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Systems of Hamilton-Jacobi Equations in Terms of Symplectic and Contact Geometry

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Abstract. *Discussed are some geometric problems concerning systems of first-order partial differential equations for the one unknown function, first of all systems of Hamilton-Jacobi equations. Let us remind that the Hamilton-Jacobi equations are imposed only on the differential of the unknown function but are invariant under its gauging by an additive constant. Physically our discussion is implied by the geometric study of the quantum-classical correspondence in terms of symplectic and contact structures. It turns out that quantum wave functions, scalar products, projectors and superpositions are in the classical limit represented by certain classical relationships in symplectic and contact spaces. Those classical concepts are rigorous limits of quantum ones, but at the same time they have very nice geometric interpretation, somehow related to the historical optico-mechanical analogy. As expected, superpositions of continuous families of wave functions are given by structures related to generalized envelopes, in a sense to the Huygens figures. They are analytically described by the "Stat" — operation introduced to mechanics by J. L. Synge and W. M. Tulczyjew [8], [10]. It is very interesting that the composition rule for the two-point characteristic function through intermediate events is in a sense a classical limit of the Feynmann path integral rule.*

1 Introduction

Let us begin with the concept of a linear sympplectic space (Π, Γ). It is a linear space Π endowed with the second-order twice-covariant skew-symmetric and non-degenerate tensor $\Gamma \in \Pi^* \wedge \Pi^* \subset \Pi^* \otimes \Pi^*$. To avoid discussion of the sense of term "non-degenerate" we assume Π to be finite-dimensional; for our purposes the passing over to the infinite dimension is not very essential, although it brings about some non-trivial problems. Therefore, analytically we have:

$$\Gamma_{ab} = -\Gamma_{ba}, \qquad \det[\Gamma_{ab}] \neq 0 \tag{1}$$

This implies obviously that the dimension of Π is even, $\dim \Pi = 2n$, $n \in \mathbb{Z}$. In analogy to the diagonalizing bases of symmetric (or complex-Hermitian) metrics, in the skew-symmetric case we are dealing with the distinguished Darboux bases such that

$$[\Gamma_{ab}] = \begin{bmatrix} 0_n & \vdots & -I_n \\ \cdots & \cdot & \cdots \\ I_n & \vdots & 0_n \end{bmatrix} \tag{2}$$

where 0_n denotes the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. The contravariant inverse $\Gamma^{-1} \in \Pi \wedge \Pi \subset \Pi \otimes \Pi$ is analitycally denoted by Γ^{ab} where

$$\Gamma^{ac}\Gamma_{cb} = \delta^a{}_b \tag{3}$$

obviously, in Darboux bases the matrix of $\left[\Gamma^{ab}\right]$ equals the minus of that for $[\Gamma_{ab}]$, but only in those bases,

$$\Gamma^{ab} = \begin{bmatrix} 0_n & \vdots & I_n \\ \cdots & . & \cdots \\ -I_n & \vdots & 0_n \end{bmatrix} \tag{4}$$

The Poisson bracket of two C^1-class functions $F, G : \Pi \to \mathbb{R}$ is definied as:

$$\{F, G\} := \Gamma^{ab} \frac{\partial F}{\partial \xi^a} \frac{\partial G}{\partial \xi^b} \tag{5}$$

where ξ^a are some linear coordinates on Π.

It is clear that the linear spaces of smooth functions, of analytic functions and of all polynomials on Π (the last two cases are different things), i.e., $C^\infty(\Pi)$, $C^\omega(\Pi)$, $C^{polynom}(\Pi)$ are Lie algebras under the Poisson bracket (5).

Obviously, by Lie-algebraic properties we mean the following ones:

- Poisson bracket is bilinear over numbers,

$$\{\alpha F + \beta G, H\} = \alpha \{F, H\} + \beta \{G, H\} \tag{6}$$

$$\{H, \alpha F + \beta G\} = \alpha \{H, F\} + \beta \{H, G\} \tag{7}$$

- It is skew-symmetric,

$$\{F, G\} = -\{G, F\} \tag{8}$$

- It satisfies the Jacobi identity,

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0 \tag{9}$$

everything for any numbers and functions in the above identifies.

And finally, in addition to the general Lie-algebraic properties of the Poisson bracket, let us quote some slightly different one, namely:

$$\{F(K_1, ..., K_l), G\} = \sum_{m=1}^{l} F_{,m}(K_1, ..., K_l)\{K_m, G\} \tag{10}$$

where $(K_1, ..., K_l), G$ are arbitrary functions on Π, F is an arbitrary function on $\mathbb{R}^l$, and comma-sign at F, $(F_{,m})$, on the right-hand side , denotes the partial derivative with respect to the m-th variable of F.

Globalizing those concepts one obtains a symplectic manifold.

Let P be a differential manifold and γ-an exterior two-form on P, i.e., a smooth skew-symmetric twice covariant tensor fields satisfying the following properties:

- At any $p \in P$, γ_p-is non-degenerate, thus

$$\det\left[\gamma_{p\,ab}\right] \neq 0 \tag{11}$$

- γ is closed, therefore,

$$d\gamma = 0, \text{ i.e., } \gamma_{ab,c} + \gamma_{bc,a} + \gamma_{ca,b} = 0 \tag{12}$$

This means that for any $p \in P$, the tangent space T_pP is a linear symplectic space (T_pP, γ_p) with the symplectic two-form γ, and locally

$$\gamma = d\omega \tag{13}$$

This means that locally the symplectic manifold may be identified with a linear symplectic space (Π, Γ), however without an additional geometric structure in P this identification is non-canonical . Nevertheless, at any $p \in P$ there exists a neighbourhood $U \ni p$ and coordinates ξ^a on U in which γ is expressed as in (2) and (4):

$$[\gamma_{ab}] = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix} \qquad \left[\gamma^{ab}\right] = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} \tag{14}$$

So, locally, there exist mappings $\varphi : U \to \Pi$ such that

$$\gamma|U = \varphi^*\Gamma.$$

The essential condition here is the local flatness, i. e., the fact that γ is closed:

$$d\gamma = 0 \tag{15}$$

Having at disposal a fixed twice covariant and non-degenerate tensor, we can define the operations of raising and lowering the tensor indices. However, γ is skew-symmetric, so, unlike the Riemannian case, there are a priori two possible conventions differing in sign. The difference is trivial, but one must fix one of them. So, we define the lowering of vector indices as follows:

$$\Pi \ni u \to u\lrcorner\Gamma \in \Pi^* \quad , \quad T_pP \ni X \to X\lrcorner\gamma_p \in T_p^*P \tag{16}$$

i.e., analytically,

$$(u\lrcorner\Gamma)_a = u^b\Gamma_{ba} = -\Gamma_{ab}u^b \quad , \quad (X\lrcorner\gamma_p)_a = X^b\,\gamma_{p\,ba} \tag{17}$$

and conversely:

$$f^a = f_b\Gamma^{ba} = -\Gamma^{ab}f_b, \quad \eta^a = \eta_b\,\gamma_p{}^{ba} = -\,\gamma_p{}^{ab}\,\eta_b \tag{18}$$

canonical transformations are defined as symmetries of the structure (P, γ), i.e., diffeomorphisms $\varphi : P \to P$ of P onto P preserving the two-form γ:

$$\varphi^* \centerdot \gamma = \gamma. \tag{19}$$

Let us remind that in the pseudo-Riemannian case, when geometry is introduced to a manifold M by some non-degenerate symmetric metric tensor g, the isometry group is always finite-dimensional. The maximal possible dimension equals $\frac{1}{2}\dim M\,(\dim M + 1)$. This is the exceptional case one is faced with in constant-curvature spaces, in particular in flat pseudo-Euclidean spaces. No differential identity satisfied by the metric tensor may increase the dimension of the isometry group above the value $\frac{1}{2}\dim M\,(\dim M + 1)$.

Unlike this, in symplectic manifolds the symmetry group is always infinite-dimensional. This is easily seen when we ask for infinitesimal symmetries.

We say that a vector field X on P ia a canonical field, or infinitesimal symmetry of (P, γ) when its local one-parameter group preserves γ. This holds when the Lie derivative of γ under X vanishes,

$$\mathfrak{L}_X \gamma = 0. \tag{20}$$

Making use of the general rule

$$\mathfrak{L}_X \gamma = X \lrcorner d\gamma + d\left(X \lrcorner \gamma\right) = 0 \tag{21}$$

and the symplectic rule (15) we obtain that

$$d\left(X \lrcorner \gamma\right) = 0, \tag{22}$$

i.e., the covector field $X \lrcorner \gamma$ is closed. In any case this holds of course when it is exact:

$$X \lrcorner \gamma = -dF \tag{23}$$

The function F is referred to as the Hamiltonian or Hamiltonian generator of the vector field X, and the latter is often denoted as X_F, therefore,

$$X_F = \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{24}$$

So, in any case it is seen that the symmetry group of (P, γ) is ruled by arbitrary functions. The integral curves of X_F satisfy the Hamilton equations with F as a Hamiltonian:

$$\frac{dq^i}{dt} = \frac{\partial F}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial F}{\partial q^i}. \tag{25}$$

2 Special cases of symplectic manifolds and spaces

It is well-known that in the usual treatment of the Hamiltonian formalism one begins with the configuration space; the even-dimensional phase space it is secondary by-product. Even if the phase space is not just derived directly from the configuration space, nevertheless the configuration-type description is alweys desirable.

The first special case is just algebraic one of the self-dual symplectic manifold. Let V be an arbitrary linear space of dimension n. Its dual, obviously of the same dimension will be denoted V^*. Let us take the Cartesian product

$$\Pi := V \times V^*. \tag{26}$$

This $2n$-dimensional linear space carries plenty of intrinsic structures. The first of them is given by the canonical bilinear form on Π:

$$\omega : \Pi \times \Pi \to \mathbb{R} \tag{27}$$

defined by the following formula:

$$\omega\left(z_1, z_2\right) = \omega\left(\left(q_1, p_1\right), \left(q_2, p_2\right)\right) := \langle p_1, q_2\rangle = p_1\left(q_2\right); \tag{28}$$

various alternative notations are used. This form has no particular (a)symmetry and is evidently degenerate. Its matrix is given by

$$[\omega_{ab}] = \begin{bmatrix} 0_n & \vdots & 0_n \\ \dots & \cdot & \dots \\ I_n & \vdots & 0_n \end{bmatrix}. \tag{29}$$

But, obviously, its symmetric and skew-symmetric parts are non-degenerate,

$$\begin{aligned} \triangle\left((q,p),(Q,P)\right) &= \langle p,Q\rangle + \langle P,q\rangle \\ 2\Gamma\left((q,p),(Q,P)\right) &= \langle p,Q\rangle - \langle P,q\rangle \end{aligned} \tag{30}$$

Γ is the natural symplectic two-form and it is the only structure which survives the transition from linear spaces to manifolds. It is not the case with the symmetric pseudo-Euclidean form Δ of the neutral signature $(n(+),n(-))$. This form has no manifold counterpart. The skew-symmetric form

$$\Gamma = \frac{1}{2}\left(\omega - \omega^T\right) = \mathsf{Asym}\omega \tag{31}$$

admits a natural reformulation to the manifold language.

Let us now replace the linear space V by a differential manifold Q, the configuration space of our system. It gives rise to the continuous family of mutually dual linear space T_qQ, $T_q{}^*Q$. The first of them is referred to as the space of generalized velocities of $q \in Q$, and the second one is the space of canonical conjugate momenta. Then, one performs the set-theoretical unios, i.e., tangent and cotangent bundles over Q:

$$P_N = TQ = \bigcup_{q\in Q} T_qQ\,, \quad P = T^*Q = \bigcup_{q\in Q} T_q{}^*Q. \tag{32}$$

Physically they are respectively spaces of Newton-Lagrange states (position and velocity) and Hamiltonian states (position and canonical momentum). The natural projections onto the configuration space will be denoted respectively by

$$\tau_Q : TQ \to Q\,, \quad \tau_Q{}^* : T^*Q \to Q; \tag{33}$$

obviously

$$\tau_Q\left(T_qQ\right) = \{q\}\,, \quad \tau_Q{}^*\left(T_q{}^*Q\right) = \{q\}. \tag{34}$$

The canonical Cartan one-form ω_Q on T^*Q is defined by the following prescription:

$$\omega_p = p \circ T\,\tau_{Q_p}{}^*, \tag{35}$$

and the symplectic two-form is its exterior differential,

$$\gamma = d\omega \tag{36}$$

Any local coordinates q^i on Q induce coordinates (q^i, v^i), (q^i, p_i) on TQ and T^*Q; for simplicity we use the same symbol q^i for Q, TQ, T^*Q, although they are all functions on different manifolds. The symbols v^i, p_i are components of vectors $v \in T_qQ$, $p \in T_q^*Q$ in coordinates q^i. It is easy to see that

$$\omega = p_i dq^i, \quad \gamma = dp_i \wedge dq^i \tag{37}$$

The Hamiltonian vector field X_F is given by the previous formula (24) and its integral curves - by (25). There is no difference in notation and the meaning of symbols in the cases of linear spaces and manifolds. For any smooth function F the elements of the one-parameter group X_F preserve γ, i.e., they are canonical transformations. The main difference between linear spaces and manifolds is that in general, in a symplectic space there exist also canonical transformations generated by the vector fields X for which the contraction $X \rfloor \gamma$ is closed,

$$d(X \rfloor \gamma) = 0\,, \quad \text{i.e.,} \quad (X \rfloor \gamma)_{a,b} - (X \rfloor \gamma)_{b,a} = 0 \tag{38}$$

but not necessarily exact.

Let us now discuss a very important point, namely, the classification of linear subspaces of a symplectic space and the resulting classification of submanifolds in a symplectic manifold. This is the key to understanding the Huygens principle and other quasiclassical relationship. The basic concept is the symplectic orthogonality, i.e., duality, of linear subspaces of the linear symplectic space (Π, Γ).

Let $\Lambda \subset \Pi$ be a linear subspace. Its Γ-dual subspace (or Γ-orthogonal subspace) $\Lambda^\perp$ consists of vectors which are Γ-dual (Γ-orthogonal) to Λ:

$$\Lambda^\perp = \{y \in \Pi : \Gamma(y, \cdot)\,|\Lambda = 0\}\,, \tag{39}$$

i.e., such ones that

$$\Gamma(y, x) = 0 \quad \text{for any} \quad x \in \Lambda. \tag{40}$$

The non-singularity of Γ implies that $\dim \Lambda^\perp = \dim \Pi - \dim \Lambda$. Therefore, $\dim \Lambda^\perp = m$ when $\dim \Lambda = 2n - m$.

The crucial point of classification of Λ-subspaces is the relationship between Λ and $\Lambda^\perp$.

In Euclidean spaces with non-degenerate positive metrics such a problem does not exist: the corresponding subspaces $\Lambda^\perp$, Λ are there complementary. In symplectic spaces all situations are possible. Let us quote the extreme special cases:

$$\begin{array}{lcl} \Lambda^\perp \subset \Lambda & - & \text{co-isotropic } \Lambda \\ \Lambda \subset \Lambda^\perp & - & \text{isotropic } \Lambda \\ \Lambda^\perp = \Lambda & - & \text{Lagrangian } \Lambda \end{array} \tag{41}$$

Obviously, in the last case $\dim \Lambda = n$. Lagrangian subspaces are simultaneously minimal co-isotropic and maximal isotropic ones.

Let us introduce the concept of internal singularity of a subspace Λ,

$$K(\Lambda) := \Lambda \cap \Lambda^\perp. \tag{42}$$

The traditional term "class" is used in the sense:

$$\mathsf{CL}\Lambda = (k, m - k)\,, \quad k = \dim K(\Lambda)\,. \tag{43}$$

One uses often the following abbreviations:

$$\mathrm{CL}\Lambda = I \quad \text{when} \quad k = m \quad , \quad \mathrm{CL}\Lambda = II \quad \text{when} \quad k = 0; \tag{44}$$

they are respectively first-class and second-class constraints.

Let us introduce an important concept of the reduced symplectic space. If $\Lambda \subset \Pi$ is a linear subspace, then the underlying linear space is given by

$$\Pi'(\Lambda) = \Lambda / K(\Lambda) = \Lambda / \left(\Lambda \cap \Lambda^{\perp}\right) \tag{45}$$

The reduced symplectic structure is given by the two-form

$$\gamma'(\Lambda) \in \Pi'(\Lambda)^{*} \wedge \Pi'(\Lambda)^{*} \quad , \quad \text{where,} \qquad \gamma \mid \Lambda = \pi_{\Lambda}{}^{*} \cdot \gamma'(\Lambda) \tag{46}$$

and $\pi_{\Lambda} : \Lambda \to \Pi'(\Lambda)$ is the canonical projection onto the quotient space. Therefore, $\gamma \mid \Lambda$ is the pull-back of $\gamma'(\Lambda)$ via π_{Λ}

$$\langle \gamma(\Lambda), u \wedge v \rangle = \langle \gamma'(\Lambda), \pi_{\Lambda} u \wedge \pi_{\Lambda} v \rangle . \tag{47}$$

If $\mathrm{Cl}\Lambda = (k, m-k)$, then obviously

$$\dim \Pi'(\Lambda) = 2\left(n - \frac{m+k}{2}\right). \tag{48}$$

If $\mathrm{Cl}\Lambda = I$, and only then,

$$\dim \Pi'(\Lambda) = 2(n-m), \tag{49}$$

i.e., there are $(n-m)$ reduced degrees of freedom

It is clear that the lowest possible co-isotropic dimension and the largest possible isotropic dimension equal $n = \frac{1}{2}\dim \Pi$. Any hyperplane $\Lambda \subset \Pi$, $\dim \Lambda = 2n-1$ is a co-isotropic subspace. Indeed, $\dim \Lambda^{\perp} = 1$ and Λ is odd-dimensional, so certainly $\Lambda^{\perp} \subset \Lambda$.

The set of all Lagrangian subspaces will be denoted by $\Delta(\Pi)$. The second-order Pfaff problem for Γ is regular, i.e., every isotropic subspace ξ^{ν} of dimension $\nu < n$ is contained in some $(\nu+1)$ -dimensional isotropic subspace $\xi^{\nu+1}$; the arbitrariness of $\xi^{\nu+1}$ does not depend on ξ^{ν}. Therefore, the knowledge of $\Delta(\Pi)$ is fully equivalent to the knowledge of the family of all isotropic subspace.

Obviously, $\Delta(\Pi)$ and the set $\Delta^{\nu}(\Pi)$ of all ν-dimensional isotropic subspaces are differential manifolds. They are submanifolds of the Grassmann manifolds $D(\Pi)$, $D^{\nu}(\Pi)$ of all n-dimensional and ν-dimensional linear subspaces of Π. One can prove that they have the following dimensions:

$$\dim \Delta(\Pi) = \frac{1}{2} n(n+1), \ \dim \Delta^{\nu}(\Pi) = \nu(2n-\nu) - \frac{1}{2}\nu(\nu-1). \tag{50}$$

$\Delta(\Pi)$ is a very important set. It defines the very symplectic two-form Γ up to a multiplier. Because of this it is a geometric "skeleton" of the symplectic space (Π, Γ). For any linear subspace $\Lambda \subset \Pi$ let $\Delta(\Lambda) \subset \Delta(\Pi)$ denote the set of Lagrangian subspaces contained in Λ. One can show that

$$\Delta(\Lambda) \neq \emptyset, \ \text{iff } \mathrm{CL}\Lambda = I, \tag{51}$$

i.e., $\Delta(\Lambda)$ is non-empty if and only if Λ is co-isotropic.

Let us give an analytic condition for thae class of a linear subspace. Let Λ be given as

$$\Lambda := \ker F_1 \cap \ker F_2 \cap \ldots \cap \ker F_m, \tag{52}$$

where $F_k \in \Pi^*$ are linear functions on Π defining Λ (they are "left-hand sides" of the linear equations for Λ). One can show that $\mathsf{CL}\Lambda = (k, m-k)$ if and only if:

$$m - k = \mathsf{Rank}\left[\{F_a, F_b\}\right]. \tag{53}$$

In non-linear description, when Λ when is given by:

$$\Lambda := \{p \in \Pi : F_a(p) = 0, \quad a = 1, \ldots, m\} \tag{54}$$

where $F_a$are arbitrary functions, we have instead the following "weak equation" in the Dirac sense:

$$m - k = \mathsf{Rank}\left[\{F_a, F_b\}\right]\Big|\Lambda. \tag{55}$$

The subspace $\Lambda^{\perp}$ is spanned by the Hamiltonian fields X_{F_k} . And the singularity $K(\Lambda) := \Lambda \cap \Lambda^{\perp}$ is spanned by such combination

$$\sum_{k=1}^{m} \lambda^k X_{F_k} \tag{56}$$

that the foolowing holds:

$$\sum_{b=1}^{m} \{F_a, F_b\}\, \lambda^b = 0 \quad , \qquad F_k \in \Pi^*. \tag{57}$$

Therefore, $\mathsf{Cl}\Lambda = I$, when

$$\{F_a, F_b\} = 0, \tag{58}$$

or at least we are dealing with the weak vanishing when the functions F_a are non-linear.

3 Nonlinear description.

Let us globalize those statements to the general symplectic manifolds (P, γ). Let M be a $(2n-m)$-dimensional constraints submanifold. Obviously, for any $p \in P$, (T_pP, γ_p) is a linear symplectic space and the above concepts may be directly applicable to the linear subspaces $T_pM \subset T_pP$. The problem is however if for different points $p \in M$ the resulting structures are smoothly compatible. Let us consider the following singular distribution on M:

$$M \ni p \to K_p(M) = T_pM \cap T_pM^{\perp}. \tag{59}$$

One can show that the distribution (59) is integrable . This follows from the Frobenius theorem and the fact that γ is closed , $d\gamma = 0$. More precisely, one obtains the integrability of (59) from the applying $d\gamma$ to the triple of vector fields k_1, k_2, u on M, where k_1, k_2 are singular, i.e., tangent to (59) and u is an arbitrary vectorfield on M. The closedness of γ implies that the Lie bracket $[k_1, k_2]$ also is singular for constraints M.

Let $K(M)$ be the integral foliation of (59). Locally there exist reduced symplectic manifolds in "not too large" open subsets of M. But globally it need not be so. However, for simplicity let us assume that the fibres of $K(M)$ are closed submanifolds of M and that the global assumptions concerning the existence of reduction are satisfied. Then, globalizing the algebraic process we obtain the reduced phase space

$$\left(P'(M), \gamma'(M)\right) = \left(M/K(M), \gamma'(M)\right) \tag{60}$$

where $\gamma'(M)$ is related to the strong restriction $\gamma||M$ by the quotient projection $\pi : M \to M/K(M)$,

$$\gamma||M = \pi^* \gamma'(M). \tag{61}$$

Obviously, $\dim P'(M) = 2\left(n - \frac{m+k}{2}\right)$ and the maximal dimension $2(n-m)$ is attained when $\mathsf{CL}M = I$. Analytically the reduction process is described as follows: We take some coordinates z^a $a = 1, ..., 2n$ in M, e.g., $\left(\ldots, q^i, \ldots; \ldots, p_i, \ldots\right)$ and describe M in terms of some $(2n-m)$ parameters u^μ, $z^a = z^a(\ldots, u^\mu, \ldots)$. Then

$$\left(\gamma||M\right)(u)_{\mu\nu} = \gamma_{ab}\left(z(u)\right) \frac{\partial z^a}{\partial u^\mu} \frac{\partial z^b}{\partial u^\nu}, \tag{62}$$

This is the restriction of γ to $M \subset P$. And now let us take some coordinates w^A, $A = 1, \ldots, 2\left(n - \frac{m+k}{2}\right)$ on $P'(M)$. The natural projection $\pi : M \to M/K(M)$ is analytically given by w^a expressed as functions of u^μ, $w^A\left(\ldots, u^\mu, \ldots\right)$. Then γ'_{AB} are given by:

$$\left(\gamma||M\right)(u)_{\mu\nu} = \gamma'_{AB}\left(w(u)\right) \frac{\partial w^A}{\partial u^\mu} \frac{\partial w^B}{\partial u^\nu}, \tag{63}$$

Let us go back for a moment to the symplectic space $(\Pi, \Gamma) = (V \times V^*, \Gamma)$ given in (30) (31). Let S be a quadratic form on V and let us consider a linear subspace $\mathfrak{m}_s \subset V \times V^*$ given by

$$\mathfrak{m}_s := \left\{(q, dS_q) : q \in V\right\} \subset V \times V^*. \tag{64}$$

And now let us take a linearly independent system of m linear functions F_k, $k = 1, \ldots, m$ on $(V \times V^*)^* \simeq V^* \times V \simeq V \times V^*$. Let $\mathfrak{m}_s$ to be a subset of $M = \ker F_1 \times \ker F_2 \times \cdots \times \ker F_m$. Therefore, S satisfies a system of linear Hamilton-Jacobi equations,

$$F_m\left(q, dS_q\right) = 0. \tag{65}$$

which obviously is consistent only if

$$\left\{F_a, F_b\right\} = 0 \quad , \quad a, b = 1, \ldots, m \tag{66}$$

This is an academic introduction. More seriously, let us consider a cotangent bundle (T^*Q, dw_Q), the potential-type Lagrange manifold

$$\mathfrak{m}_s := \left\{dS_q : q \in Q\right\} \subset T^*Q. \tag{67}$$

and let $\mathfrak{m}_S$ to be placed on constraints M given by

$$M := \left\{z \in T^*Q \ : \ F_k(z) = 0\,, \quad k = 1, \ldots, m\right\} \tag{68}$$

so that

$$\mathfrak{m}_S \subset M. \tag{69}$$

This leads to the system of non-linear Hamilton-Jacobi equations:

$$F_k\left(dS_q\right) = 0. \tag{70}$$

which is compatible only if $\mathsf{Cl}M = I$, i.e.,

$$\{F_a, F_b\}\Big| M = 0, \quad a, b = 1, \ldots m. \tag{71}$$

This system of weak conditions may be alternatively expressed as

$$\{F_a, F_b\} = C^k{}_{ab} F_k, \tag{72}$$

where $C^k{}_{ab}$ are sufficiently smooth functions.

Therefore, roughly speaking, the system of Hamilton-Jacobi equations is geometrically interpreted by the fact that some Lagrange manifold $\mathfrak{m}_s$ is a subset of some coisotropic constraints M. The fact that $\mathsf{CL}M = I$ corresponds to the compatibility conditions (71); as usual they are differential with respect to the left-hand sides of constraints equations. This is both geometrically interesting in itself, and also expresses some interesting quantum-quasiclassical relationships and the optical-mechanical analogy.

Let us begin as usual with the algebraic-symplectic version, where everything is clear; later on we turn to the manifold language. So, we go back to the symplectic space (Π, Γ) and to the manifolds of Lagrangian linear subspaces $\triangle(P)$ where Λ is a I- class linear subspace of Π.

There exists the natural mapping

$$\begin{aligned} E_\Lambda &:= \triangle(\Pi) \to \triangle(\Lambda) \\ E_\Lambda(\xi) &:= \xi \cap \Lambda + \Lambda^\perp \end{aligned} \tag{73}$$

As seen, its value on ξ is $\xi \cap \Lambda$ extended by $\Lambda^\perp$. One can easily show that $E_\Lambda(\xi)$ is a Lagrangian subspace.

This canonical operation, projecting, roughly speaking the set of Lagrangian subspaces $\triangle(\Pi)$ onto the set of solutions of the Λ-Hamilton-Jacobi equation, has a few nice properties:

- It is a retraction of $\triangle(\Pi)$ onto $\triangle(\Lambda)$:

$$E_\Lambda \Big| \triangle(\Lambda) = Id_{\triangle(\Lambda)} \tag{74}$$

  In particular, it is idempotent:

$$E_\Lambda \circ E_\Lambda = E_\Lambda \tag{75}$$

- If Λ, Φ are Poisson-compatible, i.e., $\mathsf{Cl}(\Lambda \cap \Phi) = I$, then:

$$E_\Lambda \circ E_\Phi = E_\Phi \circ E_\Lambda = E_{\Lambda \cap \Phi} \tag{76}$$

- If Λ, Φ are co-isotropic and E_Λ, E_Φ commute, then also $\Lambda \cap \Phi$ is a co-isotropic subspace (Λ, Φ are compatible, the Poisson brackets of their equations vanish, at least weakly in nonlinear description) and

$$E_{\Lambda \cap \Phi} = E_\Lambda \circ E_\Phi = E_\Phi \circ E_\Lambda \tag{77}$$

- The assignment $\Lambda \to E_\Lambda$ is symplectically covariant, i.e., for any $f \in \mathsf{Sp}(\Pi, \Gamma)$, i.e., for any linear canonical transformation, we have:

$$E_{f(\Lambda)} = F \circ E_\Lambda \circ F^{-1} \tag{78}$$

  where the mapping $F : \Delta(\Pi) \to \Delta(\Pi)$ is simply induced by f.

Let us illustrate this by a suggestive didactic example.

Let $(q_1, \ldots, q_n; p^1, \ldots, p^n)$ be a symplectic basis in Π and $(q^1, \ldots, q^n; p_1, \ldots, p_n)$ be its dual in Π^*, i.e., the induced system of symplectic coordinates in Π.

Let us take:

$$\Lambda = \ker q^1 \tag{79}$$

i.e., the linear shell of:

$$(q_2, \ldots, q_n; p^1, \ldots, p^n) \tag{80}$$

and take also the following Lagrange subset

$$\xi = \ker p_1 \cap \ldots \ker p_n, \tag{81}$$

i.e., linear shell of $(q_1, \ldots, q_n)$. So, roughly speaking, Λ is given fixation of the coordinate $q^1 = 0$, and ξ is given by fixation of all values of canonical momentum $p_i = 0$, $i = 1, \ldots, n$. Then one can show that

$$\xi \cap \Lambda = \ker q^1 \cap \ker p_1 \cap \ldots \cap \ker p_n \tag{82}$$

thus, it is a linear span $q_2, \ldots, q_n$, so that

$$E_\Lambda(\xi) = \xi \cap \Lambda + \Lambda^\perp = \mathbb{R}p^1 \oplus \mathbb{R}q_2 \oplus \ldots \oplus \mathbb{R}q_n, \tag{83}$$

This meants that (83) is a linear span of $p^1, q_2, \ldots, q_n$, and its equations have the form:

$$q^1 = 0, \quad p_2 = 0, \ldots, p_n = 0. \tag{84}$$

Therefore, if q^1 is fixed by constraints, then p_1 becomes arbitrary, completely diffused on $E_\Lambda(\xi)$, and all remaining p_a-s with $a \neq 1$ are non-affected. Fixing q^1 we make p_1 completely non-determined. This is an obvious allusion to the Heisenberg uncertainly relations. This very special example, although very simple, contains the very essence of the construction and of the quasi-classical uncertainty. Being given by n equation with (weakly) vanishing left-hand sides of Poisson brackets, Lagrangian manifolds are just quasi-classical pure states in the sense of optico-mechanical analogy and the quantum-classical correspondence.

Let us consider the general situation of the symplectic phase manifold (P, γ), e.g., $(T^*Q, d\omega)$ and M being an arbitrary I-class submanifold. And $\Delta(\Pi)$, $\Delta(\Lambda)$ are replaced by the sets (infinite-dimensional) of all possible Lagrange manifolds and all possible Lagrange manifolds placed on the first-class constraints, $\Delta(P)$, $\Delta(M)$.

The generic situation is now that either $\mathfrak{m} \in \Delta(P)$ is disjoint with M or intersects it in a clean way, like linear subspaces do:

$$p \in \mathfrak{m} \cap M: \quad T_p(\mathfrak{m} \cap M) = T_p\mathfrak{m} \cap T_pM \tag{85}$$

This gives rise to the mapping

$$\Lambda_M: \quad \Delta(P) \rightarrow \Delta(M) \tag{86}$$

where

$$\Lambda_M(\mathfrak{m}) := \bigcup_{p \in \mathfrak{m} \cap M} K_p(M), \tag{87}$$

where $K_p(M)$ is the singular fiber of $K(M)$ through the point $p \in \mathfrak{m} \cap M$. In a generic situation the dimensions agree:

$$\dim(\mathfrak{m} \cap M) + \dim K_{p \in \mathfrak{m} \cap M}(M) = n = \frac{1}{2} \dim P. \tag{88}$$

or $\Lambda_M(\mathfrak{m}) = \emptyset$ if $\mathfrak{m}$, M are disjoint. Let us remind that when we deal with linear subspaces, they are never disjoint and always intersect in a "clean" way. Reasoning in a quasiclassical/quantum way we can say that if $\mathfrak{m} \cap M = \emptyset$, this may be interpreted in such a way that the quasi-classical wave function corresponding to $\Lambda_M(\mathfrak{m})$ vanishes. And this just corresponds to the fact that the phase, information about which is contained in $\Lambda_M(\mathfrak{m})$ is not well-defined at all.

We have mentioned above about the clean intersections similar to those for linear subspaces. Let us remind however some possible exceptional situations, e.g., when M is a value-surface of an ergodic Hamiltonian. Then the fibres of $p \rightarrow K_p(M)$ are dense in M. Therefore, also $\Lambda_M(\mathfrak{m})$ is non-closed, dense in a subset of M of the same like M dimension. We have no feeling what does it meant on the quasi-classical quantum states which are represented by immersed Lagrange manifolds, not just the submanifold.

4 The two-point characteristic function, Stat - operation

The structure of quasiclassical projectors has to do with both quantum/quasiclassical projectors but also with the theory of Hamilton-Jacobi equations, their systems and the initial value problems.

Take instead Q the space-time manifold, e.g., either Galilean or Minkowskian one with space-time coordinates x^μ so that,

$$\underset{\text{Galilei}}{(x^\mu) = (t, q^i)} \quad , \quad \text{or} \quad \underset{\text{Minkowski}}{x^\mu = (x^0, x^i) = (ct, x^i)} \tag{89}$$

We assume the space-time dimension:

$$\dim X = n + 1 \tag{90}$$

not necessarily $n = 3$, unless otherwise clearly stated. Now, take the $2(n+1)$-dimensional "phase over-space" and fix the "energy hypersurface" $M \subset T^*X$ with equations, respectively:

$$F = p_t + H(t, q^a, p_a) = 0 \tag{91}$$

Galilei, p_t - t- conjugate momentum, or

$$F = \frac{1}{2m} g^{\mu\nu} (p_\mu - eA_\mu)(p_\nu - eA_\nu) - \frac{m}{2} = 0 \tag{92}$$

when we use the parametrization by the proper time. The idea of M is that there are no holonomic x^μ- constraints; equations (91)(92) depend explicitly on p_μ. Let us repeat, they are "energy equations" in the language of J. L. Synge [10]. A_μ is the electromagnetic potential co-vector.

The fibres of the singular foliation $K(M)$ are characteristic, admissible phase-space motions. Their X- projection $\pi(K(M))$ are dynamically admissible world-lines. The corresponding Hamilton-Jacobi equations have, obviously, the form: - parametrization with the proper time

$$F\left(\ldots,x^{\mu},\ldots;\ldots,\frac{\partial S}{\partial x^{\mu}},\ldots\right)=0. \tag{93}$$

For any solution S the Lagrange manifold

$$\mathfrak{m}_s:=\left\{\left(\ldots,x^{\mu},\ldots;\ldots,p_{\mu}=\frac{\partial S}{\partial x^{\mu}},\ldots\right):\quad x\in X\right\} \tag{94}$$

is foliated by $K(M,S)\subset K(M)$ - the n - parameter pencil of classical trajectories. It is the optico-mechanical analogy of quantum state, not a single trajectory, but, like Synge used to call it: the coherent family of classical solutions [10].

Let us now take any fiber T_x^*X and Λ_M- project it to $\Delta(M)$ (we can do it, T_x^*X is a Lagrangian manifold):

$$\mathfrak{m}_x:=\Lambda_M\left(T_x^*X\right) \tag{95}$$

This is a Lagrange manifold, almost all over a cross-section over an $(n+1)$-dimensional submanifold of X. This "almost all" excludes the point $x\in X$. Nevertheless, outside this dangerous region we have the potential representation:

$$\mathfrak{m}_x\cap T_y^*X=\left\{d\sigma\left(x,\cdot\right)_y\;,\quad y\in X\right\}. \tag{96}$$

The quantity $\sigma:\quad X\times X\rightarrow\mathbb{R}$ (or rather defined over some $(n+1)$- dimensional region of X) is just the "two-point characteristic function of M (of the homogeneous dynamics). Analytically, $\mathfrak{m}_x\cap T_y^*X$ has equations:

$$p_{\mu}=\frac{\partial\sigma\left(x,y\right)}{\partial y^{\mu}}. \tag{97}$$

The quantity σ,or, to be more precise σ_M, may be analytically expressed as:

$$\sigma_M\left(x,y\right)=\int\limits_{l(x,y)}\omega=\int\limits_{l(x,y)}p_{\mu}dx^{\mu}, \tag{98}$$

where $l(x,y)$ denotes a characteristic trajectory from $K(M)$ joining the ${\tau^*}^{-1}(x)$, ${\tau^*}^{-1}(y)$. Let us remind that if x,y are sufficiently close to each other (and causally related), there exists exactly one $l(x,y)$. If it does not exist, σ_{μ} is not defined, i.e., the corresponding quasiclassical amplitude, the phase of which is σ_{μ}, does vanish. If at "large separation" there are a few ones, then σ_{μ} is multivalued and the corresponding quasiclassical propagator is a sum of a few terms.

As mentioned $\sigma_M(x,\cdot)$ is the Hamilton-Jacobi propagator at $x\in X$. Let $\Sigma\subset X$ be a Cauchy hypersurface for initial data, and let $f:\quad\Sigma\rightarrow\mathbb{R}$ be some Cauchy data for the Hamilton-Jacobi equation. Then the stationary values

$$S\left(x\right)=\mathsf{Stat}_{q\in\Sigma}\left(f(q)+\sigma_M\left(q,x\right)\right), \tag{99}$$

represent the solution of the f-Cauchy problem:

$$F\left(\ldots,x^{\mu},\ldots;\ldots,\frac{\partial S}{\partial x^{\mu}},\ldots\right)=0,\quad F\Big|\Sigma=f. \tag{100}$$

We must explain the meaning of the Synge-Tulczyjew Stat-symbol [8] [10]. For simplicity, let us assume that the function $\Psi : Q \to \mathbb{R}$ on a differential manifold Q has exactly one stationary point $a \in Q$, where its differential does vanish. The quantity $\Psi(a)$ is just identified with $\mathsf{Stat}\Psi$

$$\mathsf{Stat}\Psi := \{\Psi(a) : \quad d\Psi_a = 0\}. \tag{101}$$

If there is a discrete family of stationary points, then $\mathsf{Stat}\Psi$ is the set of values at them. If they form a connected continuum $Y \subset Q$, then it follows that $\Psi\big|Y = \mathsf{const}$ and this constant is just $\mathsf{Stat}\Psi$. And obviously all situations between. One can show that if

$$\Lambda_M \mathfrak{m}_s = \mathfrak{m}_{s'}, \tag{102}$$

then

$$S'(x) = \mathsf{Stat}_y \left(S(y) + \sigma_M(y, x)\right). \tag{103}$$

The idempotence property of Λ_M implies that

$$\sigma_M(x, y) = \mathsf{Stat}_z \left(\sigma_M(x, z) + \sigma_M(z, y)\right), \tag{104}$$

And more generally, we have the "Feynmann rule":

$$\sigma_M(x, y) = \mathsf{Stat}_{(z_1,\ldots,z_k)} \left(\sigma_M(x, z_1) + \sigma_M(z_1, z_2) + \ldots + \sigma_M(z_k, y)\right). \tag{105}$$

Let us consider as an example a free Galilean particle:

$$\sigma_M(x, y) = S(a, z : q, t) = \frac{m}{2(t-z)} g_{ij} \left(q^i - a^i\right)\left(q^j - a^j\right). \tag{106}$$

The corresponding unimodular factor for the Schrödinger propagation for

$$\hbar\imath \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi, \tag{107}$$

i.e.,

$$\exp\left(\frac{\imath}{\hbar}\sigma_M(x, y)\right) = \exp\left(\imath \frac{m}{2\hbar(t-z)} g_{ij}\left(q^i - a^i\right)\left(q^j - a^j\right)\right) \tag{108}$$

agrees with that for rigorous quantum formula.

The family:

$$\{S(a, 0; \cdot, \cdot) : a \in Q\} \tag{109}$$

is a complete integral of free Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m} g^{kl} \frac{\partial S}{\partial q^k}\frac{\partial S}{\partial q^l} = 0 \tag{110}$$

As usual, the corresponding Van Vleck determinant

$$\det\left[\frac{\partial^2 S}{\partial q^i \partial a^j}\right] \tag{111}$$

gives the quasiclassical probability term at $\exp\left(\frac{\imath}{\hbar}\sigma_M(x, y)\right)$:

$$\sqrt{\det\left[\frac{\partial^2 \sigma}{\partial q^i \partial a^j}\right]} \exp\left(\imath \frac{m}{2\hbar(t-z)} g_{ij}\left(q^i - a^i\right)\left(q^j - a^j\right)\right) \tag{112}$$

After modifying the constant normalization term, it becomes just the rigorous quantum Schrödinger propagator:

$$\mathcal{K}\left(\tau,\xi\right)=\left(\frac{m}{2\pi hi\tau}\right)^{\frac{n}{2}}\exp\left(\frac{i}{\hbar}\frac{m}{2\tau}g_{kl}\xi^{k}\xi^{l}\right). \tag{113}$$

where $\tau = t - z$, $\xi^i = q^i - a^i$. Such a compatibility between classical, quasiclassical and purely quantum rules does appear quite often in high-symmetry problems, like geodetic, oscillatory and Columb motion. In any case, it is quite interesting that the rigorous quantum formula may be found quite often within the framework of the purely classical concepts.

It is interesting that the classical, but nevertheless the rigorous quantum formulas (112) (113) were obtained with the help of zeroth/first order WKB approximation. The density expression, i.e., the Van Vleck determinant was obtained from the zeroth-order phase approximation. Therefore, in the sense of optico-mechanical analogy it corresponds to something like Fresnel approximation.

Let us also mention that in the homogeneous, e.g., relativistic mechanics the proper description of quasi-classical phenomena will be based on the tensor expression

$$\mathcal{V}=D^{\mu}dx^{0}\wedge\ldots\wedge\mu\wedge\ldots dx^{n}\otimes da^{1}\wedge\ldots\wedge da^{n}. \tag{114}$$

Contracting this with $t(a)=\frac{\partial}{\partial a^1}\wedge\ldots\wedge\frac{\partial}{\partial a^n}$ we obtain the differential form

$$j\left(t(a)\right)=D^{\mu}dx^{0}\wedge dx^{1}\wedge\ldots\wedge\mu\wedge\ldots dx^{n} \tag{115}$$

Obviously D^μ is the minor of the matrix $\left[\frac{\partial^2 S}{\partial x^\mu \partial a^j}\right]$ obtained by the removing of the μ-th column. The symbol μ used in the exterior product means that the differential form dx^μ is rejected. The structure of Van Vlect objects implies that

$$\frac{\partial j^{\mu}}{\partial x^{\mu}}=0\,,\quad j^{\mu}=(-1)^{\mu}D^{\mu}. \tag{116}$$

It is clear that any singular vector field k of the dynamical constraints $M \subset T^*X$ is tangent to submanifolds $\mathfrak{m}_a = \mu^{-1}(a)$, and because of this it induces on them the vector fields $k(a)$. Those, projected onto X give rise to the vector fields $v(a)$. One can easily show that

$$v^{\mu}\left(x,a\right)\frac{\partial^{2}S}{\partial x^{\mu}\partial a^{i}}=0\,,\quad k\left(a\right)\rfloor j\left(t_{a}\right)=0. \tag{117}$$

where v^μ are components of the vector field v.

Obviously, according to the standard rule,

$$\mathcal{L}_{k(a)}j\left(t_{a}\right)=k\left(a\right)\rfloor dj\left(t_{a}\right)+d\left(k\left(a\right)\rfloor j\left(t_{a}\right)\right). \tag{118}$$

But according to (117) we have then

$$\mathcal{L}_{k(a)}j\left(t_{a}\right)=0, \tag{119}$$

i.e., $j(t_{a})$ is obtained by the pull-back of a fixed differential form on the Lagrangian submanifold $\pi_M(\mathfrak{m}_a)$. This is, roughly speaking, the conservation of probability.

5 Contact spaces and their geometric meaning. Huygens superpositions and envelopes

The symplectic Pfaff problem is quadratic: it demands us to find subspaces and submanifolds on which the two-form γ (or Γ) does vanish. Its solution is $\Delta(P)$ or $\Delta(\Pi)$, the set of Lagrange manifold and of their submanifold. However, working with the quadratic Pfaff problem is inconvenient, it is much more easy to deal with the linear one. And what is more important, there are certain quasiclassical structures but one: superposition. What is the way out? To increase the dimension.

In non-conservative, time-dependent Hamilton mechanics one introduces the odd-dimensional evolution space parametrized by the variables $(t, q^i; p_i)$. Then we put:

$$\Omega_H = p_i dq^i - H(t, q, p)\, dt = P_i dQ^i - dz. \tag{120}$$

The last expression is fundamental for the traditional way of introducing and discussing canonical transformations.

The corresponding singular two-form is given by

$$\Gamma_H = d\Omega_H = dp_i \wedge dq^i - dH \wedge dt \tag{121}$$

Realistic trajectories are then integral curves of the singular vector field of Γ_H, normalized according to:

$$X_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{122}$$

Obviously, this is a contact space in the sense of (120). But strictly speaking, the contact structure in a manifold C (contact manifold) is given when C is a principal fiber bundle over the phase space (symplectic manifold (P, γ)) with the one-dimensional structure group, additive $\mathbb{R}$ or multiplicative $U(1)$, and with the connection one-form Ω for which the curvature two-form Γ is a pull-back of γ under the projection $\pi : C \rightarrow P$,

$$\Gamma = d\Omega = \pi^* \cdot \gamma. \tag{123}$$

The local description is given by

$$\Omega = p_i dq^i - dz \tag{124}$$

so locally just like in (120), and the principal vector field is given by:

$$k = -\frac{\partial}{\partial z}. \tag{125}$$

Integral surfaces of Ω are horizontal or Legendre submanifolds of C. Their set will be denoted by $\mathcal{H}(C)$. For any $\mathfrak{m} \in \Delta(P)$, $\pi^{-1}(\mathfrak{m}) \subset C$ is foliated by its horizontal lift.

Any constraints $M \subset P$ give rise to the contact space constraints $\pi^{-1}(M)$. Their singular foliation is then lifted to

$$K^{\Omega}(M) = \text{hor lift} K(M) \tag{126}$$

Let us take two Lagrange manifolds $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathcal{H}(C)$. We assume for a moment that $\mathfrak{M}_1, \mathfrak{M}_2$ project onto $\mathfrak{m}_1, \mathfrak{m}_2$ interesting along a single point or along some connected and simply-connected region in P. Then there is exactly one element t of the structure group, such that $\mathfrak{m}_1, \mathfrak{m}_2 \in \Delta(p)$. Then we tell that the scalar product $[\mathfrak{M}_1|\mathfrak{M}_2]$ is given by:

$$[\mathfrak{M}_1|\mathfrak{M}_2] = t \tag{127}$$

And more generally we say that the Huygens scalar product or vertical distance of $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathcal{H}(C)$ is defined as the subset $[\mathfrak{M}_1|\mathfrak{M}_2]$ of the structure group such that $\mathfrak{M}_2 \cap g_t\mathfrak{M}_1 \neq \emptyset$, i.e., is non-empty. If $[\mathfrak{M}_1|\mathfrak{M}_2] = \emptyset$, then we say that $\mathfrak{M}_1$, $\mathfrak{M}_2$ are orthogonal. Then their projections $\mathfrak{m}_1$, $\mathfrak{m}_2 \subset P$ are disjoint, $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \emptyset$.

Special contact transformations are defined as symmetries of the contact space, i.e., such one-to-one mappings $\varphi : C \to C$ which preserve Ω,

$$\varphi^*\Omega = \Omega. \tag{128}$$

Then automatically everything is conserved, e.g., the singular fibers and the action of structure group.

Moreover, for any special contact transformation $\varphi : C \to C$ there exists exactly one canonical mapping $\underline{\varphi} P \to P$ of P onto itself such that $\underline{\varphi} \circ \pi = \pi \circ \varphi$, where, let us remind,

$$\underline{\varphi}^*\gamma = \gamma. \tag{129}$$

One can easily show that the special contact transformations are "unitary" in the sense of preserving the above-introduced "scalar product",

$$[\varphi\mathfrak{M}_1|\varphi\mathfrak{M}_2] = [\mathfrak{M}_1|\mathfrak{M}_2]\,. \tag{130}$$

Let us consider the potential Legendre manifolds:

$$\mathfrak{M}_S := \{(dS_q, S(q)) : q \in Q\} \subset T^*Q \times \mathbb{R}. \tag{131}$$

Then one can easily show that

$$[\mathfrak{M}_{S_1}|\mathfrak{M}_{S_2}] = \mathsf{Stat}\,(S_2 - S_1) \tag{132}$$

One can show, using the WKB-method that it is not the formal analogy but just the classical limit of the phase of the scalar product. Namely, if we take the wave functions on Q,

$$\Psi_1 = \sqrt{D_1}\exp\left(\frac{i}{\hbar}S_1\right)\ , \quad \Psi_2 = \sqrt{D_2}\exp\left(\frac{i}{\hbar}S_2\right) \tag{133}$$

and their scalar product

$$\langle\Psi_1|\Psi_2\rangle = \int \overline{\Psi}_1\,(q)\,\Psi_2(q)d_nq = \sqrt{D}\exp\left(\frac{i}{\hbar}\varphi\right). \tag{134}$$

then in the limit $\hbar \to 0$, the method of stationary phase tells us that

$$\varphi = \mathsf{Stat}\,(S_2 - S_1) \tag{135}$$

just like in (132). Therefore, the vertical distance between $\mathfrak{M}_{S_1}$ and $\mathfrak{M}_{S_2}$ really tells us what is the quasiclassical phase of the scalar product. Nevertheless, the expression (135) is just based on the contact geometry, without any appealing to this limit transition. If the projections $\mathfrak{m}_1, \mathfrak{m}_2$ of $\mathfrak{M}_1, \mathfrak{M}_2$ to (P, γ) intersect along some connected submanifold

$\mathfrak{m}_1 \cap \mathfrak{m}_2$, then the vertical distance between $\mathfrak{M}_1, \mathfrak{M}_2$ is nevertheless unique. If $\mathfrak{m}_1 \cap \mathfrak{m}_2$ consists of a few connected components, then $[\mathfrak{M}_1|\mathfrak{M}_2]$ is a subset of the structure group and the classical scalar product is the sum of a few expressions (135). If $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \emptyset$, then $[\mathfrak{M}_1|\mathfrak{M}_2] = \emptyset$ and this corresponds just to orthogonality in the classical limit.

Let us now introduce the concept of superposition of Legendre manifolds, to be more precise, of their continuous family. It turns out that the concept is strictly related to the Huygens envelope prescription, as it should be expected. Let us begin with taking a differential submanifold $N \subset C$. We say that its determinant set, or the characteristic set $\Sigma(N)$ is the subset which consists of such points $z \in N$ that

$$\Omega_z \big| T_z N = 0, \tag{136}$$

i.e., Ω_z vanishes on the tangent spaces at z.

Take a family of Legendre subsets:

$$\{\mathfrak{M}_a : a \in A\} \subset \mathcal{H}(C) \tag{137}$$

The superposition of $\mathfrak{M}_a$-s denoted by

$$\mathfrak{M} = \underset{a \in A}{\boldsymbol{E}} \, \mathfrak{M}_a \tag{138}$$

is the maximal element of $\mathcal{H}(C)$ contained in the characteristic set of the union of $\mathfrak{M}_a$-s,

$$\Sigma\left(\bigcup_{a \in A} \mathfrak{M}_a\right) \tag{139}$$

Examples make this clear:

Example 1)

In $C = T^*Q \times \mathbb{R}$ take a family of Legendre manifolds with definite positions,

$$\mathfrak{M}_q := \left(T_q^* Q, 0\right). \tag{140}$$

And take the potential Legendre manifold:

$$\mathfrak{M}_S := \{(dS_q, S(q)) : q \in Q\}. \tag{141}$$

Obviously, it has equations:

$$p_i = \frac{\partial S}{\partial q^i} \quad , \quad i = 1, \ldots, n \quad , \quad z = S(q) \tag{142}$$

Then

$$\mathfrak{M}_S = \underset{q \in Q}{\boldsymbol{E}} \, [S(q)] \, \mathfrak{M}_q \tag{143}$$

where $[t]\,\mathfrak{M}$ denotes the raising of $\mathfrak{M}$ by t in the z- direction.

Example 2)

In $C = T^*Q \times \mathbb{R}$ take the family of Legendre manifolds:

$$\mathfrak{M}_a = \mathfrak{M}_{S(\cdot,a)} = \{(dS(\cdot,a)_q, S(q,a)) : q \in Q\}; \tag{144}$$

where

$$S : Q \times A \to \mathbb{R} \tag{145}$$

is a differentiable function and A is a parameter set. Take a coefficent function $f : A \to \mathbb{R}$ and assume:

$$\underset{a \in A}{\boldsymbol{E}} \left[f\left(a\right) \right] \mathfrak{M}_a = \left\{ \left(dS_q, S\left(q\right) \right) : q \in Q \right\} \tag{146}$$

Then:

$$S(q) = \underset{a \in A}{\mathsf{Stat}} \left(S\left(q, a\right) + f\left(a\right) \right) \tag{147}$$

This is a generic situation. What it means geometrically?

Take projections of $\mathfrak{M}$-s onto $Q \times \mathbb{R}$:

$$\begin{aligned} \xi_S &:= \{(q, S(q)) : \quad q \in Q\} \subset Q \times \mathbb{R} \\ \xi_a &:= \xi_{S(\cdot,a)} = \{(q, S(q,a)) : \quad q \in Q\} \subset Q \times \mathbb{R} \end{aligned} \tag{148}$$

Therefore, ξ_S, the diagram of S, is the usual envelope of the $f(a)$- moved in the z - direction ξ_a-s, i.e., the diagrams of $S\left(\cdot, a\right) + f(a)$.

This is the generic, regular situation. In the previous example we were dealing with the singular situation where the graph of S was the "envelope" of the n- parameter family of the null-dimensional manifolds $(q, S(q))$ in $Q \times \mathbb{R}$. Nevertheless, in the phase-space language that situation was just as regular as the previous one.

Example 3)

Now let Q be an n- dimensional linear space V, Therefore T^*Q and C become respectively $V \times V^*$ and $V \times V^* \times \mathbb{R}$ (or $V \times V^* \times U(1)$). Take the Legendre manifolds of the fixed position and momenta:

$$\mathfrak{M}\left[x\right] = \left\{ \left(x, p, 0\right) : \, p \in V^* \right\} \quad , \quad \mathfrak{M}\left[p\right] = \left\{ \left(x, p, \langle p, x \rangle\right) : \, x \in V \right\} \tag{149}$$

Then we have:

$$\mathfrak{M}\left[p\right] = \underset{x \in V}{\boldsymbol{E}} \left[\langle p, x\rangle\right] \mathfrak{M}\left[x\right] \quad , \quad \mathfrak{M}\left[x\right] = \underset{p \in V}{\boldsymbol{E}} \left[-\langle p, x\rangle\right] \mathfrak{M}\left[p\right] \tag{150}$$

and for any function $S : \; V \to \mathbb{R}$,

$$\mathfrak{M}_S = \underset{x \in V}{\boldsymbol{E}} \left[S(x)\right] \mathfrak{M}\left[x\right] = \underset{p \in V^*}{\boldsymbol{E}} \left[\widehat{S}(p)\right] \mathfrak{M}\left[p\right], \tag{151}$$

where the following Fourier- envelope relations hold:

$$\widehat{S}[p] = \underset{x \in V}{\mathsf{Stat}} \left(S(x) - \langle p, x\rangle\right), \quad S[x] = \underset{p \in V^*}{\mathsf{Stat}} \left(\widehat{S}(p) + \langle p, x\rangle\right) \tag{152}$$

Let $F : \; \mathcal{H}\left(C\right) \to \mathcal{H}\left(C\right)$ -be a mapping of the set of Legendre transformations induced by STC $f : \; C \to C$. It is "envelope-linear" (Huygens-linear):

$$F \underset{a \in A}{\boldsymbol{E}} \left[t_a\right] \mathfrak{M}_a = \underset{a \in A}{\boldsymbol{E}} \left[t_a\right] F\mathfrak{M}_a. \tag{153}$$

It is also "envelope-unitary"

$$\left[F\mathfrak{M}_1 \middle| F\mathfrak{M}_2\right] = \left[\mathfrak{M}_1 \middle| \mathfrak{M}_2\right] \tag{154}$$

Now, we shall lift the operation $\Lambda_M : \; \Delta(P) \to \Delta(M)$ to some operation in the contact space

$$\Pi_M : \; \mathcal{H}(C) \to \mathcal{H}_M(C). \tag{155}$$

$\mathcal{H}_M(C)$- is the family of Legendre submanifolds on $\pi^{-1}(M)$. This operation,

$$\Pi_M : \mathcal{H}(C) \to \mathcal{H}_M(C)\ ,\ CLM = I \tag{156}$$

π- projects onto $\Lambda_M : \Delta(P) \to \Delta(M)$

$$\Pi \circ \Pi_M = \Lambda_M \circ \Pi \tag{157}$$

where $\Pi : \mathcal{H}(C) \to \Delta(P)$ is the natural π - projection. And:

$$\mathfrak{M} \cap \pi^{-1}(M) = \mathfrak{M} \cap (\Pi_M \mathfrak{M})\,. \tag{158}$$

Obviously, $\Pi_M(\mathfrak{M})$ is a one of horizontal lifts of $\Lambda_M(\mathfrak{m})$, $\mathfrak{m} = \pi(\mathfrak{M})$, such one which has a non-trivial intersection with $\mathfrak{M}$.

Let us quote the properties of Π_M:

1. Retraction/projection:
$$\Pi_M \circ \Pi_M = \Pi_M \tag{159}$$

2. If M, N - compatible co-isotropic (I-class), then:
$$\Pi_M \circ \Pi_N = \Pi_N \circ \Pi_M = \Pi_{M\cap N} \tag{160}$$

3. If
$$\Pi_M \circ \Pi_N = \Pi_N \circ \Pi_M, \tag{161}$$
then N, M - compatible and the last line equals $\Pi_{M\cap N}$.

4. If F- a mapping of $\mathcal{H}(C)$ induced by the special contact transformation f, then
$$\Pi_{f(M)} = F \circ \Pi_M \circ F^{-1} \tag{162}$$

5. Projections are "envelope-linear" (Hughens-linear)
$$\Pi_M \underset{a\in A}{\boldsymbol{E}} [T_a]\, \mathfrak{M}_a = \underset{a\in A}{\boldsymbol{E}} [T_a]\, \Pi_M \mathfrak{M}_a. \tag{163}$$

If $\{\mathfrak{M}_a :\ a \in A\} \subset \mathcal{H}(C)$ is such that the projections $\mathfrak{m}_a = \pi(\mathfrak{M}_a)$ do foliate (P,γ) ("polarization"), then

$$\mathfrak{M} = \underset{a\in A}{\boldsymbol{E}} [\mathfrak{M}_a \mid \mathfrak{M}]\, \mathfrak{M}_a \tag{164}$$

(under certain additional conditions). As $\mathfrak{m}_a \cap \mathfrak{m}_b = \emptyset$, this is an orthonormal "basis".

Let $\{\mathfrak{M}_q :\ q \in Q\}$ be such a basis and let a special contact transformation U of C be such that its symplectic projection $\underline{U}$ to P transforms $\mathfrak{m}_q$-s so that $\underline{U}(\mathfrak{m}_q)$ intersects other $\mathfrak{m}_{q'}$ - s pointwisely:

$$\underline{U}(\mathfrak{m}_q) \cap \mathfrak{m}_{q'} - \text{a one-element set.} \tag{165}$$

Then the following holds:

$$U\mathfrak{M}_q = \underset{q'\in Q}{\boldsymbol{E}} U(q',q)\, \mathfrak{M}_{q'}\ ,\ U(q',q) = [\mathfrak{M}_{q'} \mid U\mathfrak{M}_q] \tag{166}$$

and for any Lagrange manifold

$$\mathfrak{M} = \underset{q\in Q}{\boldsymbol{E}} [S(q)]\, \mathfrak{M}_q \tag{167}$$

we have:

$$U\mathfrak{M} = \underset{q\in Q}{\boldsymbol{E}}\,[S(q)]\,U\mathfrak{M}_q = \underset{q\in Q}{\boldsymbol{E}}\,[S'(q)]\,\mathfrak{M}_q. \tag{168}$$

where

$$S'(q) = \underset{q'\in Q}{\mathsf{Stat}}\left(U\left(q,q'\right) + S\left(q'\right)\right) \tag{169}$$

These are "Huygens matrix elements" - the W- type genereting function of $\underline{U}$.

Let us now consider a systems of Hamilton-Jacobi equations so, we take constraints

$$M \subset T^{*}X \quad - \quad \text{I-class submanifold}$$
$$\text{(co-isotropic of co-dimension m)}$$

Analytical description of M is as follows

$$M:\ F_a\left(x^1,\ldots,x^N;p_1,\ldots,p_N\right) = 0\ ,\ a = 1,\ldots,m \tag{170}$$

with the co-isotropy property:

$$\{F_a,F_b\}\mid M = ,\ \text{i.e.,}\ \{F_a,F_b\} = C^k{}_{ab}F_k. \tag{171}$$

The corresponding compatible system of Hamilton-Jacobi equations has the form

$$F_a\left(x^1,\ldots,x^N;\frac{\partial S}{\partial x^1},\ldots,\frac{\partial S}{\partial x^N}\right) = 0\ ,\ a = 1,\ldots,m. \tag{172}$$

This means that Lagrange and Legendre manifolds

$$\begin{aligned} \mathfrak{m}_s &:= \{dS_x:\ x\in X\}\quad;\quad p_\mu = \frac{\partial S}{\partial x^\mu} \\ \mathfrak{M}_s &:= \{(dS_x,S(x)):\ x\in X\}\quad;\quad p_\mu = \frac{\partial S}{\partial x^\mu},\ z = S(x) \end{aligned} \tag{173}$$

belong recpectively to $M\subset P$, $\pi^{-1}(M)\subset C$.

Let us consider the concept of the complete integral. We take an $(N-m)$- parameter family of solutions:

$$S\left(\ldots,x^\mu,\ldots;a^1,\ldots,a^{N-m}\right) \tag{174}$$

such that the Legendre manifolds

$$\mathfrak{M}_a := \left\{\left(dS\left(\cdot,a\right)_x, S\left(x,a\right)\right):\ x\in X\right\} \tag{175}$$

fit together so that $\bigcup_{a\in\mathbb{R}^{N-m}}\mathfrak{M}_a$ is an image of a cross-section of C over M.

At least locally there are many complete integrals. But a general solution is to depend on arbitrary functions. But in a sense $\mathfrak{M}_a$ is a "basis". The general solution is ruled by a function f of $(N-m)$ variables:

$$\begin{aligned} S\left(x;f\right) &= \underset{a\in\mathbb{R}^{N-m}}{\mathsf{Stat}}\left(S\left(x,a\right) + f\left(a\right)\right), \\ \mathfrak{M}\left(f\right) &= \underset{a\in\mathbb{R}^{N-m}}{\boldsymbol{E}}\left[f\left(a\right)\right]\mathfrak{M}_a \end{aligned} \tag{176}$$

Roughly speaking, every solution of H-J- equations may be obtained in this way by "superpositing" elements of a complete integral. One can show that those purely contact-geometric concepts follow also from $\hbar\to 0$.

6 Explanation in terms of WWMV (Weyl-Wigner-Moyal-Ville) formalism for quanta

Let us represent the operator Ain the integral form:

$$(\mathbb{A}\Psi)(q) = \int A[\bar{q}, q'] \Psi(\bar{q}') d_n \bar{q}' \tag{177}$$

Futher on, we introduce the phase space description:

$$\begin{aligned} \mathbb{A}[\bar{q}, \bar{q}'] &= \int \exp\left(\frac{\imath}{\hbar}\bar{p}\cdot(q - \bar{q}')\right) A\left(\frac{1}{2}(\bar{q} + \bar{q}'), \bar{p}\right) \frac{d_n\bar{p}}{(2\pi h)^n} \\ \mathbb{A}[\bar{q}, p] &= \int \exp\left(-\frac{\imath}{h}\bar{p}\cdot\bar{\alpha}\right) A\left[q + \frac{\bar{\alpha}}{2}, \bar{q} - \frac{\bar{\alpha}}{2}\right] d_n\bar{\alpha} \end{aligned} \tag{178}$$

Then the multiplication of operators is represented by the Weyl-Wigner-Moyal-Ville composition of functions A, B:

$$\begin{aligned} (A * B)(z) &= 2^{2n} \int \exp\left(\frac{2\imath}{\hbar}\Gamma(z - z_1, z - z_2)\right) A(z_1) B(z_2) d\mu(z_1) d\mu(z_2) \\ (\boldsymbol{A}\Psi)(q) &= \frac{1}{(2\pi h)^n} \int \exp\left(\frac{\imath}{h}p\cdot(\bar{q} - \bar{q}')\right) A\left(\frac{1}{2}(\bar{q} + \bar{q}', p)\right) \Psi(\bar{q}') d_n\bar{q}' d_n p \end{aligned} \tag{179}$$

In particular, for the phase-space distributions for pure states we obtain:

$$\rho(\bar{q}, p) = \frac{1}{(2\pi)^n} \int \Psi\left(\bar{q} - \frac{1}{2}h\tau\right) \exp(-\imath\bar{\tau}\bar{p}) \Psi\left(\bar{q} + \frac{1}{2}\hbar\bar{\tau}\right) d_n\tau \tag{180}$$

For the diagonal elements of the H^+ - algebraic "bases" describing the states localized in positions and momenta,

$$\rho_{xx}(\bar{q}, \bar{p}) = \delta(q - \bar{x}) \ , \ \rho_{\bar{y}\bar{y}}(q, p) = \delta(\bar{p} - y). \tag{181}$$

This is the special case of:

$$\begin{aligned} \rho_{\bar{q}_1, \bar{q}_2}(q, \bar{p}) &= \delta\left(q - \frac{1}{2}(\bar{q}_1 + \bar{q}_2)\right) \exp\left(\frac{\imath}{\hbar}\bar{p}\cdot(q_1 - \bar{q}_2)\right) \\ \rho_{\bar{p}_1, \bar{p}_2}(q, p) &= \delta\left(p - \frac{1}{2}(\bar{p}_1 + \bar{p}_2)\right) \exp\left(\frac{\imath}{\hbar}(\bar{p}_1 - p_2)\cdot\bar{q}\right) \end{aligned} \tag{182}$$

They had their suports on Lagrange submanifolds. It is no longer the case for the general pure states. Nevertheless, it is still true in the quasi-classical limit. Take

$$\Psi(q) = \sqrt{D(q)} \exp\left(\frac{\imath}{\hbar}S(q)\right). \tag{183}$$

Then, taking distribution-sense limit, we obtain

$$\rho_{CL}[D, S] = \lim_{h \to 0} \rho[D, S] = D(\bar{q}) \delta\left(p_1 - \frac{\partial S}{\partial q^1}\right) \cdots \delta\left(p_n - \frac{\partial S}{\partial q^n}\right) \tag{184}$$

Similarly, for the classical limit with wave functions

$$\Psi = \sqrt{D(q)} \exp\left(\frac{\imath}{\hbar}S(q)\right)$$

we obtain in the WKB-limit of the Schtödinger equation

$$\hbar i \frac{\partial \Psi}{\partial t} = \mathbb{H}\Psi$$

the following classical equation

$$\begin{aligned} \frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}, t\right) &= 0 \\ \frac{\partial D}{\partial t} + \frac{\partial}{\partial q^i} j^i &= 0 \end{aligned}$$

where

$$j^i = Dv\,[H,S]^i = D\frac{\partial H}{\partial p_i}\left(q^i, \frac{\partial S}{\partial q^i}, t\right)$$

Geometrically,

$$\frac{\partial D}{\partial t} + \mathfrak{L}_{v[H,S]} D = 0$$

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