cdot\boldsymbol{m}(\boldsymbol{z})\,d\boldsymbol{z}+\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla_z\zeta(z)\right|^2 d\boldsymbol{z}. \end{aligned} \tag{1}$$

The integrand $W_\varepsilon$ in the last relation is a density of the internal magnetoelastic energy of a weakly compressible body. The constants $\kappa$ and $\alpha$ stand for material properties. The subsequent terms in the above equation represent the nonlocal effect (surface elastic energy effect), magneto-elastic energy, exchange magnetic energy, external energy and magnetostatic energy, respectively. The magnetic potential $\zeta$ satisfies the following equations:

$$\mathrm{div}_z\left(\nabla_z\zeta(z)-\chi_{\boldsymbol{y}(\Omega)}\boldsymbol{m}(z)\right)=0,\ \boldsymbol{z}\in\mathbb{R}^3, \tag{2}$$

where $\chi_{\boldsymbol{y}(\Omega)}$ is a characteristic function of domain $\boldsymbol{y}(\Omega)$. We use the same definition as in [4]:

$$\int\limits_{\Omega}\left|\nabla^2\boldsymbol{y}\left(\boldsymbol{x}\right)\right|^2 d\boldsymbol{x}=\int\limits_{\Omega}\left[\sum_{i,j=1}^{3}\left|\frac{\partial^2\boldsymbol{y}\left(\boldsymbol{x}\right)}{\partial x_i\partial x_j}\right|^2\right]d\boldsymbol{x}.$$

For our purposes it will be convenient to consider the magnetization $\boldsymbol{m}(\boldsymbol{y}(\cdot))\colon\Omega\to\mathbb{R}^3$ in the reference configuration.

Let us split the total energy functional as follows:

$$J_\varepsilon\left(\boldsymbol{y},\boldsymbol{m}\right)=G_\varepsilon\left(\boldsymbol{y},\boldsymbol{m}\right)+\Phi(\boldsymbol{y},\boldsymbol{m}), \tag{3}$$

where (with all terms rewritten in the reference configuration)

$$\begin{aligned}G_\varepsilon\left(\boldsymbol{y},\boldsymbol{m}\right)&=\int\limits_{\Omega}\kappa\left|\nabla^2\boldsymbol{y}(\boldsymbol{x})\right|^2 d\boldsymbol{x}\\&+\int\limits_{\Omega}W_\varepsilon\left(\nabla\boldsymbol{y}(x),\boldsymbol{m}(\boldsymbol{y}(x))\right)\,d\boldsymbol{x}\\&+\int\limits_{\Omega}\alpha\left|\nabla_y\boldsymbol{m}\left(\boldsymbol{y}(\boldsymbol{x})\right)\right|^2\det\nabla\boldsymbol{y}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\end{aligned} \tag{4}$$

and

$$\begin{aligned}\Phi(\boldsymbol{y},\boldsymbol{m})\equiv&-\int\limits_{\Omega}\boldsymbol{h}\left(\boldsymbol{y}\left(\boldsymbol{x}\right)\right)\cdot\boldsymbol{m}\left(\boldsymbol{y}(x)\right)\det\nabla\boldsymbol{y}\left(\boldsymbol{x}\right)d\boldsymbol{x}\\&+\frac{1}{2}\int\limits_{\mathbb{R}^3}\left|\nabla_{\boldsymbol{z}}\zeta(z)\right|^2 d\boldsymbol{z}.\end{aligned}$$

**Remark 3.1.** The functional $\Phi(\boldsymbol{y},\boldsymbol{m})$ is a continuous functional, so it may be treated as a perturbation functional in the sense of the $\Gamma$-convergence, cf. property 3 in the Theorem 1.2.

The proof of the continuity of the functional is contained in the Theorem 4.1 of the paper [4].

The nonconvex magnetic anisotropy density $W_\varepsilon\left(\boldsymbol{F},\boldsymbol{m}\right)$ is a continuous function of the deformation gradient $\boldsymbol{F}\in\mathbb{M}^{3\times3}$ (the set of $3\times3$ matrices), $\boldsymbol{F}=\nabla\boldsymbol{y}$ and the magnetization $\boldsymbol{m}\in\mathbb{R}^3$.

The energy $W_\varepsilon$ is split into two parts i.e., we assume that the anisotropy free energy density takes the form:

$$W_\varepsilon\left(\boldsymbol{F},\boldsymbol{m}\right)=\widehat{W}\left(\boldsymbol{F},\boldsymbol{m}\right)+\frac{1}{\varepsilon}\left(\psi\left(\det\boldsymbol{F}\right)-\psi\left(1\right)\right), \tag{5}$$

where $\varepsilon\in(0,1)$ is a small parameter.

The last term in (5) is referred to as a penalization term introduced in a similar way as in [5, 11–13].

Additional assumptions on the function $\psi$ are:

$$\begin{cases}\psi\left(a\right)-\psi\left(1\right)=0\ \Leftrightarrow\ a=1,\\ \psi'(1)=0,\\ \psi''(a)\geq c_0>0.\end{cases} \tag{6}$$

Moreover, the functions $\widehat{W}$ and $\psi$ satisfy the following growth conditions:

1. function $\psi\colon\ (0,\infty)\to\mathbb{R}$ is continuous, convex and is such that for some $q>2$
$$c_L\left(a^{-q}+a^q\right)\leq\psi\left(a\right)\ \text{ for all }\ a\in(0,+\infty), \tag{7}$$
2. continuous function $\widehat{W}\colon\mathbb{M}^{3\times3}\times\mathbb{R}^3\to\mathbb{R}$ is such that
$$\begin{aligned}C_L\left(\left|\boldsymbol{F}\right|^2-1\right)&\leq\widehat{W}\left(\boldsymbol{F},\boldsymbol{m}\right)\\&\leq C_U\left(\left|\boldsymbol{F}\right|^r+1\right)\quad\text{for}\quad 2\leq r<6,\end{aligned} \tag{8}$$
where the positive constants $C_L,C_U$ are such that $0<C_L<C_U$.

Let us notice, that assumptions (6) imply the strong convergence in $L^2(\Omega)$ of $\det\nabla\boldsymbol{y}\to1$, because the following inequality holds

$$\psi(a)-\psi(1)\geq\frac{c_0}{2}(a-1)^2. \tag{9}$$

Similarly as in (5) we define

$$G_\varepsilon(\boldsymbol{y},\boldsymbol{m})=\widehat{G}(\boldsymbol{y},\boldsymbol{m})+\frac{1}{\varepsilon}\int_\Omega(\psi(\det\nabla\boldsymbol{y})-\psi(1))\mathrm{d}\boldsymbol{x}.$$

For deformation satisfying the constraint of incompressibility $\det\nabla\boldsymbol{y}=1$ the functional $\widehat{G}$ reads as

$$\begin{aligned}\widehat{G}(\boldsymbol{y},\boldsymbol{m})&=\int\limits_{\Omega}\kappa\left|\nabla^2\boldsymbol{y}(\boldsymbol{x})\right|^2 d\boldsymbol{x}\\&+\int\limits_{\Omega}\widehat{W}(\nabla\boldsymbol{y},\boldsymbol{m}(\boldsymbol{y}(x)))\,d\boldsymbol{x}+\int\limits_{\Omega}\alpha\left|\nabla\boldsymbol{m}(\boldsymbol{x})\right|^2\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{10}$$

## 4. Kinematically admissible functions

Here we introduce important sets for our considerations.

**Definition 4.1.** We denote by $\mathcal{A}$ the set of the following functions, called the set of admissible functions

$$\begin{aligned}\mathcal{A}=\Bigg\{&(\boldsymbol{y},\boldsymbol{m})\in W^{2,2}\left(\Omega;\mathbb{R}^3\right)\\&\times W^{1,2}\left(\boldsymbol{y}(\Omega);\mathbb{R}^3\right):\boldsymbol{y}\left(\boldsymbol{x}\right)=\boldsymbol{y}_0\left(\boldsymbol{x}\right)\\&\text{for}\quad\boldsymbol{x}\in\partial\Omega_1,\qquad\det\nabla\boldsymbol{y}>0,\\&\text{and}\quad\int\limits_{\Omega}\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\leq\mathrm{vol}(\boldsymbol{y}(\Omega))\Bigg\}.\end{aligned} \tag{11}$$

Here $\partial\Omega_1$ means a part of the boundary $\partial\Omega$ with a positive measure. Additionally, we require that the magnetization vector satisfies the *saturation condition* given as

$$\left|\boldsymbol{m}\left(\boldsymbol{y}\left(\boldsymbol{x}\right)\right)\right|\det\nabla\boldsymbol{y}(\boldsymbol{x})=m_s=const\quad\text{a. e. in }\Omega. \tag{12}$$

**Remark 4.2.** We note that the set of admissible functions given above is equivalent to the definition of the same set introduced in [4]. Namely, the conditions of impermeability and the Ciarlet-Nečas condition

$$\int\limits_{\Omega}\det\nabla\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\leq\mathrm{vol}(\boldsymbol{y}(\Omega)) \tag{13}$$

guarantees the injectivity of mapping $\boldsymbol{y}$, see e.g. [14].

Since $\boldsymbol{y} \in W^{2,2}(\Omega, \mathbb{R}^3)$ so by an embedding theorem on Sobolev spaces, cf. [15–17], we deduce that $\boldsymbol{y} \in W^{1,s}(\Omega, \mathbb{R}^3)$, for $s \in [1, 6]$ but the compact embedding occurs for $s \in [1, 6)$. Taking $s > 3$ we may use the Ciarlet-Nečas result, cf. [15, 18], which guarantees the closure of the set $\mathcal{A}$.

Let us define the set $\mathcal{A}_0$ of kinematically admissible functions for the incompressible problem as follows.

**Definition 4.3.**

$$\mathcal{A}_0 = \{(\boldsymbol{y}, \boldsymbol{m}) \in W^{2,2}(\Omega; \mathbb{R}^3) \times W^{1,2}(\boldsymbol{y}(\Omega); \mathbb{R}^3) \colon \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}) \ \forall\, \boldsymbol{x} \in \partial\Omega_1, \quad \det\nabla\boldsymbol{y} = 1 \text{ a. e. in } \Omega, \quad |\Omega| \le \mathrm{vol}(\boldsymbol{y}(\Omega))\}. \tag{14}$$

In what follows we assume that $\mathcal{A}_0 \neq \emptyset$. The weak convergence in the set $\mathcal{A}$ of admissible functions, given below, is defined as in [4].

**Definition 4.4.** The sequence $\{(\boldsymbol{y}_n, \boldsymbol{m}_n)\} \subset \mathcal{A}$ converges weakly to $(\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{A}$ if and only if the following conditions hold:

$$\begin{gathered} \boldsymbol{y}_n \rightharpoonup \boldsymbol{y} \text{ in } W^{2,2}(\Omega; \mathbb{R}^3), \\ \chi_{\boldsymbol{y}_n(\Omega)} \boldsymbol{m}_n \to \chi_{\boldsymbol{y}(\Omega)} \boldsymbol{m} \quad \text{in} \quad L^2(\mathbb{R}^3; \mathbb{R}^3), \\ \chi_{\boldsymbol{y}_n(\Omega)} \nabla \boldsymbol{z} \boldsymbol{m}_n \rightharpoonup \chi_{\boldsymbol{y}(\Omega)} \nabla \boldsymbol{z} \boldsymbol{m} \quad \text{in} \quad L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3}). \end{gathered} \tag{15}$$

For each $\varepsilon \in (0, 1)$ let us formulate the following minimization problems for energy functionals.

## 5. Γ-limit of the sequence of energies

We assume (cf. [5]) that for every $\varepsilon > 0$ there exist an element $(\boldsymbol{y}_1, \boldsymbol{m}_1) \in \mathcal{A}$ and an element $(\boldsymbol{y}_2, \boldsymbol{m}_2) \in \mathcal{A}_0$, such that $G_\varepsilon(\boldsymbol{y}_1, \boldsymbol{m}_1)$ and $\widehat{G}(\boldsymbol{y}_2, \boldsymbol{m}_2)$ are finite.

The problem to be considered now is to prove that the sequence of solutions of slightly compressible problems converges to a solution of the incompressible problem when the compressibility tends to zero ($\varepsilon \to 0$).

In accordance to the notion of the Γ-convergence let us introduce the topological space $X$ with its norm topology, namely:

$$X \equiv [L^2(\Omega)]^3 \times [L^2(\Omega)]^3 \tag{16}$$

and the following functionals

$$\overline{G}_\varepsilon,\ \overline{G}_0 \colon X \to \overline{\mathbb{R}},$$

such that

$$\overline{G}_\varepsilon(\boldsymbol{y}, \boldsymbol{m}) = \begin{cases} G_\varepsilon(\boldsymbol{y}, \boldsymbol{m}) & \text{if } (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{A}, \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\overline{G}_0(\boldsymbol{y}, \boldsymbol{m}) = \begin{cases} \widehat{G}(\boldsymbol{y}, \boldsymbol{m}) & \text{if } (\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{A}_0, \\ +\infty & \text{otherwise.} \end{cases}$$

The functionals $G_\varepsilon(\boldsymbol{y}, \boldsymbol{m})$ and $\widehat{G}(\boldsymbol{y}, \boldsymbol{m})$ are given by (4) and (10), respectively.

Additionally, for each $\varepsilon \in (0, 1)$, the minimum of the functional $G_\varepsilon(\boldsymbol{y}, \boldsymbol{m})$ is attainable, cf. [4] therefore there exists at least one point in the set $\mathcal{A}$, denoted by $(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \in \mathcal{A}$ such that

$$\inf_{(\boldsymbol{y}, \boldsymbol{m}) \in \mathcal{A}} G_\varepsilon(\boldsymbol{y}, \boldsymbol{m}) = G_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon). \tag{17}$$

**Theorem 5.1.** Let $\overline{G}_\varepsilon$ be a functional given above. Then we have $\Gamma\text{-}\lim\limits_{\varepsilon\to 0} \overline{G}_\varepsilon = \overline{G}_0$.

The proof of the theorem is based on the following lemma.

**Lemma 5.2.** Let us assume that the sequence $(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon)$ is convergent to the limit $(\boldsymbol{y}_0, \boldsymbol{m}_0)$. Then we have

$$\liminf_{\varepsilon\to 0} \overline{G}_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \ge \overline{G}_0(\boldsymbol{y}_0, \boldsymbol{m}_0).$$

(a) First, we assume that $(\boldsymbol{y}_0, \boldsymbol{m}_0) \in \mathcal{A}_0$ then

$$\liminf_{\varepsilon\to 0} \overline{G}_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \ge \liminf_{\varepsilon\to 0} G_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \ge \liminf_{\varepsilon\to 0} \widehat{G}(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \ge \widehat{G}(\boldsymbol{y}_0, \boldsymbol{m}_0).$$

The last inequality is a consequence of a lower semi-continuity of the functional $\widehat{G}$.
Further, it is trivial that *ex definitione*

$$\widehat{G}(\boldsymbol{y}_0, \boldsymbol{m}_0) \equiv \overline{G}_0(\boldsymbol{y}_0, \boldsymbol{m}_0).$$

(b) Now, let us take $(\boldsymbol{y}_0, \boldsymbol{m}_0) \notin \mathcal{A}_0$ than

1° $\liminf\limits_{\varepsilon\to 0} \overline{G}_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) = \limsup\limits_{\varepsilon\to 0} \overline{G}_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) = +\infty$

2° $\overline{G}_0(\boldsymbol{y}_0, \boldsymbol{m}_0) = +\infty$,

so the proof is finished.

We note that a constant sequence $(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) = (\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}})$ has the following properties

(i) $(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \xrightarrow{L^2} (\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}})$

(ii) $\limsup\limits_{\varepsilon\to 0} \overline{G}_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \le \overline{G}_0(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}})$.

what is easy to check. The property (i) is obvious. To check (ii) let us consider two cases

(a) $(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}}) \in \mathcal{A}_0$
(b) $(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}}) \notin \mathcal{A}_0$.

From (a) it follows that:

$$\limsup_{\varepsilon\to 0} \overline{G}_\varepsilon(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}}) \equiv \limsup_{\varepsilon\to 0} \overline{G}_0(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}}) = \overline{G}_0(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}}).$$

From (b) it follows that

$$\overline{G}_0(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}}) = +\infty$$

and

$$\limsup_{\varepsilon\to 0} \overline{G}_\varepsilon(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}}) = \limsup_{\varepsilon\to 0} \Big[\widehat{G}(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{m}}) + \frac{1}{\varepsilon} \int_\Omega [\psi(\det\nabla\widehat{\boldsymbol{y}}) - \psi(1)] dx\Big] = +\infty.$$

The proof of the theorem 5.1 is now finished by virtue of the Theorem 1.2, point 2.

## 6. The solution of the incompressible problem

For small $\varepsilon > 0$ let us define the set

$$X_0 = \{(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) : \inf_X \overline{G}_\varepsilon = \overline{G}_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon)\}_{\varepsilon>0},$$

which is a subset of the set $\mathcal{A}$.

**Lemma 6.1.** The set $X_0$ is a compact set in $X$.

Since $\mathcal{A}_0 \subset \mathcal{A}$ we have

$$\begin{aligned}\overline{G}_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) &= \inf_{\mathcal{A}} \overline{G}_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \leq \inf_{\mathcal{A}_0} \overline{G}_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \\ &= \inf_{\mathcal{A}_0} \overline{G}_0(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \leq \overline{G}_0(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) \leq C < +\infty.\end{aligned} \tag{18}$$

Following the proof of the theorem in [5] we conclude that

$$\|\nabla^2 \boldsymbol{y}_\varepsilon\|_{L^2} \leq C,$$

$$\|\nabla \boldsymbol{y}_\varepsilon\|_{L^2} \leq C,$$

$$\|\det \nabla \boldsymbol{y}_\varepsilon\|_{L^q} \leq C,$$

$$\|\nabla \boldsymbol{m}_\varepsilon\|_{L^2} \leq C,$$

where $C$ is a constant independent of $\varepsilon$ and $q > 2$. The convergence of $\det\nabla \boldsymbol{y}_\varepsilon$ follows from properties of nonlinear Nemytski operator $\det(\cdot)$, cf. [4, 5, 19].

Since every sequence of minimizers $(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon)$ from the definition is such that $\inf_{X_0} \overline{G}_\varepsilon = \inf_X \overline{G}_\varepsilon$ (for all $\varepsilon > 0$) then, due to Theorem 2.3 and Theorem 5.1 we have $\inf_X \overline{G}_0 = \lim_{\varepsilon\to 0}(\inf_X \overline{G}_\varepsilon)$. Moreover, we have $\lim_{\varepsilon\to 0} \overline{G}_\varepsilon(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon) = \lim_{\varepsilon\to 0} \inf_X \overline{G}_\varepsilon$, also directly from the definition. So, by virtue of the fundamental property of the Γ-convergence, Theorem 2.3, we conclude that the limit of the sequence $(\boldsymbol{y}_\varepsilon, \boldsymbol{m}_\varepsilon)$ minimizes $\overline{G}_0$ on $X$.

Taken into account the Remark 3.1 we conclude that there exists

$$\Gamma - \lim J_\varepsilon(\boldsymbol{y}, \boldsymbol{m})$$

and is equal to:

$$\widehat{J}\,(\boldsymbol{y}, \boldsymbol{m}) = \widehat{G}\,(\boldsymbol{y}, \boldsymbol{m}) + \Phi(\boldsymbol{y}, \boldsymbol{m}), \tag{19}$$

over the set $\mathcal{A}_0 \subset \mathcal{A}$. Moreover, there exists a minimizer of $\widehat{J}$ which is equal to minimizer of $\widehat{G}$.

## 7. Final remarks

We have shown that the free energy of the incompressible material, cf. (10), taking finite values on the admissible set $\mathcal{A}_0$, see (14), is arbitrarily close to the free energy of the sufficiently slightly compressible material. Moreover, the existence theorem for the incompressible problem (19) has been solved, which means that the main result of [5] is recovered by means of a different approach, i.e. the Γ-convergence.

The numerical formulation of a boundary value problem for magnetostrictive incompressible material requires an analytical form of a penalization function which is the volumetric part of the free energy.

In [3, 20] the isochoric-volumetric different forms of decoupling of the elastic strain energy function (cf. (5) in the paper) are analyzed. The focus is on the analytical form of the volumetric part. The authors discuss a few of the forms.

We conclude that all the penalty functionals fulfill all abstract mathematical assumption posed in the paper (6)–(7). They can be used in the a numerical model of a weakly compressible magnetostrictive material.

In particular, the Γ-convergence is employed to obtain the macroscopic behaviour of elastic composite materials interacting with thermo-electro-magnetic fields, cf. [21].

The obtained result may be useful in more complex situations, when the composite magnetostrictive materials are considered. The homogenization theory of composites offers different methods to determine the link between properties of components, their geometrical configuration and macroscopic properties of composites, e.g. [22–25]. As we already have noticed at present, it is crucial to control and to design composite properties made of a polymer matrix and a giant magnetostrictive material, due to its wide range of applications in smart devices. The polymer material is modelled in mechanics as nonlinear and incompressible while the magnetostrictive metal may be modeled as elastically linear and compressible. It is well known that the composite possesses magnetostrictive properties and is elastically nonlinear. A certain basic problem to apply the homogenization methods to magnetostrictive composites has been studied in [26], cf also [27].

Results obtained in the paper may have play a crucial role in homogenisation process of magnetostrictive composites with incompressible (or nearly incompressible) nonlocal elastic behaviour.

There is a need to conduct further studies in that direction for define effective properties of magnetostrictive composite structural elements with incompressible (or nearly incompressible) nonlocal elastic behaviour.

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