SO(4,$\mathbb{R}$), RELATED GROUPS AND THREE-DIMENSIONAL TWO-GYROSCOPIC PROBLEMS

J.J. Sławianowski[†], B. Gołubowska[‡], E.E. Rożko[§]

Institute of Fundamental Technological Research, Polish Academy of Sciences
Pawińskiego 5B, 02-106 Warszawa, Poland

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Discussed are some problems of two (or more) mutually coupled systems with gyroscopic degrees of freedom. First of all, we mean the motion of a small gyroscope in the non-relativistic Einstein Universe $\mathbb{R} \times S^3(0, R)$; the second factor denoting the Euclidean 3-sphere of radius R in $\mathbb{R}^4$. But certain problems concerning two-gyroscopic systems in Euclidean space $\mathbb{R}^3$ are also mentioned. The special stress is laid on the relationship between various models of the configuration space like, *e.g.*, SU(2)$\times$SU(2), SO(4, $\mathbb{R}$), SO(3, $\mathbb{R}$) $\times$ SO(3, $\mathbb{R}$) *etc.* They are locally diffeomorphic, but globally different. We concentrate on classical problems, nevertheless, some quantum aspects are also mentioned.

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1. Some geometry of SU(2) and of its byproducts

When working within the realm of low-dimensional Lie groups and Lie algebras, one is often faced with various identifications or other links between them [1,2,3]. Some of those links are quite obvious, some rather not directly visible, just hidden, in any case non-expected from a perhaps naive point of view. They have no analogues in higher dimensions and it is difficult to decide if they are "accidental", or just "mysterious", "profound". What concerns the second possibility, there are speculations which resemble the anthropic principle, and namely in that perhaps the space and space-time dimensions three and four are not accidental in the "Best of All Possible Worlds" [3, 4, 5, 6].

[†] jslawian@ippt.gov.pl
[‡] bgolub@ippt.gov.pl
[§] erozko@ippt.gov.pl

The universal covering groups of $\mathrm{SO}(3,\mathbb{R}) \subset \mathrm{GL}(3,\mathbb{R})$ and $\mathrm{SO}(1,3)^{\uparrow} \subset \mathrm{GL}(4,\mathbb{R})$ are isomorphic respectively with $\mathrm{SU}(2) \subset \mathrm{GL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C}) \subset \mathrm{GL}(2,\mathbb{C})$. The prescription for the corresponding $2:1$ epimorphisms has a very natural and lucid structure. The coverings $\mathrm{Spin}(n)$ of $\mathrm{SO}(\mathrm{n},\mathbb{R})$ groups become very simple and well-known classical groups. The groups $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{GL}(2,\mathbb{R})$, $\mathrm{SO}(1,2) \subset \mathrm{GL}(3,\mathbb{R})$, $\mathrm{SU}(1,1) \subset \mathrm{GL}(2,\mathbb{C})$ have the same Lie algebras [3, 4, 5, 6]. The special pseudounitary group $\mathrm{SU}(2,2) \subset \mathrm{GL}(4,\mathbb{C})$ is isomorphic with the universal covering group of the Minkowskian conformal group $\mathrm{CO}(1,3)$. The structure of the covering epimorphism is here rather obscure in comparison with those for the groups $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{SO}(1,3)^{\uparrow}$.

The special orthogonal group in four dimensions, $\mathrm{SO}(4,\mathbb{R})$, and the Cartesian product $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$ have isomorphic Lie algebras. Incidentally, $n = 4$ is the only exceptional case among all $\mathrm{SO}(n,\mathbb{R})$ with $n > 2$ when the semisimplicity breaks down. Let us stress here an important point that, globally $\mathrm{SO}(4,\mathbb{R})$ is not the Cartesian product of two copies of $\mathrm{SO}(3,\mathbb{R})$. The situation here is more complicated. Namely, the covering group of $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$ is obviously given by $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The two-element center of $\mathrm{SU}(2)$ will be denoted by $Z_2 = \{\mathrm{I}, -\mathrm{I}\}$; I denotes the 2×2 identity matrix. Obviously, $\mathrm{SU}(2)/Z_2 = \mathrm{SO}(3,\mathbb{R})$. The four-element center of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is given by the Cartesian product

$$G = Z_2 \times Z_2 = \{(\mathrm{I},\mathrm{I}), (\mathrm{I},-\mathrm{I}), (-\mathrm{I},\mathrm{I}), (-\mathrm{I},-\mathrm{I})\} \ . \tag{1}$$

It contains three two-element subgroups, in particular,

$$H = \{(\mathrm{I},\mathrm{I}), (-\mathrm{I},-\mathrm{I})\} \ . \tag{2}$$

It is clear that

$$\left(\mathrm{SU}(2) \times \mathrm{SU}(2)\right)/G = \mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})\,, \tag{3}$$

but

$$\left(\mathrm{SU}(2) \times \mathrm{SU}(2)\right)/H = \mathrm{SO}(4,\mathbb{R})\,. \tag{4}$$

The subgroup H, is, so to speak, entangled with respect to the Cartesian-product-structure. Because of this, $\mathrm{SO}(4,\mathbb{R})$ is not globally isomorphic with $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$ nor with any Cartesian product, although their Lie algebras are both identical with that of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

There exist also "non-entangled" quotient structures, the left and right ones, given respectively by the division by groups

$$H(r) = \mathrm{I} \times Z_2\,, \qquad H(l) = Z_2 \times \mathrm{I}\,, \tag{5}$$

i.e.,

$$\begin{aligned}\left(\mathrm{SU}(2) \times \mathrm{SU}(2)\right)/H(r) &= \mathrm{SU}(2) \times \mathrm{SO}(3,\mathbb{R})\,,\\ \left(\mathrm{SU}(2) \times \mathrm{SU}(2)\right)/H(l) &= \mathrm{SO}(3,\mathbb{R}) \times \mathrm{SU}(2)\,.\end{aligned} \tag{6}$$

There are certain important kinships between various real forms of the same complex Lie group, *e.g.*, $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{U}(n)$ as matrix subgroups of $\mathrm{GL}(n,\mathbb{C})$; similarly, $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SU}(n)$ are different real forms of $\mathrm{SL}(n,\mathbb{C})$. These kinships are independent of the dimension n, and in this sense they are less mysterious. Nevertheless, for any fixed n, thus *e.g.* also for its physical value $n = 3$, taking them seriously, one might perhaps suspect that there exists some physical relationship between models of internal degrees of freedom ruled, *e.g.*, by $\mathrm{GL}(3,\mathbb{R})$ and $\mathrm{U}(3)$. One can speculate about some unifying framework provided by $\mathrm{GL}(3,\mathbb{C})$. May the three "colours" of fundamental strongly interacting particles have something to do with affinely deformable bodies in the three-dimensional space? But let us stop here with such speculations and "prophecies" which at this stage cannot be concluded; neither accepted nor rejected.

In quantum-mechanical applications of 2×2 matrices it is commonly accepted to use the Pauli matrices as basis elements. This choice is also convenient in certain problems concerning geometry of the three-dimensional rotation group.

According to the standard, historical convention

$$\begin{aligned} \sigma_0 &= I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \sigma_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ \sigma_2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, & \sigma_3 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned} \tag{7}$$

It is convenient to use the "relativistic" convention of Greek and Latin indices, $\mu = 0, 1, 2, 3$, $i = 1, 2, 3$. Roughly speaking, σ_i the "proper" Pauli matrices, are "space-like", and σ_0 is "time-like". The Latin elements σ_i are basic trace-less 2×2 matrices. A non-careful use of analytical matrix conventions may obscure the geometric meaning of symbols. For example, all second-order tensors, *i.e.*, mixed, twice contravariant, and twice covariant ones, are analytically represented by 2×2 matrices. Overlooking of this fact leads very easily to confusions and wrong, even just meaningless statements. One must be careful if matrices represent linear endomorphisms or bilinear/sesquilinear forms. Pauli matrices may represent both some basic linear mappings of $\mathbb{C}^2$ into itself, or some basic sesquilinear hermitian forms on $\mathbb{C}^2$. As linear mappings they are $\mathbb{C}$-basic in $\mathrm{L}(\mathbb{C}^2)$; as sesquilinear forms they are $\mathbb{R}$-basic in the real linear space $\mathrm{Herm}(\mathbb{C}^{2*} \otimes \mathbb{C}^{2*})$ of Hermitian forms on $\mathbb{C}^2$. As linear mappings they are also $\mathbb{R}$-basic in the real linear space of Hermitian linear mappings of $\mathbb{C}^2$ into itself, $\mathrm{Herm}(\mathbb{C}^{2*} \otimes \mathbb{C}^{2*}, \delta)$; the symbol δ denotes here the standard scalar product on $\mathbb{C}^2$,

$$\delta\,(u, v) = \delta_{ab}\overline{u^a}v^b = \sum_{a=1,2} \overline{u^a}v^a\,. \tag{8}$$

In the theory of non-relativistic and relativistic spinors, the analytical σ-matrices are used in both meanings. When dealing with the Lie algebras, it is more convenient to use another normalisation and take as basic linear mappings the τ_μ given by

$$\tau_0 = \frac{1}{2}\sigma_0 = \frac{1}{2}I_2\,, \qquad \tau_a = \frac{1}{2\imath}\sigma_a\,, \qquad a = 1,2,3\,. \tag{9}$$

This is a merely cosmetic custom, we are just used to the Levi-Civita symbol as a system of structure constants of SU(2) or SO(3, $\mathbb{R}$)

$$[\tau_a, \tau_b] = \varepsilon_{ab}{}^{c}\tau_c\,; \tag{10}$$

the shift of indices is meant here in the trivial sense of the Kronecker "delta".

In certain problems it is more convenient to use as basic linear mappings the matrices

$$\theta_\mu = \frac{1}{2\imath}\sigma_\mu\,; \tag{11}$$

obviously, $\theta_a = \tau_a$, $a = 1,2,3$, $\theta_0 = \frac{1}{2\imath}\sigma_0 = \frac{1}{2\imath}I_2 = -\imath\tau_0$.

The matrices $\tau_a = \theta_a$ are basic exp-generators of SU(2), and θ_μ are basic generators of U(2); obviously, "basic" is meant here over reals $\mathbb{R}$. And clearly τ_μ are basic (over reals) exp-generators of the group $\mathbb{R}^+\mathrm{SU}(2) = \exp(\mathbb{R})\mathrm{SU}(2)$.

It must be stressed that when the σ-matrices are interpreted as an analytic representation of linear endomorphisms, then the above "relativistic" notation is a bit artificial and misleading. The point is that the identity matrix is invariant under inner automorphisms, *i.e.*, similarity transformations

$$x \to axa^{-1}\,. \tag{12}$$

And it is just this transformation rule which applies to matrices interpreted as an analytical description of linear endomorphisms. Therefore, the $\mathbb{R}$-one-dimensional subspaces $\mathbb{R}\tau_0$, $\mathbb{R}\theta_0$ and the $\mathbb{C}$-one-dimensional subspace $\mathbb{C}\tau_0 = \mathbb{C}\theta_0$ are all invariant under the above similarity transformation and so are the corresponding one-dimensional groups. There is nothing like the "relativistic" mixing of τ_0/θ_0 with τ_a/θ_a. This mixing occurs only when the Pauli matrices are used as an analytical representation of Hermitian sesquilinear forms or their contravariant dual counterparts. Depending on their contravariant or covariant character, we have respectively the following transformation rules instead of the above similarity

$$x \to axa^{+}\,, \qquad x \to {a^{-1}}^{+}xa^{-1}\,. \tag{13}$$

And these transformation rules result in the "relativistic" mixing of σ_μ-matrices, *e.g.*

$$a\sigma_\mu a^+ = |\det a|\, \sigma_\nu L^\nu{}_\mu\,, \tag{14}$$

where L is a restricted Lorentz transformation matrix

$$\eta_{\mu\nu} = \eta_{\alpha\beta} L^\alpha{}_\mu L^\beta{}_\nu\,, \qquad [\eta_{\mu\nu}] = \operatorname{diag}(1,-1,-1,-1)\ . \tag{15}$$

It is so for any $a \in \mathrm{GL}(2,\mathbb{C})$; if a runs over the special linear group $\mathrm{SL}(2,\mathbb{C})$ ($\det a = 1$), then the above assignment $a \to L$ describes the $2:1$ universal covering of $\mathrm{SO}(1,3)^\uparrow$ by $\mathrm{SL}(2,\mathbb{C})$. In the sequel we do not deal with those "relativistic" aspects of the quadruplet of matrices σ_μ. Below we are dealing only with the SU(2) subgroup of $\mathrm{SL}(2,\mathbb{C})$, *i.e.*, with the $1 \oplus \mathrm{SO}(3,\mathbb{R})$ subgroup of $\mathrm{SO}(1,3)^\uparrow$. Nevertheless, the "relativistic" quadruplet of matrices τ_μ does occur in the exponential formula for SU(2)

$$u\left(\overline{k}\right) = \exp\left(k^a \tau_a\right) = x^\mu\left(\overline{k}\right)\left(2\tau_\mu\right)\ , \tag{16}$$

where

$$x^0 = \cos\frac{k}{2}\,, \qquad x^a = \frac{k^a}{k}\sin\frac{k}{2} = n^a \sin\frac{k}{2}\,, \qquad a = 1,2,3\,. \tag{17}$$

The quantity k, the length of the vector $\overline{k}$ runs over the range $[0,2\pi]$. The quantities k^a are known as canonical coordinates of the first kind on SU(2); in applications $\overline{k}$ is known as the rotation vector. It is meant here in the sense of the universal, thus double, covering of $\mathrm{SO}(3,\mathbb{R})$. Because of this, the range of k is doubled in comparison with the usual range $[0,\pi]$ of the rotation angle (assuming, of course, that the range of the rotation axis unit vectors $\bar{n}$ is complete). At the center $Z_2 = \{\mathrm{I}_2, -\mathrm{I}_2\}$ of SU(2) the rotation unit vector $\bar{n}$ is not well-defined. More precisely, for any unit vector $\bar{n}$ the following holds

$$u\left(\overline{O}\right) = u(O\bar{n}) = \mathrm{I}_2\,, \qquad u(2\pi\bar{n}) = -\mathrm{I}_2\,. \tag{18}$$

Any coset projecting onto a given element of $\mathrm{SO}(3,\mathbb{R})$ has the form $\{u,-u\}$. Its elements u, $-u$ are placed on a one-dimensional subgroup, *i.e.*, straight-line through $\bar{k} = 0$ (identity element I in SU(2)). They are remote by the parameter distance 2π along the mentioned straight-line in $\mathbb{R}^3$. More precisely, $-u(\bar{k}) = u(\bar{l})$, where $\left|\bar{k} - \bar{l}\right| = 2\pi$ and $\bar{k} \times \bar{l} = 0$. The covering projection from SU(2) onto $\mathrm{SO}(3,\mathbb{R})$ is given by

$$\mathrm{SU}(2) \ni v \mapsto R \in \mathrm{SO}(3,\mathbb{R})\,, \qquad \text{where} \qquad vu\left(\bar{k}\right)v^{-1} = u\left(R\bar{k}\right)\ . \tag{19}$$

Explicitly, any $v(\bar{k}) \in \mathrm{SU}(2)$ is then mapped onto $R(\bar{k}) \in \mathrm{SO}(3,\mathbb{R})$, where

$$R\left(\bar{k}\right) = \exp\left(k^a \mathcal{E}_a\right)\ , \qquad \left(\mathcal{E}_a\right)^b{}_c := -\varepsilon_a{}^b{}_c\,. \tag{20}$$

This time $\bar{k}$ as a parameter of $R(\bar{k})$ is the usual rotation vector and its magnitude runs over the range $[0, \pi]$. On the surface $k = \pi$ in $\mathbb{R}^3$ there is antipodal identification and for any unit vector $\bar{n}$ we have $R(\pi\bar{n}) = R(-\pi\bar{n})$; all such elements are square roots of the group identity, $R(\pi\bar{n})R(\pi\bar{n}) = \mathrm{I}_3$.

It is seen that the parameters x^μ in (16) are constrained by the condition

$$\left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 = 1 \tag{21}$$

in the four-dimensional linear space $\mathbb{R}$-spanned by the matrices τ_μ, *i.e.*, in $\mathbb{R}\tau_0 \otimes \mathbb{R}\tau_1 \otimes \mathbb{R}\tau_2 \otimes \mathbb{R}\tau_3$. Moreover, one can show that every point of the unit sphere (21) corresponds to exactly one point of $\mathrm{SU}(2)$. So, one tells roughly that $\mathrm{SU}(2) \simeq S^3(0,1) \subset \mathbb{R}^4$.

Let us mention also some other parametrisations of $\mathrm{SU}(2)$ and its quotient $\mathrm{SO}(3,\mathbb{R})$. One of them are spherical variables in the space of rotation vector, (k, ϑ, φ), where obviously,

$$k^1 = k \sin\vartheta \cos\varphi\,, \qquad k^2 = k \sin\vartheta \sin\varphi\,, \qquad k^3 = k\cos\vartheta\,. \tag{22}$$

Canonical coordinates of the second kind on $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$ are practically not used. This is strange, incidentally. On $\mathrm{SU}(2)$ those coordinates, (α, β, γ) are defined by

$$u\left\{\alpha, \beta, \gamma\right\} = \exp(\alpha\tau_1)\ \exp(\beta\tau_2)\ \exp(\gamma\tau_3)\,, \tag{23}$$

and similarly on $\mathrm{SO}(3,\mathbb{R})$. The popularly used Euler angles are neither first-kind nor second-kind canonical variables. They appear via the product of one-parameter subgroups, however, two of those subgroups coincide.

It was told above that $\mathrm{SU}(2)$ may be canonically identified with the unit sphere in $\mathbb{R}^4$. The metric on $\mathrm{SU}(2)$ induced from $\mathbb{R}^4$ by the restriction of the usual Euclidean metric

$$dS^2 = \left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \tag{24}$$

to that sphere $S^3(0,1)$ is proportional to the Killing metric of $\mathrm{SU}(2)$. More precisely, the Killing metric is negatively definite ($\mathrm{SU}(2)$ is compact) and equals the induced metric from $\mathbb{R}^4$ multiplied by (-2). Taking this into account one can show that the $\mathrm{SU}(2)$-metric is proportional to one underlying the arc element

$$ds^2 = dk^2 + 4\sin^2\frac{k}{2}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) = dk^2 + 4\sin^2\frac{k}{2}\ d\bar{n}\cdot d\bar{n}\,, \tag{25}$$

where $\bar{n}(\vartheta, \varphi)$ denotes the versor of $\bar{k}$ as a function of angular coordinates. Using more sophisticated terms we can say that the Killing metric is the (-2)-multiple of

$$g = dk \otimes dk + 4\sin^2\frac{k}{2}\ \delta_{AB}\ dn^A \otimes dn^B\,. \tag{26}$$

It is invariant under the left and right regular translations in SU(2)

$$\mathrm{SU}(2) \ni x \mapsto kxl \in \mathrm{SU}(2)\,, \qquad k, l \in \mathrm{SU}(2)\,. \tag{27}$$

This action preserves also the metrics of all concentric spheres $S^3(0, R) \subset \mathbb{R}^4$, and therefore, the Euclidean metric of $\mathbb{R}^4$. But it is seen that $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acting as above is not just the rotation group $\mathrm{SO}(4, \mathbb{R})$, but its $2:1$ universal covering, because Z_2 acts trivially on SU(2), and therefore also on $\mathbb{R}^4$. Indeed, $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts non-effectively and it is just Z_2 that is the center of non-effectiveness. Taking in the last formula $k = l = -\mathrm{I}_2$, we obtain the identity transformation of spheres $S^3(0, R)$. This is just the root of the global distinction between $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{SO}(4, \mathbb{R})$ (4). But their Lie algebras are isomorphic with each other and with the Lie algebra of $\mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R})$

$$\mathrm{SU}(2)' \times \mathrm{SU}(2)' = \mathrm{SO}(4, \mathbb{R})' = \mathrm{SO}(3, \mathbb{R})' \times \mathrm{SO}(3, \mathbb{R})'\,. \tag{28}$$

To see this we should use the standard basis of $\mathrm{SO}(4, \mathbb{R})'$

$$\begin{aligned}
M_1 &= \mathcal{E}^{32} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad M_2 = \mathcal{E}^{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \\
M_3 &= \mathcal{E}^{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_1 = \mathcal{E}^{01} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
N_2 &= \mathcal{E}^{02} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_3 = \mathcal{E}^{03} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{29}$$

and replace it by the system of linear combinations

$$X_i = \frac{1}{2}\left(M_i + N_i\right)\,, \qquad Y_i = \frac{1}{2}\left(M_i - N_i\right)\,, \qquad i = 1, 2, 3\,. \tag{30}$$

It is seen that

$$[X_i, X_j] = \varepsilon_{ij}{}^{k} X_k\,, \qquad [Y_i, Y_j] = \varepsilon_{ij}{}^{k} Y_k\,, \qquad [X_i, Y_j] = o\,, \tag{31}$$

i.e., one obtains a pair of independent relations (10). This fact enables one to reduce the problem of finding the unitary irreducible representations of $\mathrm{SO}(4, \mathbb{R})$ and of its universal covering, to operating on representations of $\mathrm{SO}(3,\mathbb{R})$ and of its covering $\mathrm{SU}(2)$. Using appropriate complexification procedure, one constructs irreducible representations of the Lorentz group $\mathrm{SO}(1,3)^{\uparrow}$ and of its covering $\mathrm{SL}(2,\mathbb{C})$ from unitary irreducible representations of $\mathrm{SU}(2)$.

The Killing metric (25)/(26) on $\mathrm{SU}(2)$ is invariant under the action (27) of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ through $\mathrm{SO}(4,\mathbb{R})$. And so is its contravariant inverse

$$g^{-1}=\frac{\partial}{\partial k}\otimes\frac{\partial}{\partial k}+\frac{1}{4\sin^2\frac{k}{2}}\,\delta^{AB}D_A\otimes D_B\,, \tag{32}$$

where the contravariant vectors D_A are identical with the generators of inner automorphisms in $\mathrm{SU}(2)$

$$u\mapsto vuv^{-1}\,, \tag{33}$$

therefore

$$D_A=\varepsilon_{AB}{}^{C}k^B\frac{\partial}{\partial k^C}\,. \tag{34}$$

The following expressions correspond in a suggestive way to the usual duality rules between basic vector and covector fields

$$\begin{aligned}\left\langle dk,\frac{\partial}{\partial k}\right\rangle &= 1\,, \qquad \langle dk,D_A\rangle=0\,,\\ \left\langle dn^A,\frac{\partial}{\partial k}\right\rangle &= 0\,, \qquad \left\langle dn^A,D_B\right\rangle=\varepsilon^{A}{}_{BC}n^C\,.\end{aligned} \tag{35}$$

The usual coordinate expression for g^{-1} reads as follows

$$g^{ij}=\frac{k^2}{4\sin^2\frac{k}{2}}\,\delta^{ij}+\left(1-\frac{k^2}{4\sin^2\frac{k}{2}}\right)n^in^j\,. \tag{36}$$

Indeed,

$$g^{ik}g_{kj}=\delta^i{}_j\,, \tag{37}$$

where g_{ij} are the usual covariant components of (25)

$$g_{ij}=\frac{4}{k^2}\sin^2\frac{k}{2}\,\delta_{ij}+\left(1-\frac{4}{k^2}\sin^2\frac{k}{2}\right)n_in_j\,. \tag{38}$$

In all formulas n^i are $\mathbb{R}^3$-components of the unit radius-vector $\bar{n}=\bar{k}/k$ and the shift of its index is meant in the Kronecker-delta sense.

It is important to quote expressions for the basic vector fields ${}^{\mathrm{l}}E_A$, ${}^{\mathrm{r}}E_A$ generating respectively the left and right regular translations (27) on SU(2). They are respectively the basic right- and left-invariant vector fields on SU(2). One can show that they are given by

$$\begin{aligned} {}^{\mathrm{l}}E_A &= n_A\frac{\partial}{\partial k} - \frac{1}{2}\cot\frac{k}{2}\varepsilon_{ABC}n^B D^C + \frac{1}{2}D_A\,, \\ {}^{\mathrm{r}}E_A &= n_A\frac{\partial}{\partial k} - \frac{1}{2}\cot\frac{k}{2}\varepsilon_{ABC}n^B D^C - \frac{1}{2}D_A\,, \end{aligned} \tag{39}$$

so that the following holds

$${}^{\mathrm{l}}E_A - {}^{\mathrm{r}}E_A = D_A\,. \tag{40}$$

They satisfy the following structure commutation rules

$$\begin{aligned} \left[{}^{\mathrm{l}}E_A, {}^{\mathrm{l}}E_B\right] &= -\varepsilon_{AB}{}^C\,{}^{\mathrm{l}}E_C\,, & \left[{}^{\mathrm{r}}E_A, {}^{\mathrm{r}}E_B\right] &= \varepsilon_{AB}{}^C\,{}^{\mathrm{r}}E_C\,, \\ \left[{}^{\mathrm{l}}E_A, {}^{\mathrm{r}}E_B\right] &= 0\,, & \left[D_A, D_B\right] &= -\varepsilon_{AB}{}^C\,D_C\,. \end{aligned} \tag{41}$$

The corresponding dual Maurer–Cartan forms ${}^{\mathrm{l}}E^A$, ${}^{\mathrm{l}}E^B$, defined by

$$\left\langle {}^{\mathrm{l}}E^A, {}^{\mathrm{l}}E_B\right\rangle = \delta^A{}_B\,, \qquad \left\langle {}^{\mathrm{r}}E^A, {}^{\mathrm{r}}E_B\right\rangle = \delta^A{}_B \tag{42}$$

are given by the following expressions

$$\begin{aligned} {}^{\mathrm{l}}E^A &= n^A dk + 2\sin^2\frac{k}{2}\varepsilon^{ABC}n_B dn_C + \sin k\; dn^A\,, \\ {}^{\mathrm{r}}E^A &= n^A dk - 2\sin^2\frac{k}{2}\varepsilon^{ABC}n_B dn_C + \sin k\; dn^A\,. \end{aligned} \tag{43}$$

The Killing metric field (divided by (-2)) may be expressed as

$$g = \delta_{AB}{}^{\mathrm{l}}E^A \otimes {}^{\mathrm{l}}E^B = \delta_{AB}{}^{\mathrm{r}}E^A \otimes {}^{\mathrm{r}}E^B\,, \tag{44}$$

and its contravariant inverse is given by

$$g^{-1} = \delta^{AB\mathrm{l}}E_A \otimes {}^{\mathrm{l}}E_B = \delta^{AB\mathrm{r}}E_A \otimes {}^{\mathrm{r}}E_B\,. \tag{45}$$

There is a good way of visualising the global distinction between groups $\mathrm{SU}(2)\times\mathrm{SU}(2)$, $\mathrm{SO}(4,\mathbb{R}) \approx (\mathrm{SU}(2)\times\mathrm{SU}(2))/H$, $\mathrm{SU}(2)\times\mathrm{SO}(3,\mathbb{R}) \approx (\mathrm{SU}(2)\times\mathrm{SU}(2))/H(r)$, $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SU}(2) \approx (\mathrm{SU}(2)\times\mathrm{SU}(2))/H(l)$, and finally $\mathrm{SO}(3,\mathbb{R})\times\mathrm{SO}(3,\mathbb{R}) \approx (\mathrm{SU}(2)\times\mathrm{SU}(2))/G = \mathrm{SU}(2)/Z_2\times\mathrm{SU}(2)/Z_2$. It is so to speak a quantum-mechanical way of starting from the simply-connected group $\mathrm{SU}(2)\times\mathrm{SU}(2)$, using the Peter–Weyl theorem about expanding functions into power-series of matrix elements of unireps, and then restricting the function space by conditions imposed on the expansion coefficients [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Those conditions are equivalent to the division procedures quoted above. In a sense, this resembles the Sikorski language of differential spaces. Namely, any function on $\mathrm{SU}(2)$ and on $\mathrm{SU}(2)\times\mathrm{SU}(2)$ may be expanded as follows

$$\Psi(u) = \sum_{jmk} c^{j}{}_{km} D^{j}{}_{mk}(u) = \sum_{j} \mathrm{Tr}\left(c^{j} D^{j}(u)\right)\,, \tag{46}$$

$$\Psi(u,v) = \sum_{\substack{ls\\ mk\ rn}} c^{l}{}_{km}{}^{s}{}_{nr} D^{l}{}_{mk}(u) D^{s}{}_{rn}(v)\,. \tag{47}$$

Here, summation over j, l, s is extended over all non-negative integers and half-integers, and for any fixed values of l, s, the quantities k, m and n, r run over all integers or half-integers respectively from $-l$ to l and from $-s$ to s, jumping by one. Clearly, D^j are $(2j+1)\times(2j+1)$-matrices of unitary irreducible representations of $\mathrm{SU}(2)$. On $\mathrm{SU}(2)$ and $\mathrm{SU}(2)\times\mathrm{SU}(2)$ the C-coefficients are arbitrary and restricted only by the demand that the function series do converge. To obtain functions on $\mathrm{SO}(4,\mathbb{R}) = (\mathrm{SU}(2)\times\mathrm{SU}(2))/H$ one must assume that in (47) the C-coefficients do vanish when s, j have different "halfness", *i.e.*, when $2s$, $2j$ have a different parity. More precisely, $2s$ and $2j$ in (47) must be simultaneously even or simultaneously odd. To obtain a general function on $\mathrm{SU}(2)\times\mathrm{SO}(3,\mathbb{R}) \approx (\mathrm{SU}(2)\times\mathrm{SU}(2))/H(r)$ one must forbid in (47) the half-integer j. Similarly, on $\mathrm{SO}(3,\mathbb{R})\times\mathrm{SU}(2) \approx (\mathrm{SU}(2)\times\mathrm{SU}(2))/H(l)$ one must assume that C do vanish for half-integer values of s. And finally, on $\mathrm{SO}(3,\mathbb{R})\times\mathrm{SO}(3,\mathbb{R})$ the half-integer values of s and j are excluded.

Obviously, the group representation property

$$D^{j}(u_1 u_2) = D^{j}(u_1) D^{j}(u_2)\,, \qquad D^{j}(\mathrm{I}_2) = \mathrm{I}_{2j+1} \tag{48}$$

implies that

$$D^{j}\left(u\left(\bar{k}\right)\right) = \exp\left(\frac{i}{\hbar}\, k^{a} \mathcal{S}^{j}{}_{a}\right)\,, \tag{49}$$

where $\mathcal{S}^{j}{}_{a}$ are the $(2j+1)\times(2j+1)$ matrices of the j-th angular momentum. All of them satisfy the Poisson rule

$$\frac{1}{\hbar i}\left[\mathcal{S}_a, \mathcal{S}_b\right] = \varepsilon_{ab}{}^{c} \mathcal{S}_c\,. \tag{50}$$

Therefore, on the infinitesimal level

$$\begin{aligned} \frac{\hbar}{i}\,{}^{\mathrm{l}}E_A D^j &= \mathcal{S}^j{}_A D^j\,, \qquad \frac{\hbar}{i}\,{}^{\mathrm{r}}E_A D^j = D^j \mathcal{S}^j{}_A\,, \\ \frac{\hbar}{i} D_A D^j &= \left[\mathcal{S}^j{}_A, D^j\right]\,, \end{aligned} \tag{51}$$

and the Casimir rule holds

$$-\hbar^2 \sum_A {}^{\mathrm{l}}E_A{}^{\mathrm{l}}E_A D^j = -\hbar^2 \sum_A {}^{\mathrm{r}}E_A{}^{\mathrm{r}}E_A D^j = \hbar^2 j(j+1) D^j. \tag{52}$$

All these rules hold in SU(2) and $\mathrm{SO}(3,\mathbb{R})$ and the difference concerns only the range of angular variables and the spectra of operators (half-integer and integer in SU(2) and only integer in $\mathrm{SO}(3,\mathbb{R})$).

2. Small rigid body in Einstein universe

Now let us turn to mechanical interpretation. $\mathrm{SO}(3,\mathbb{R})$ is the configuration space of a rigid body with the non-moving center of mass. The Cartesian product $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$ may be interpreted as the configuration space of a pair of such bodies. But it is clear that the covering spaces SU(2), $\mathrm{SU}(2) \times \mathrm{SU}(2)$ may be also interpreted in such terms, especially in quantum problems of small bodies, first of all in some attempts of describing internal degrees of freedom. The same concerns all models based on the quotient groups (3)–(6). All of them are locally isomorphic with $\mathrm{SO}(4,\mathbb{R})$ or $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$, but of course the global differences between them may be quite astonished and worth to be investigated.

We are here interested in some particular, slightly different problem of motion of a small rigid body in the spherical Einstein world. It is clear that the Einstein universe, *i.e.*, three-dimensional sphere of radius R in $\mathbb{R}^4$, $S^3(0,R) \subset \mathbb{R}^4$ is diffeomorphic with SU(2) and has the isometry group $\mathrm{SO}(4,\mathbb{R}) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/H$. To be more precise, Einstein universe is the four-dimensional space-time manifold metrically diffeomorphic with $\mathbb{R} \times S^3(0,R)$. We mean the empty, matter-free and non-relativistic space-time and often identify it simply with the spatial factor $S^3(0,R)$, sphere of radius R in $\mathbb{R}^4$. And infinitesimal gyroscope moving translationally in $S^3(0,R)$ has in addition the internal configuration space ruled by the group $\mathrm{SO}(3,\mathbb{R})$. This gives us together the configuration space $\mathrm{SU}(2) \times \mathrm{SO}(3,\mathbb{R})$, *i.e.*, (6). Taking its covering space, *e.g.*, in quantum models, one obtains $\mathrm{SU}(2)\times\mathrm{SU}(2)$, *i.e.*, some kind of kinematical resonance between translational and internal degrees of freedom.

A few more general remarks are necessary, or at least mostly welcome here. A "small" gyroscope moving in a Riemann space (M, g) is described by curves $\gamma : \mathbb{R} \to F(M, g)$ in the principal fibre bundle $F(M, g)$ of g-orthonormal linear frames in M. The point $x \in M$ describes the instantaneous position of the body in M, and the orthonormal frame $e = (\ldots, e_A, \ldots)$ at x represents the instantaneous orientation of co-moving axes frozen into the body. The vectors e_A are mutually orthogonal and normalised elements of $T_x M$, the tangent space at $x \in M$. More precisely, at least in the classical theory, instead $F(M, g)$ one must use one of its connected components. If x^i are coordinates in M and $e^i{}_A$ are the corresponding components of vectors e_A, then the following holds

$$g_x\left(e_A, e_B\right) = g(x)_{ij}\ e^i{}_A e^j{}_B = \delta_{AB}\,. \tag{53}$$

Generalised velocity along the curve $\gamma : \mathbb{R} \to F(M, g)$ has in the manifold $F(M)$ of all (not necessarily orthonormal) frames in M the components

$$\left(\frac{dx^i}{dt}, \frac{d}{dt}\ e^i{}_A\right). \tag{54}$$

Clearly, unlike $\frac{dx^i}{dt}$, $\frac{d}{dt}\ e^i{}_A$ are not tensor components in M, and because of this it is better to use the covariant internal velocities

$$\boldsymbol{V}^i{}_A = \frac{D}{Dt}\ e^i{}_A = \frac{d}{dt}\ e^i{}_A + \Gamma^i{}_{jk}\ e^j{}_A \frac{dx^k}{dt}\,, \tag{55}$$

where $\Gamma^i{}_{jk}$ are components of the Levi-Civita affine connection built of g_{ij}. By analogy to extended rigid body in a flat space we have the following expressions for the kinetic energy [13, 14, 15, 16, 17, 18, 19]

$$\begin{aligned} T &= T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2}\ g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{1}{2}\ g_{ij}\left(\frac{D}{Dt}\ e^i{}_A\right)\left(\frac{D}{Dt}\ e^j{}_B\right) J^{AB} \\ &= \frac{m}{2}\ g_{ij} v^i v^j + \frac{1}{2}\ \delta_{KL}\widehat{\Omega}^K{}_A \widehat{\Omega}^L{}_B J^{AB} \\ &= \frac{m}{2}\ \delta_{AB}\hat{v}^A \hat{v}^B + \frac{1}{2}\ \delta_{KL}\widehat{\Omega}^K{}_A \widehat{\Omega}^L{}_B J^{AB}\,. \end{aligned} \tag{56}$$

The meaning of symbols used here is as follows

$$\Omega^i{}_j = \left(\frac{D}{Dt}\ e^i{}_A\right) e^A{}_j\,, \qquad \widehat{\Omega}^A{}_B = e^A{}_i \frac{D}{Dt}\ e^i{}_B = e^A{}_i\ \Omega^i{}_j\ e^j{}_B \tag{57}$$

are spatial and co-moving components of angular velocity, e^A is the dual co-basis of e_A, and $\hat{v}^A = e^A{}_i v^i = e^A{}_i \frac{dx^i}{dt}$ are co-moving components of translational velocity. The quantities J^{AB} are co-moving, thus constant, components of the internal tensor, or more precisely — of the quadrupole momentum of the mass distribution within the body. Obviously, the angular velocity in both representations, and translational co-moving velocity $\hat{v}^A$ are non-holonomic velocities. Angular velocity is skew-symmetric in the metrical sense

$$\Omega^i{}_j = -g_{jk}\Omega^k{}_l g^{li} = -\Omega_j{}^i\,, \qquad \widehat{\Omega}^A{}_B = -\delta_{BC}\delta^{AD}\widehat{\Omega}^C{}_D = -\widehat{\Omega}_B{}^A\,. \tag{58}$$

There are only $\frac{1}{2}n\,(n-1)$ independent components of $\Omega^i{}_j$ and so for $\widehat{\Omega}^A{}_B$ and obviously so for n^2 components of $e^i{}_A$ constrained by $\frac{1}{2}n\,(n+1)$ conditions (53). Therefore, certainly $e^i{}_A$ are not independent generalised coordinates of gyroscopic motion. In our opinion, the most convenient way of introducing generalised coordinates of gyroscopic motion consists in using some non-holonomic reference frame in M, $(\ldots, E_A, \ldots)$ and expressing each moving orthonormal gyroscopic basis $(\ldots, e_A, \ldots)$ in terms of E

$$e_A\,(x(t)) = E_B\,(x(t))\,L^B{}_A(t)\,. \tag{59}$$

Here $\left[L^B{}_A\right]$ is an orthogonal $n\times n$ matrix parameterised in terms of some fixed coordinates in $\mathrm{SO}(n,\mathbb{R})$, *e.g.*, the skew-symmetric tensor, bivector, of canonical coordinates of first kind. The peculiarity of dimension $n=3$ is that the angular velocities and bivectors of canonical coordinates may be identified with axial pseudovectors.

If M is an n-dimensional semisimple Lie group with the Killing metric g, then $F(M,g)$, or rather its connected component may be canonically identified with the Cartesian product $G\times\mathrm{SO}(n,\mathbb{R})$. Any choice of the Killing-orthonormal basis $(\ldots, E_A, \ldots)$ in the Lie algebra G' gives rise to two such canonical identifications. Namely, $(\ldots, E_A, \ldots)$ may be extended to the global right- and left-invariant orthonormal systems of vector fields $(\ldots, {}^{\mathrm{l}}E_A, \ldots)$, $(\ldots, {}^{\mathrm{r}}E_A, \ldots)$ as described above. They generate the left and right regular translations in G. And then, at any point $x\in M$, any orthonormal frame $e=(\ldots, e_A, \ldots)$, $e_A\in T_xM$, may be expressed as follows

$$e_A = {}^{\mathrm{l}}E_{Bx}\,{}^{\mathrm{l}}L^B{}_A\,, \qquad e_A = {}^{\mathrm{r}}E_{Bx}\,{}^{\mathrm{r}}L^B{}_A\,, \tag{60}$$

${}^{\mathrm{l}}L, {}^{\mathrm{r}}L\in\mathrm{SO}(n,\mathbb{R})$. It is only a matter of convention if we choose the "left" or "right" representation. With both conventions, any $e\in F(G,g)$ is represented by a pair of independent labels, $x\in G$, $L\in\mathrm{SO}(n,\mathbb{R})$. In this way, $F(G,g)$ becomes the Cartesian product $G\times\mathrm{SO}(n,\mathbb{R})$.

We are dealing in this paper with the special case when $M=G=\mathrm{SU}(2)\simeq S^3(0,1)\simeq S^3(0,R)\subset\mathbb{R}^4$. Therefore, $F(M,g)$ becomes $M\times\mathrm{SO}(3,\mathbb{R})$. In any case, for any fixed "radius of Universe" R, there exists some

R-dependent identification of our configuration space with $\mathrm{SU}(2)\times\mathrm{SO}(3,\mathbb{R})$. It is not simply connected; its $2:1$ universal covering is $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Now, it becomes clear what is meant by the relationship with the two-gyroscopic system in Euclidean space $\mathbb{R}^3$.

Kinetic energy is given by (56); moreover, we assume the simplified version of the spherical rigid body, when

$$J^{AB} = \mathrm{I}\delta^{AB}\,, \tag{61}$$

I denoting the scalar moment of inertia.

In a general differential manifold M the angular velocity (57) splits under the representation (59) as follows

$$\widehat{\varOmega}^{A}{}_{B} = \widehat{\varOmega}(\mathrm{rl})^{A}{}_{B} + \widehat{\varOmega}(\mathrm{dr})^{A}{}_{B}\,, \tag{62}$$

where $\widehat{\varOmega}(\mathrm{rl})$, $\widehat{\varOmega}(\mathrm{dr})$ denote respectively the "relative", *i.e.*, internal angular velocity, and the "drive" term in the sense of representation through the fixed reference field E. They are given by

$$\widehat{\varOmega}(\mathrm{rl})^{A}{}_{B} = L^{-1A}{}_{C}\frac{d}{dt}\,L^{C}{}_{B} = L^{-1A}{}_{C}\;\varOmega(\mathrm{rl})^{C}{}_{D}\;L^{D}{}_{B}\,, \tag{63}$$

where we use the symbols:

$$\varOmega(\mathrm{rl})^{C}{}_{D} = \frac{dL^{C}{}_{E}}{dt}L^{-1E}{}_{D}\,, \tag{64}$$

and

$$\widehat{\varOmega}(\mathrm{dr})^{A}{}_{B} = L^{-1A}{}_{K}\,\varGamma^{K}{}_{LM}\,L^{L}{}_{B}\,L^{M}{}_{N}\,\hat{v}^{N}\,. \tag{65}$$

In the last formula $\hat{v}^N$ are co-moving components of translational velocity,

$$\hat{v}^{N} = e^{N}{}_{i}\,v^{i}\,, \tag{66}$$

and $\varGamma^{K}{}_{LM}$ are E-nonholonomic components of the Levi-Civita connection $\varGamma^{i}{}_{jk}$ built of the metric g, thus

$$\begin{aligned}
\varGamma^{A}{}_{BC} &= E^{A}{}_{i}\left(\varGamma^{i}{}_{jk} - \varGamma_{\mathrm{tel}}(E)^{i}{}_{jk}\right)E^{j}{}_{B}\,E^{k}{}_{C}\,,\\
\varGamma_{\mathrm{tel}}(E)^{i}{}_{jk} &= E^{i}{}_{A}\,\frac{\partial}{\partial x^{k}}\,E^{A}{}_{j}\,,\\
\varGamma^{i}{}_{jk} &= \tfrac{1}{2}\,g^{im}\left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right)\,.
\end{aligned} \tag{67}$$

One can easily show that, in the special case we are interested in, namely, when $M = G = \mathrm{SU}(2) \simeq \mathrm{SO}(3,\mathbb{R}) \subset \mathbb{R}^4$, the non-holonomic coefficients of the Killing–Levi-Civita connection are given by

$$\varGamma^{A}{}_{BC} = -\tfrac{1}{2}\,\varepsilon^{A}{}_{BC}\,. \tag{68}$$

When one copy of SU(2) in SU(2) × SU(2) is to be interpreted as a manifold of translational positions, the Einstein sphere of radius R, then it is more convenient to use the rescaled variables $\bar{r}$ instead $\bar{k}$. Namely, the both versors $\bar{r}/r$, $\bar{k}/k$ are to be identical, but the new length r will be given by

$$r = R\,k/2\,. \tag{69}$$

Then, at the antipole/"South Pole" $k = 2\pi$ we have $r = R\pi$ and the total around length of the meridian from the North Pole via South Pole back to the North Pole equals $2\pi R$, just as it should be on the R-sphere. The Killing arc element is then renormalised as

$$\begin{aligned} ds^2 &= dr^2 + R^2 \sin^2 \frac{r}{R} \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2\right) \\ &= dr^2 + R^2 \sin^2 \frac{r}{R}\; d\bar{n} \cdot d\bar{n}\,. \end{aligned} \tag{70}$$

More geometrically, the underlying metric tensor g is then expressed as

$$\begin{aligned} g(R) &= dr \otimes dr + R^2 \sin^2 \frac{r}{R} \delta_{AB} dn^A \otimes dn^B \\ &= \delta_{AB}\, {}^{\mathrm{l}}E(R)^A \otimes {}^{\mathrm{l}}E(R)^B = \delta_{AB}\, {}^{\mathrm{r}}E(R)^A \otimes {}^{\mathrm{r}}E(R)^B\,, \end{aligned} \tag{71}$$

where ${}^{\mathrm{l}}E(R)^A$, ${}^{\mathrm{r}}E(R)^A$ denote renormalised co-bases and ${}^{\mathrm{l}}E(R)_A$, ${}^{\mathrm{r}}E(R)_A$ are their dual bases. Therefore, they are given by

$$\begin{aligned} {}^{\mathrm{l}}E(R)^A &= n^A dr + R \sin^2 \frac{r}{R} \varepsilon^A{}_{BC} n^B dn^C + \frac{R}{2} \sin \frac{2r}{R} dn^A\,, \\ {}^{\mathrm{r}}E(R)^A &= n^A dr - R \sin^2 \frac{r}{R} \varepsilon^A{}_{BC} n^B dn^C + \frac{R}{2} \sin \frac{2r}{R} dn^A\,, \\ {}^{\mathrm{l}}E(R)_A &= n^A \frac{\partial}{\partial r} - \frac{1}{R} \cot \frac{r}{R}\; \varepsilon_{ABC} n^B D^C + \frac{1}{R} D^A\,, \\ {}^{\mathrm{r}}E(R)_A &= n^A \frac{\partial}{\partial r} - \frac{1}{R} \cot \frac{r}{R}\; \varepsilon_{ABC} n^B D^C - \frac{1}{R} D^A\,, \end{aligned} \tag{72}$$

and, obviously,

$$D_A = {}^{\mathrm{l}}E_A - {}^{\mathrm{r}}E_A = \varepsilon_{AB}{}^C r^B \frac{\partial}{\partial r^C}\,. \tag{73}$$

Tensor indices in all those expressions are raised and lowered with the use of "Kronecker delta".

Let us notice that the formulas (72) may be written in the following index-free form, when systems of co-vectors and vectors are represented as three-dimensional $\mathbb{R}^3$-vectors

$$
\begin{aligned}
{}^{\mathrm{l}}\overline{E} &= \bar{n}\frac{\partial}{\partial r} - \frac{1}{R}\cot\frac{r}{R}\bar{n}\times\overline{D} + \frac{1}{2}\overline{D}\,,\\
{}^{\mathrm{r}}\overline{E} &= \bar{n}\frac{\partial}{\partial r} - \frac{1}{R}\cot\frac{r}{R}\bar{n}\times\overline{D} - \frac{1}{2}\overline{D}\,,\\
{}^{\mathrm{l}}\underline{E} &= \bar{n}dr + R\sin^2\frac{r}{R}\bar{n}\times d\bar{n} + \frac{R}{2}\sin\frac{2r}{R}\bar{n}\,,\\
{}^{\mathrm{r}}\underline{E} &= \bar{n}dr - R\sin^2\frac{r}{R}\bar{n}\times d\bar{n} + \frac{R}{2}\sin\frac{2r}{R}\bar{n}\,.
\end{aligned}
\tag{74}
$$

Here ${}^{\mathrm{l}}\overline{E}$, ${}^{\mathrm{r}}\overline{E}$ denote the systems of "vectors", and ${}^{\mathrm{l}}\underline{E}$, ${}^{\mathrm{r}}\underline{E}$ are "co-vectors", both with the A-indices.

Let us stress that all those analytical expressions may be geometrically interpreted in such a way that the unit SU(2)-sphere in $\mathbb{R}^4$ is submitted to the dilatation extending its radius to R. The resulting metric (70), (71) is then obtained as a pull-back of the Euclidean $\mathbb{R}^4$-metric to the injected sphere submanifold $S^3(0,R)$. Parametrising this sphere by coordinates (r,ϑ,φ) given by

$$
\begin{aligned}
x^1 &= R\sin\frac{r}{R}\sin\vartheta\cos\varphi\,, \qquad & x^2 = R\sin\frac{r}{R}\sin\vartheta\sin\varphi\,,\\
x^3 &= R\sin\frac{r}{R}\cos\vartheta\,, & x^4 = R\cos\frac{r}{R}\,,
\end{aligned}
\tag{75}
$$

and substituting this to the $\mathbb{R}^4$ Euclidean metric

$$
dS^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2\,,
\tag{76}
$$

one obtains just (70), (71). The vector fields ${}^{\mathrm{l}}E(R)$, ${}^{\mathrm{r}}E(R)$ are Killing vectors of the isometry group SO$(4,\mathbb{R})$ of $S^3(0,R)$.

It is clear that the R-gauged vector fields ${}^{\mathrm{l}}E(R)$, ${}^{\mathrm{r}}E(R)$ satisfy the following commutation rules

$$
\begin{aligned}
\left[{}^{\mathrm{l}}E(R)_A, {}^{\mathrm{l}}E(R)_B\right] &= -\frac{2}{R}\,\varepsilon_{AB}{}^{C}\,{}^{\mathrm{l}}E(R)_C\,,\\
\left[{}^{\mathrm{r}}E(R)_A, {}^{\mathrm{r}}E(R)_B\right] &= \frac{2}{R}\,\varepsilon_{AB}{}^{C}\,{}^{\mathrm{r}}E(R)_C\,,\\
\left[{}^{\mathrm{r}}E(R)_A, {}^{\mathrm{r}}E(R)_B\right] &= 0\,.
\end{aligned}
\tag{77}
$$

In the limit $R\to\infty$ these commutators do vanish, and both ${}^{\mathrm{l}}E(R)$, ${}^{\mathrm{r}}E(R)$ become $\bar{r}$-translation operators,

$$
\lim_{R\to\infty}{}^{\mathrm{l}}E(R)_A = \lim_{R\to\infty}{}^{\mathrm{r}}E(R)_A = \frac{\partial}{\partial x^A}\,,
\tag{78}
$$

and dually,

$$\lim_{R\to\infty} {}^{\mathrm{l}}E(R)^{A} = \lim_{R\to\infty} {}^{\mathrm{r}}E(R)^{A} = dx^{A}\,, \tag{79}$$

at any fixed value of $\bar{r}$. And similarly, for any fixed $\bar{r}$ we have

$$\lim_{R\to\infty} g(R)_{ij} = \delta_{ij}\,. \tag{80}$$

On the fibers of $F(M,g)$, all identified with $\mathrm{SO}(3,\mathbb{R})$, or with its covering $\mathrm{SU}(2)$ (the identification based on the fixed choice of ${}^{\mathrm{l}}E(R)$ and ${}^{\mathrm{r}}E(R)$), one introduces the canonical coordinates $\bar{\varkappa}$ (analogue of $\bar{k}$ in (39)) and the right/left-invariant vector fields ${}^{\mathrm{l}}E_{A}$ and ${}^{\mathrm{r}}E_{A}$ and their dual forms ${}^{\mathrm{l}}E^{A}$ and ${}^{\mathrm{r}}E^{A}$. They are given just by (39) with $\bar{k}$ replaced by $\bar{\varkappa}$. The six components of $\bar{k}$, $\bar{\varkappa}$ are our generalised coordinates, respectively translational in $M \simeq \mathrm{SU}(2) \simeq S^{3}(0,R)$, and internal, *i.e.* rotational in $\mathrm{SU}(2)$ or $\mathrm{SO}(3,\mathbb{R})$. In three dimensions the tensors of angular velocities, both in spatial and co-moving representation, are represented by axial vectors, *i.e.*,

$$\varOmega_{\mathrm{tr}}(R)^{D} = {}^{\mathrm{l}}E^{D}{}_{i}(R,\bar{r})\frac{dr^{i}}{dt}\,, \qquad \varOmega^{A}_{\mathrm{int}} = {}^{\mathrm{l}}E^{A}{}_{i}(\bar{\varkappa})\frac{d\varkappa^{i}}{dt}\,. \tag{81}$$

Strictly speaking, the first of those expressions gives us the translational velocity in M, which however, may be in three dimensions interpreted as a kind of angular velocity. In expressions for the covector fields we have indicated the independent variables symbols. Similarly, using the co-moving representations we would have

$$\widehat{\varOmega}_{\mathrm{tr}}(R)^{D} = {}^{\mathrm{r}}E^{D}{}_{j}(R,\bar{r})\frac{dr^{j}}{dt}\,, \qquad \widehat{\varOmega}^{D}_{\mathrm{int}} = {}^{\mathrm{r}}E^{D}{}_{j}(\bar{\varkappa})\frac{d\varkappa^{j}}{dt}\,. \tag{82}$$

Combining the formula (56) with (81), (82), (72), (39), (64), (65) we obtain after some relatively complicated but in principle simple calculations the following expression for the total kinetic energy

$$\begin{aligned} T \;=\; & \frac{1}{2}\left(m + \frac{\mathrm{I}}{R^{2}}\right)\delta_{AB}\varOmega_{\mathrm{tr}}(R)^{A}\varOmega_{\mathrm{tr}}(R)^{B} - \frac{\mathrm{I}}{R}\,\delta_{AB}\varOmega^{A}_{\mathrm{int}}\varOmega_{\mathrm{tr}}(R)^{B} \\ & + \frac{\mathrm{I}}{2}\,\delta_{AB}\varOmega^{A}_{\mathrm{int}}\varOmega^{B}_{\mathrm{int}}\,. \end{aligned} \tag{83}$$

This is a geodetic Lagrangian. For potential systems without magnetic field, Lagrangian has the shape

$$L = T - V(\bar{r},\bar{\varkappa})\,; \tag{84}$$

when the magnetic fields is present, there are also terms linear in generalised velocities $\frac{d\bar{r}}{dt}$, $\frac{d\bar{\varkappa}}{dt}$.

For Lagrangians (84) the Legendre transformation may be easily expressed in non-holonomic terms as follows

$$\begin{aligned} S_{\mathrm{tr}}(R)_A &= \frac{\partial T}{\partial \varOmega_{\mathrm{tr}}(R)^A} = {}^{\mathrm{l}}E^i{}_A(R,\bar{r})p_i\,, \\ S_{\mathrm{int}\ A} &= \frac{\partial T}{\partial \varOmega^A_{\mathrm{int}}} = {}^{\mathrm{l}}E^i{}_A(\bar{\varkappa})\pi_i\,, \end{aligned} \tag{85}$$

or, equivalently,

$$\begin{aligned} \widehat{S}_{\mathrm{tr}}(R)_A &= \frac{\partial T}{\partial \widehat{\varOmega}_{\mathrm{tr}}(R)^A} = {}^{\mathrm{r}}E^i{}_A(R,\bar{r})p_i\,, \\ \widehat{S}_{\mathrm{int}\ A} &= \frac{\partial T}{\partial \widehat{\varOmega}^A_{\mathrm{int}}} = {}^{\mathrm{r}}E^i{}_A(\bar{\varkappa})\pi_i\,. \end{aligned} \tag{86}$$

Let us remind that ${}^{\mathrm{l}}E_A(R,\bar{r})$ are dual to ${}^{\mathrm{l}}E^A(R,\bar{r})$, ${}^{\mathrm{r}}E_A(R,\bar{r})$ are dual to ${}^{\mathrm{r}}E^A(R,\bar{r})$. Similarly, $S_{\mathrm{tr}}(R)_A$, $S_{\mathrm{int}\ A}$ are dual to $\varOmega_{\mathrm{tr}}(R)^A$, $\varOmega^A_{\mathrm{int}}$, $\widehat{S}_{\mathrm{tr}}(R)_A$, $\widehat{S}_{\mathrm{int}\ A}$ are dual respectively to $\widehat{\varOmega}_{\mathrm{tr}}(R)^A$, $\widehat{\varOmega}^A_{\mathrm{int}}$, and p_i, π_i are holonomic canonical momenta conjugate to r^i, $\varkappa^i$ respectively. The basic Poisson brackets have the following geometrically legible form

$$\begin{aligned} \{S_{\mathrm{tr}}(R)_A, S_{\mathrm{tr}}(R)_B\} &= \frac{2}{R}\varepsilon_{AB}{}^C S_{\mathrm{tr}}(R)_C\,, \\ \left\{\widehat{S}_{\mathrm{tr}}(R)_A, \widehat{S}_{\mathrm{tr}}(R)_B\right\} &= -\frac{2}{R}\varepsilon_{AB}{}^C \widehat{S}_{\mathrm{tr}}(R)_C\,, \\ \left\{S_{\mathrm{tr}}(R)_A, \widehat{S}_{\mathrm{tr}}(R)_B\right\} &= 0\,, \\ \{S_{\mathrm{int}\ A}, S_{\mathrm{int}\ B}\} &= \varepsilon_{AB}{}^C S_{\mathrm{int}\ C}\,, \\ \left\{\widehat{S}_{\mathrm{int}\ A}, \widehat{S}_{\mathrm{int}\ B}\right\} &= -\varepsilon_{AB}{}^C \widehat{S}_{\mathrm{int}\ C}\,, \\ \left\{S_{\mathrm{int}\ A}, \widehat{S}_{\mathrm{int}\ B}\right\} &= 0\,, \\ \{S_{\mathrm{tr}}(R)_A, S_{\mathrm{int}\ B}\} &= 0\,, \qquad etc\,. \end{aligned} \tag{87}$$

One shows easily that for the potential systems the Legendre transformation has the following explicit form

$$\begin{aligned} S_{\mathrm{tr}}(R)_A &= \left(m + \frac{\mathrm{I}}{R^2}\right)\varOmega_{\mathrm{tr}}(R)_A - \frac{\mathrm{I}}{R}\,\varOmega_{\mathrm{int}\ A}\,, \\ S_{\mathrm{int}\ A} &= -\frac{\mathrm{I}}{R}\,\varOmega_{\mathrm{tr}}(R)_A + \mathrm{I}\,\varOmega_{\mathrm{int}\ A}\,. \end{aligned} \tag{88}$$

Inverting it, we obtain

$$\begin{aligned} \Omega_{\rm tr}(R)^A &= \frac{1}{m}\, S_{\rm tr}(R)^A + \frac{1}{mR}\, S_{\rm int}{}^A\,, \\ \Omega^A_{\rm int} &= \frac{1}{mR}\, S_{\rm tr}(R)^A + \frac{\mathrm{I}+mR^2}{\mathrm{I}mR^2}\, S_{\rm int}{}^A\,, \end{aligned} \tag{89}$$

where on the right-hand sides of (88), (89) the indices are moved in the trivial sense of Kronecker symbol.

It is well-known that for the Hamiltonian dynamical systems, the time evolution of any phase-space function F satisfies the following equation

$$\frac{dF}{dt} = \{F, H\}\ . \tag{90}$$

Taking as F the functions $S_{\rm tr}(R)_A$, $S_{\rm int\ A}$, we obtain the following system of equations of motion

$$\begin{aligned} \frac{d}{dt}\, S_{\rm tr}(R)_A &= \frac{2}{mR^2}\, \varepsilon_A{}^{BC} S_{\rm int\ B} S_{\rm tr}(R)_C + F_A\,, \\ \frac{d}{dt}\, S_{\rm int\ A} &= \frac{1}{mR}\, \varepsilon_A{}^{BC} S_{\rm tr}(R)_B S_{\rm int\ C} + N_A\,, \end{aligned} \tag{91}$$

respectively for the translational (orbital) and internal (spin) motion. In (91) F_A and N_A denote respectively the pseudovectors of translational force and rotational torque

$$\begin{aligned} F_A &= \{S_{\rm tr}(R)_A, V\} = -{}^1E(R,\bar r)_A V(\bar r, \bar\varkappa)\,, \\ N_A &= \{S_{\rm int\ A}, V\} = -{}^1E(\bar\varkappa)_A V(\bar r, \bar\varkappa)\,. \end{aligned} \tag{92}$$

In this form these expressions are valid for the potential forces, nevertheless, they have also a more general applicability, including *e.g.* friction. Of course, they must have then a different, in general velocity-dependent structure non-derivable from a single function V on the configuration space. A typical situation then is that F_A and N_A in (91) are given by a combination of (92) and some non-conservative velocity-dependent term. Nevertheless, formally (91) is still valid [20,21,22,23].

It is clear that even for "simple" (or rather simply-looking) form of the potentials $V(\bar r, \bar\varkappa)$, it will be difficult to say without a detailed analysis anything about solutions of (91). The question is what may be said about the solution of geodetic equations, *i.e.*, ones with $V = 0$. The dynamics for the system of state variables $S_{\rm tr}(R)_A$, $S_{\rm int\ A}$ becomes then autonomous and ruled only by the mixed term $\overline{S}_{\rm tr}(R)\cdot\overline{S}_{\rm int}$ of the kinetic energy, or more precisely, of the geodetic Hamiltonian

$$\mathcal{T} = \frac{1}{2m}\overline{S}_{\rm tr}(R)\cdot\overline{S}_{\rm tr}(R) + \frac{1}{mR}\overline{S}_{\rm tr}(R)\cdot\overline{S}_{\rm int} + \frac{\mathrm{I}+mR^2}{2\mathrm{I}mR^2}\overline{S}_{\rm int}\cdot\overline{S}_{\rm int} \tag{93}$$

which is obtained from (83) by substituting the inverse Legendre transformation (89). It is convenient to use here the standard symbols of the three-dimensional vector calculus, like scalar products and vector products. The resulting dynamical equations for $\overline{S}_{\rm tr}(R)$, $\overline{S}_{\rm int}$ have the form

$$\begin{aligned} \frac{d}{dt}\,\overline{S}_{\rm tr}(R) &= \frac{2}{mR^2}\,\overline{S}_{\rm int}\times\overline{S}_{\rm tr}(R)\,, \\ \frac{d}{dt}\,\overline{S}_{\rm int} &= \frac{1}{mR}\,\overline{S}_{\rm tr}(R)\times\overline{S}_{\rm int}\,. \end{aligned} \tag{94}$$

This system does not depend on the inertial momentum I. Nevertheless, the full dynamical system for our twelve state variables $\left(\bar{r},\bar{\varkappa},\overline{S}_{\rm tr},\overline{S}_{\rm int}\right)$ is evidently I-dependent and ruled by all three terms of the kinetic energy (79), (93). As expected, in the limit of infinite radius, $R\to\infty$, the quantities $\overline{S}_{\rm tr}(R)$, $\overline{S}_{\rm int}$ are conserved. For a finite R equations (94) imply that the quantity

$$\overline{J} := \frac{R}{2}\,\overline{S}_{\rm tr}(R)+\overline{S}_{\rm int} \tag{95}$$

is a vector constant of motion. And obviously, the lengths of its constituents are so as well; the quantities

$$\overline{S}_{\rm tr}(R)\cdot\overline{S}_{\rm tr}(R)\,,\qquad \overline{S}_{\rm int}\cdot\overline{S}_{\rm int} \tag{96}$$

are constants of motion in virtue of (94).

Therefore, in geodetic motion we have five independent constants of motion (95), (96) in the six-dimensional space of angular momenta $\overline{S}_{\rm tr}(R)$, $\overline{S}_{\rm int}$. The two-dimensional plane determined by vectors $\overline{S}_{\rm tr}(R)$, $\overline{S}_{\rm int}$ rotates around the direction given by (95). The lengths of $\overline{S}_{\rm tr}(R)$, $\overline{S}_{\rm int}$ and the angle between these vectors are constants of motion. With fixed values of the mentioned constants of motion, the only time-dependent parameter is the angle between the plane spanned by $\overline{S}_{\rm tr}(R)$, $\overline{S}_{\rm int}$ and a fixed plane containing the vector $\overline{J}$.

This nice geodetic picture breaks down when some potential $V(\bar{r},\bar{\varkappa})$ is introduced.

In the model discussed above, the coupling between two kinds of angular momenta (translational one and spin) is realized exclusively by the second $\left(\sim\overline{S}_{\rm tr}(R)\cdot\overline{S}_{\rm int}\right)$ term of the kinetic energy (93). Being purely geometric, and built algebraically (bilinearly) of generators, it enables one to perform the above qualitative discussion of solutions. Even for relatively simple structure of the potential $V(\bar{r},\bar{\varkappa})$, it is in general practically impossible to deduce anything, even on the purely qualitative level as done above.

Let us observe that the discussion carried out here concerned a rather general situation when the translational and internal angular velocities, or rather — the translational and spin angular momenta, were superposed with the use of coefficients involving three arbitrary constants R, I, m. In any case, we had no so simple relationships like (30), (31).

What concerns the global classical problems, some strange and surprising phenomena may appear, when instead of working in the simply connected configuration space $S^3(0,R) \times \mathrm{SU}(2) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$, we perform the division to $\mathrm{SU}(2) \times \mathrm{SO}(3,\mathbb{R})$, $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$ *etc.* In any case, it is more safe to consider dynamical problems in $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and to look carefully what results when the quotient procedure is performed.

When discussing the quantised problem [24], one uses the Peter–Weyl expansion (47) on $\mathrm{L}^2\left(\mathrm{SU}(2) \times \mathrm{SU}(2)\right)$, or rather, its version with the k-variable in the "left" SU(2) replaced by $r = Rk/2$ (69). Then the classical angular momenta (85) are replaced by differential operators

$$\begin{aligned}
\boldsymbol{S}_{\mathrm{tr}}\left(R\right)_A &= \frac{\hbar}{\imath}\, {}^{\mathrm{l}}E^m{}_A\left(R, \overline{r}\right) \frac{\partial}{\partial r^m}\,, \\
\boldsymbol{S}_{\mathrm{int}\,A} &= \frac{\hbar}{\imath}\, {}^{\mathrm{l}}E^m{}_A\left(\overline{\kappa}\right) \frac{\partial}{\partial \kappa^m}\,,
\end{aligned} \tag{97}$$

and similarly for their co-moving version

$$\begin{aligned}
\widehat{\boldsymbol{S}}_{\mathrm{tr}}\left(R\right)_A &= \frac{\hbar}{\imath}\, {}^{\mathrm{r}}E^m{}_A\left(R, \overline{r}\right) \frac{\partial}{\partial r^m}\,, \\
\widehat{\boldsymbol{S}}_{\mathrm{int}\,A} &= \frac{\hbar}{\imath}\, {}^{\mathrm{r}}E^m{}_A\left(\overline{\kappa}\right) \frac{\partial}{\partial \kappa^m}\,.
\end{aligned} \tag{98}$$

Their quantum Poisson brackets, *i.e.*, commutators divided by $\hbar\imath$ have the form

$$\begin{aligned}
\frac{1}{\hbar\imath}\left[\boldsymbol{S}_{\mathrm{tr}}\left(R\right)_A, \boldsymbol{S}_{\mathrm{tr}}\left(R\right)_B\right] &= \frac{2}{R}\,\varepsilon_{AB}{}^C \boldsymbol{S}_{\mathrm{tr}}\left(R\right)_C\,, \\
\frac{1}{\hbar\imath}\left[\widehat{\boldsymbol{S}}_{\mathrm{tr}}\left(R\right)_A, \widehat{\boldsymbol{S}}_{\mathrm{tr}}\left(R\right)_B\right] &= -\frac{2}{R}\,\varepsilon_{AB}{}^C \widehat{\boldsymbol{S}}_{\mathrm{tr}}\left(R\right)_C\,, \\
\frac{1}{\hbar\imath}\left[\boldsymbol{S}_{\mathrm{tr}}\left(R\right)_A, \widehat{\boldsymbol{S}}_{\mathrm{tr}}\left(R\right)_B\right] &= 0\,,
\end{aligned} \tag{99}$$

and similarly for $\boldsymbol{S}_{\mathrm{int}}$, $\widehat{\boldsymbol{S}}_{\mathrm{int}}$, but without the $2/R$-multipliers on the right-hand side

$$\begin{aligned}
\frac{1}{\hbar\imath}\left[\boldsymbol{S}_{\mathrm{int}\,A}, \boldsymbol{S}_{\mathrm{int}\,B}\right] &= \varepsilon_{AB}{}^C\, \boldsymbol{S}_{\mathrm{int}\,C}\,, \\
\frac{1}{\hbar\imath}\left[\widehat{\boldsymbol{S}}_{\mathrm{int}\,A}, \widehat{\boldsymbol{S}}_{\mathrm{int}\,B}\right] &= -\varepsilon_{AB}{}^C\, \widehat{\boldsymbol{S}}_{\mathrm{int}\,C}\,, \\
\frac{1}{\hbar\imath}\left[\boldsymbol{S}_{\mathrm{int}\,A}, \widehat{\boldsymbol{S}}_{\mathrm{int}\,B}\right] &= 0\,.
\end{aligned} \tag{100}$$

It is clear that all translational quantities do Poisson-commute with the all internal ones. The quantum counterpart of (93), *i.e.*, the operator of kinetic energy, is given by

$$\begin{aligned} \boldsymbol{T} &= \frac{1}{2m}\,\delta^{AB}\,\boldsymbol{S}_{\mathrm{tr}}(R)_{A}\,\boldsymbol{S}_{\mathrm{tr}}(R)_{B} + \frac{1}{mR}\,\delta^{AB}\,\boldsymbol{S}_{\mathrm{tr}}(R)_{A}\,\boldsymbol{S}_{\mathrm{int}\,B} \\ &\quad + \frac{1}{2}\left(\frac{1}{I} + \frac{1}{mR^{2}}\right)\delta^{AB}\,\boldsymbol{S}_{\mathrm{int}\,A}\,\boldsymbol{S}_{\mathrm{int}\,B}\,. \end{aligned} \tag{101}$$

All operators act in the Hilbert space of wave functions on $\mathrm{SU}(2)\times\mathrm{SU}(2)$, or on some of the quotient groups, (3)–(6). More precisely, they act on $S^3(0,R)\times\mathrm{SU}(2)$, or on a quotient manifold. All mentioned manifolds are compact, and the Hilbert structure in functions spaces over them is meant in the sense of the natural scalar product

$$\langle \Psi_1 \mid \Psi_2\rangle = \int \overline{\Psi_1(u,v)}\Psi_2(u,v)d\mu_R(u)d\mu(v)\,, \tag{102}$$

where, obviously, the measures μ_R, μ are meant in the sense of the Killing metrics on $\mathrm{S}^3(0,R)$, $\mathrm{SU}(2)$,

$$\begin{aligned} d\mu_R\left(u\left(R,\bar r\right)\right) &= R^2\sin^2\frac{r}{R}\sin\vartheta dr d\vartheta d\varphi = \frac{R^2}{r^2}\sin^2\frac{r}{R}d_3\bar r\,, \\ d\mu\left(v\left(\bar\kappa\right)\right) &= 4\sin^2\frac{\kappa}{2}\sin\vartheta d\kappa d\vartheta d\varphi = \frac{4}{\kappa^2}\sin^2\frac{\kappa}{2}d_3\bar\kappa\,. \end{aligned} \tag{103}$$

This is not the Killing normalisation nor one used often with finite or compact groups, when the group volume equals one by definition. Normalisation in (103) is one suited to coordinates used, *i.e.*, in our coordinates the density of measure at the neutral element equals one. The group volumes are given by

$$\mu\left(S^3(0,R)\right) = 2\pi^2R^3\,, \qquad \mu\left(\mathrm{SU}(2)\right) = 16\pi^2\,, \qquad \mu\left(\mathrm{SO}(3,\mathbb{R})\right) = 8\pi^2\,. \tag{104}$$

Clearly, it is a nice thing to have the true definition of volume of $S^3(0,R)$ and to remember that SU(2) is "twice larger" than $\mathrm{SO}(3,\mathbb{R})$. Nevertheless, one must remember that there are problems and standard formulas based on the normalisation of volume to unity. Forgetting this fact one can introduce mistakes based on the bad normalisation, *e.g.*, in the Clebsch–Gordan formulas for multiplication of matrix elements of unitary irreducible representations. However, in this paper we do not deal with such problems.

If we use the expansion (46), (47), then, obviously, the action of $\boldsymbol{S}_{\mathrm{tr}\;A}$ on wave functions is represented by the following algebraic action on expansion coefficients

$$\left[C^{\mathrm{l}}{}_{km}{}^{s}{}_{nr}\right] \mapsto \left[\frac{2}{R}\,C^{\mathrm{l}}{}_{kp}{}^{s}{}_{nr}\;S^{\mathrm{l}}{}_{pm}\right]; \tag{105}$$

obviously, the summation over p is meant here. And similarly, the action of spin operators $\boldsymbol{S}_{\mathrm{int}\ A}$ is in the language of C-coefficients represented by

$$\left[C^{\mathrm{l}}{}_{km}{}^{s}{}_{nr}\right] \mapsto \left[C^{\mathrm{l}}{}_{km}{}^{s}{}_{np}\ S^{s}{}_{pr}\right], \tag{106}$$

again the summation convention over the matrix index p is assumed.

The $2/R$ -factor in (105) is very important. Namely, when $R \to \infty$, the distances between energy levels with fixed quantum numbers tend to zero. Because of this, the spectrum becomes, roughly speaking, continuous, just as it is in $\mathbb{R}^3$.

All terms of the kinetic energy operator (101) do commutate with the operators

$$(\boldsymbol{S}_{\mathrm{tr}})^2 = \delta^{AB}\boldsymbol{S}_{\mathrm{tr}\ A}\boldsymbol{S}_{\mathrm{tr}\ B}\,, \qquad (\boldsymbol{S}_{\mathrm{int}})^2 = \delta^{AB}\boldsymbol{S}_{\mathrm{int}\ A}\boldsymbol{S}_{\mathrm{int}\ B}\,. \tag{107}$$

Therefore, in geodetic problems the quantum numbers s, j are "good quantum numbers" which may be used to label the basic stationary states of (101)

$$\boldsymbol{T}\Psi = E\Psi\,. \tag{108}$$

Those basic states, labelled partially by s, J satisfy the following system of algebraic eigenequations obtained by substituting the above data to (108)

$$\delta^{AB}C^{\mathrm{l}}{}_{kp}{}^{s}{}_{nq}\ S_{A}{}^{\mathrm{l}}{}_{pm}S_{B}{}^{s}{}_{qr} = \lambda C^{\mathrm{l}}{}_{km}{}^{s}{}_{nr}\,, \tag{109}$$

where, let us remind, $S^{\mathrm{l}}{}_{A}$, $S^{s}{}_{B}$ are matrices of the A-th and B-th component of angular momenta within the l-th and s-th unitary irreducible representations of SU(2). The eigenvalues λ are related to the energy eigenvalues E as follows

$$E = \frac{2}{mR^2}\left(\lambda + l\ (l+1)\ \hbar^2\right) + \frac{1}{2}\left(\frac{1}{\mathrm{I}} + \frac{1}{mR^2}\right) s\ (s+1)\ \hbar^2\,. \tag{110}$$

It is clear that λ and $l(l+1)\hbar^2$ are R-independent, and so is the first of s-terms, one proportional to $\frac{1}{\mathrm{I}}$. With any fixed values of quantum numbers l, s, there is a complete degeneracy with respect to the quantum numbers k, n in (109). This degeneracy is $(2k+1)(2n+1)$-fold one. Unlike this, the values of λ, as seen in (109) are somehow linked to the second, *i.e.*, right quantum numbers p, q (or (m,r)). We do not get into details here. As seen from (110), when $R \to \infty$, the spectrum of "translational" quantum numbers in $S^3(0,R)$ becomes "almost continuous" in the sense that for fixed quantum numbers, the transition frequencies tend to zero. Translational energy levels for fixed quantum numbers become closer and closer. Obviously, it is not so for the second term of (110), where they become asymptotically the energy levels of the spherical top.

Of course, when some $(\bar{r}, \bar{\varkappa})$-dependent potentials are admitted, then everything becomes catastrophically more complicated. The absolute values of translational and internal angular momenta are no longer constants of motion and their quantum numbers l, s cease to be "good quantum numbers" any longer. A proper description is then based on the use of Clebsh–Gordan coefficients for $\mathrm{SU}(2)$ or $\mathrm{SO}(3, \mathbb{R})$.

3. A pair of rigid bodies in Euclidean space

Let us finish with a few remarks concerning another aspect of $\mathrm{SO}(4, \mathbb{R})$, or rather of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and its quotients (3)–(6). Namely, we mean a system of two gyroscopes in the flat Euclidean space identified with $\mathbb{R}^3$. Their primary configuration space is the simply connected $\mathrm{SU}(2) \times \mathrm{SU}(2)$, when translational motion is not taken into account. When translational degrees of freedom are admitted, the configuration space is given by the semidirect product $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R}^3 \times \mathbb{R}^3$, or again by the corresponding quotient group. In any case, it is convenient to begin with $\mathrm{SU}(2) \times \mathrm{SU}(2)$, especially when dealing with quantum problems. $\mathrm{SU}(2)$ is a spinorial extension of the configuration space $\mathrm{SO}(3, \mathbb{R})$ of a single rigid body without translational motion. The kinetic energy of the classical rigid body is then given by a left-invariant metric tensor on $\mathrm{SU}(2)$ or $\mathrm{SO}(3, \mathbb{R})$, *i.e.*, explicitly, by

$$T = \frac{1}{2} \sum_{A=1}^{3} \mathrm{I}_A \, \widehat{\Omega}^{A2} , \qquad \widehat{\Omega}^A = {}^{\mathrm{r}}E^A{}_j(\bar{\varkappa}) \frac{d\varkappa^j}{dt} , \tag{111}$$

with the meaning of symbols as above. Therefore, $\bar{\varkappa}$ is the rotation vector, I_A are co-moving components of the internal tensor, and $\widehat{\Omega}^A$ are co-moving components of the angular velocity vector. When one deals with a pair of rigid bodies with configurations described by the rotation vectors $\bar{\varkappa}$, $\bar{\lambda}$, then, obviously, the kinetic energy is given by the sum

$$T = \frac{1}{2} \sum_{A=1}^{3} \mathrm{I}_A(1) \widehat{\Omega}[\bar{\varkappa}]^{A2} + \frac{1}{2} \sum_{A=1}^{3} \mathrm{I}_A(2) \widehat{\Omega} \left[\bar{\lambda}\right]^{A2} , \tag{112}$$

where, obviously,

$$\widehat{\Omega}[\bar{\varkappa}]^A = {}^{\mathrm{r}}E^A{}_j(\bar{\varkappa}) \frac{d\varkappa^j}{dt} , \qquad \widehat{\Omega} \left[\bar{\lambda}\right]^A = {}^{\mathrm{r}}E^A{}_j \left(\bar{\lambda}\right) \frac{d\lambda^j}{dt} . \tag{113}$$

The quantities $\mathrm{I}_A(1)$, $\mathrm{I}_A(2)$ in (112) are, obviously, the co-moving main inertial moments. The quadratic form (112) is left-invariant on $\mathrm{SU}(2) \times$

SU(2) (or on $\mathrm{SO}(3,\mathbb{R})\times\mathrm{SO}(3,\mathbb{R})$). For the potential systems, after Legendre transformation one obtains for (112) the following canonical expression

$$\mathcal{T} = \sum_{A=1}^{3} \frac{1}{2\mathrm{I}_A(1)} \widehat{S}[\bar{\varkappa}]_A{}^2 + \frac{1}{2}\sum_{A=1}^{3} \frac{1}{2\mathrm{I}_A(2)} \widehat{S}\left[\bar{\lambda}\right]_A{}^2, \tag{114}$$

where

$$\widehat{S}\left[\bar{\varkappa}\right]_A = \mathrm{I}_A(1)\widehat{\Omega}\left[\bar{\varkappa}\right]_A\,, \qquad \widehat{S}\left[\bar{\lambda}\right]_A = \mathrm{I}_A(2)\widehat{\Omega}\left[\bar{\lambda}\right]_A\,, \tag{115}$$

and the tensor indices are shifted with the use of Kronecker-delta.

It is clear that $\widehat{S}[\bar{\varkappa}]_A$, $\widehat{S}[\bar{\lambda}]_B$ are in the Poisson-involution with each other and have the usual Poisson brackets for co-moving components

$$\left\{\widehat{S}[\bar{\varkappa}]_A, \widehat{S}[\bar{\varkappa}]_B\right\} = -\varepsilon_{AB}{}^C\widehat{S}[\bar{\varkappa}]_C\,, \qquad \left\{\widehat{S}\left[\bar{\lambda}\right]_A, \widehat{S}\left[\bar{\lambda}\right]_B\right\} = -\varepsilon_{AB}{}^C\widehat{S}\left[\bar{\lambda}\right]_C\,. \tag{116}$$

In the quantised theory those spin components are represented by the operators

$$\widehat{\boldsymbol{S}}[\bar{\varkappa}]_A = \frac{\hbar}{i}\,{}^{\mathrm{r}}E^a{}_A(\bar{\varkappa})\frac{\partial}{\partial\varkappa^a}\,, \qquad \widehat{\boldsymbol{S}}\left[\bar{\lambda}\right]_A = \frac{\hbar}{i}\,{}^{\mathrm{r}}E^a{}_A\left(\bar{\lambda}\right)\frac{\partial}{\partial\lambda^a} \tag{117}$$

and obey the quantum Poisson brackets identical with (116).

Using the expansion (46), (47) we easily find that the operators $\widehat{\boldsymbol{S}}[\bar{\varkappa}]_A$, $\widehat{\boldsymbol{S}}[\bar{\lambda}]_A$ acting on the wave amplitudes Ψ result in the following action on coefficients

$$\begin{aligned}\left[C^{\mathrm{l}}{}_{km}{}^{s}{}_{nr}\right] &\mapsto \left[S^{\mathrm{l}}{}_{kp}C^{\mathrm{l}}{}_{pm}{}^{s}{}_{nr}\right]\,,\\ \left[C^{\mathrm{l}}{}_{km}{}^{s}{}_{nr}\right] &\mapsto \left[S^{s}{}_{np}C^{\mathrm{l}}{}_{km}{}^{s}{}_{pr}\right]\,.\end{aligned} \tag{118}$$

Therefore, the kinetic energy operator (114) acts algebraically in this representation, multiplying the corresponding $(2l+1)\times(2l+1)$ and $(2s+1)\times(2s+1)$ submatrices of C^{ls} on the left, respectively by

$$\sum_{A=1}^{3}\frac{1}{2\mathrm{I}_A(1)}\widehat{S}^{\mathrm{l}}\left[\bar{\varkappa}\right]_A^2\,, \qquad \sum_{A=1}^{3}\frac{1}{2\mathrm{I}_A(2)}\widehat{S}^{s}\left[\bar{\lambda}\right]_A^2 \tag{119}$$

and summing the results. In particular, when the both tops are spherical, this consists in multiplying by the Casimir invariants of S^{l}, S^{s} matrices

$$\left[C^{\mathrm{l}}{}_{km}{}^{s}{}_{nr}\right] \mapsto \left[\hbar^2\left(l(l+1)+s(s+1)\right)C^{\mathrm{l}}{}_{km}{}^{s}{}_{nr}\right]\,. \tag{120}$$

One can also admit the gyroscopic coupling of angular momenta, *i.e.*, introduce to (114) the term bilinear in $\widehat{S}[\bar{\varkappa}]_A$, $\widehat{S}[\bar{\lambda}]_B$, so as to obtain

$$\begin{aligned}\mathcal{T} \;&= \sum_{A=1}^{3} \frac{1}{2\mathrm{I}_A(1)} \widehat{S}[\bar{\varkappa}]_A{}^2 + \frac{1}{2} \sum_{A=1}^{3} \frac{1}{2\mathrm{I}_A(2)} \widehat{S}\left[\bar{\lambda}\right]_A{}^2 \\ &+ \sum_{A,B=1}^{3} \frac{1}{2\mathrm{I}_{AB}(1,2)} \widehat{S}[\bar{\varkappa}]_A \widehat{S}\left[\bar{\lambda}\right]_B \,, \end{aligned} \tag{121}$$

or its quantum operator version. The matrix labels l, s are then still good quantum numbers and everything reduces to the (l, s)-subspace of wave functions. This breaks down when some potential terms $V(u, v)$ are admitted and the l, s-quantities cease to be constants of motion. And the more so when the potential energy depends on all configuration variables $V(x, y; u, v)$; the vectors x, y refer to the positions of the centers of mass.

If two rigid bodies are spherical and identical, *i.e.*,

$$\mathrm{I}_A(1) = \mathrm{I}_A(2) = \mathrm{I}\,, \qquad A = 1, 2, 3\,, \tag{122}$$

and there is no term of gyroscopic interaction (the third term in (121)), then again the problem reduces locally to the doubly-invariant (left- and right-invariant) geodetic problem on $\mathrm{SU}(2) \times \mathrm{SU}(2)$ or on $\mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R})$ or $\mathrm{SO}(4, \mathbb{R})$.

Using the four-dimensional language as in (29)–(31) we find that the geodetic part of Hamiltonian is given by

$$\mathcal{T} = \frac{1}{4\mathrm{I}} \left(\overline{M} \cdot \overline{M} + \overline{N} \cdot \overline{N} \right) \,, \tag{123}$$

where

$$M_A = \widehat{S}[\bar{\varkappa}]_A + \widehat{S}\left[\bar{\lambda}\right]_A \,, \qquad N_A = \widehat{S}[\bar{\varkappa}]_A - \widehat{S}\left[\bar{\lambda}\right]_A \,. \tag{124}$$

The expression (123) is proportional to the second-order Casimir invariant of $\mathrm{SO}(4, \mathbb{R})$,

$$\mathcal{T} \simeq g^{\alpha\mu} g^{\beta\nu} \varepsilon_{\alpha\beta} \varepsilon_{\mu\nu} \,, \tag{125}$$

where $\varepsilon_{\alpha\beta}$ are $\mathrm{SU}(2) \times \mathrm{SU}(2)$ counterparts of (29).

This is the geodetic model suggested by the three-dimensional geometry. However, in $\mathbb{R}^4$ there exists also another second-order Casimir invariant. It is obtained as the square-root of a fourth-order Casimir, namely

$$\det\left[\varepsilon_{\mu\nu}\right] \simeq \left(\frac{1}{4} \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{\mu\nu} \varepsilon_{\alpha\beta} \right)^2 \simeq \left(\overline{M} \cdot \overline{N} \right)^2 \,. \tag{126}$$

However, from the point of view of three-dimensional geometry of $\mathbb{R}^3$, the quantity $\overline{M} \cdot \overline{N}$ is not a scalar, it is a pseudoscalar. It changes its sign under the $\mathbb{R}^3$-reflection. The reason is that the four-dimensional Levi-Civita symbol is used in the construction of $\overline{M} \cdot \overline{N}$. Nevertheless, it is not excluded that the geodetic model combining the two expressions might be used,

$$\mathcal{T} = \frac{1}{4\mathrm{I}} \left(\overline{M} \cdot \overline{M} + \overline{N} \cdot \overline{N} \right) + \frac{1}{4K} \, \overline{M} \cdot \overline{N} \, , \tag{127}$$

K being an additional inertial parameter. But, unfortunately, the second term of (127) is not positively definite. The same concerns the operator quantum version. But of course, one can speculate about the fourth-order "kinetic energy"

$$\mathcal{T} = \frac{1}{4\mathrm{I}} \left(\overline{M} \cdot \overline{M} + \overline{N} \cdot \overline{N} \right) + \frac{1}{4L} \left(\overline{M} \cdot \overline{N} \right)^2 . \tag{128}$$

It is evidently positive (if both I, L are so), but its structure is rather far from the physical intuition, both on the classical and quantum level. And the more so when some potentials are added to the geodetic term (128).

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Appendix

Some geometric remarks

Let us finish with an appendix concerning some geometry of SU(2) and $\mathrm{SO}(3,\mathbb{R})$. It is not directly connected with the problems investigated above, nevertheless it sheds some light on them. Take the Killing metric (70), (71) on the spherical world, and introduce instead r a new "radial" variable of the dimension of length,

$$\xi = R \tan \frac{r}{2R} \, , \tag{129}$$

and keeping the same angular variables $\bar{n} = \bar{r}/r = \bar{\xi}/\xi$. It is clear that this is the conformal mapping of $S^3(0,R)$ onto $\mathbb{R}^3$; the South Pole $r = \pi R$ explodes to infinity, $\xi = \infty$, and

$$ds^2 = \frac{4}{\left(1 + \xi^2/R^2\right)^2} \left(d\xi^2 + \xi^2 \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \right) \, , \tag{130}$$

or, in analogy to (70), (71),

$$ds^2 = \frac{4}{\left(1+\xi^2/R^2\right)^2}\left(d\xi^2 + \xi^2 d\bar{n}\cdot d\bar{n}\right)\,, \tag{131}$$

or, using the tensorial way of writing, the metric is

$$g = \frac{4}{\left(1+\xi^2/R^2\right)^2}\left(d\xi\otimes d\xi + \xi^2\delta_{AB}\; dn^A\otimes dn^B\right)\,. \tag{132}$$

The $\mathrm{SO}(3,\mathbb{R})$-points correspond to

$$r = \pi R/2, \qquad \xi = R \tag{133}$$

taken with the antipodal identification if we are to get the "elliptic space".

It is interesting to see what results if the corresponding transformation is performed just on $\mathrm{SU}(2)$, by introducing a new radial variable,

$$\rho = a\tan k/4\,, \tag{134}$$

where a denotes an arbitrary dimension-less positive constant. Then, the squared arc element becomes

$$ds^2 = \frac{16}{a^2\left(1+\rho^2/a^2\right)^2}\left(d\rho^2 + \rho^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right)\,, \tag{135}$$

again the conformal mapping of $\mathrm{SU}(2)$ onto $\mathbb{R}^3$. The point of $\mathrm{SO}(3,\mathbb{R})$ is given by

$$k = \pi\,, \qquad \rho = a\,, \tag{136}$$

again with the antipodal identification.

This conformal mapping of $S^3(0,R)$, $\mathrm{SU}(2)$ onto $\mathbb{R}^3$ suggests us certain explicitly integrable models of potentials for the "small" rigid body in Einstein space.

Let us also mention about the projective mapping of $\mathrm{SO}(3,\mathbb{R})$ onto $P\mathbb{R}^3$, given by

$$\theta = 2\tan\frac{k}{2}\,, \qquad \bar{\theta}/\theta = \bar{k}/k = \bar{n}\,. \tag{137}$$

It transforms the Killing metric on $\mathrm{SO}(3,\mathbb{R})$ in an apparently non-interesting way, resulting in

$$ds^2 = \frac{16}{\left(4+\theta^2\right)^2}d\theta^2 + \frac{4}{\left(4+\theta^2\right)^2}\theta^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\,. \tag{138}$$

Nevertheless, using the Hamilton–Jacobi equation one can show that it is really projective, *i.e.*, transforms the Killing geodetics of $\mathrm{SO}(3,\mathbb{R})$ onto straight lines in $\mathbb{R}^3$. Moreover, one can show that it establishes a correspondence between Bertrand systems on $\mathbb{R}^3$ and ones in $\mathrm{SO}(3,\mathbb{R})$ (or, in a sense in $\mathrm{SU}(2)$). The resulting Bertrand systems on the rotation/unitary group correspond, respectively, to the isotropic oscillator and Coulomb–Kepler problems

$$V_{\mathrm{osc}} = 2\varkappa \tan^2 \frac{k}{2}\,, \qquad V_{\mathrm{co}} = -\frac{\alpha}{2}\cot^2 \frac{k}{2}\,. \tag{139}$$

All their orbits are closed. Because of this integrability and complete degeneracy they might be perhaps useful in certain potential models mentioned in this paper.

The projective correspondence between the phase portraits for

$$\begin{aligned} L &= T - V \\ &= \frac{\mathrm{I}}{2}\left(\frac{16}{(4+\theta^2)^2}\left(\frac{d\theta}{dt}\right)^2 + \frac{4}{4+\theta^2}\theta^2\left(\left(\frac{d\vartheta}{dt}\right)^2 + \sin^2\vartheta\left(\frac{d\varphi}{dt}\right)^2\right)\right) - V(\vartheta)\,, \end{aligned} \tag{140}$$

where V is given by (139), and the phase portraits for the material point,

$$L = T - V = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\left(\frac{d\vartheta}{dt}\right)^2 + \sin^2\vartheta\left(\frac{d\varphi}{dt}\right)^2\right)\right) - V(r)\,, \tag{141}$$

where V is the usual $\mathbb{R}^3$-Bertrand potential,

$$V_{\mathrm{osc}} = \frac{\varkappa}{2}r^2\,, \qquad V_{\mathrm{co}} = -\frac{\alpha}{r} \tag{142}$$

may be easily seen. One has to use the planar Hamilton–Jacobi equation with the "radial" variable given respectively by

$$w = \frac{2}{\theta}\,, \qquad w = \frac{2}{r}\,. \tag{143}$$

It may be shown that the mapping (137) establishes the one-to-one relationship between phase portraits for the Bertrand-type potentials on $\mathrm{SU}(2)$ and $\mathbb{R}^3$. Nevertheless, the difference in topology of $\mathrm{SU}(2)$ and $\mathbb{R}^3$ implies that orbits are mapped onto orbits, however, they are swept with different velocities.

Let us mention that this relationship may be related to the Beltrami theorem which establishes a link between geodesics of different constant-curvature spaces. We close this paper with some rough remarks. We considered here mainly (although not exclusively) the special case of geodetic motion in the principal fibre bundle of orthonormal frames $F(S^3(0,R),g)$. One of the reasons was our interest in the exceptional case of the resonance I= mR^2 between translational and internal motion. But, as mentioned above, in the beginning of Sec. 2, the same may be done in a more general case. The geometric couplings between linear momentum and torsion and the one between spin and curvature are just like in this special case. *cf.*, *e.g.*, [13,14,15,16,17]. It is more interesting to ask about the true generally-relativistic model. In our opinion, this may be done in a similar way, in a principal fibre bundle of orthonormal frames over the space-time manifold, and may be with the use of variational principles like those suggested by H.P. Künzle. In any case, except some adaptation to the four-dimensional language, the structure of equations of motion and of the couplings torsion-linear momentum, curvature-spin is expected to remain structurally like in the non-relativistic model.

The topological structure of the three-dimensional space does not mean essentially. It is only true that the mentioned resonance between translations and internal rotations does not occur. Some remarks concerning the general spatial case, *e.g.*, in the Lobachevski space, are given in [13,14,15,16,17].

It is important that the main terms of the classical equations of motion of structured bodies are essentially geometric and have to do with Bianchi identities (the coupling: linear momentum-torsion and spin-curvature). In a sense they explain all experimental tests like the Gravity Probe B [25].

There is a natural question concerning the relationship between our model and the spin connection used in quantum theory of relativistic spinning particles. To compare them, one must first formulate the generally-covariant (generally-relativistic) version of our model, it is not yet ready. We expect a good compatibility on the "geodetic level". Let us mention, however that in classical field theories based on the gauge idea, spin is a primary characteristic of the particle. In our model, it is an aspect of the quantised internal motion. There are more degrees of freedom and it is quite possible that our model may predict some new phenomena in comparison with the purely gauge model. Nevertheless, in generally-relativistic model of particles with internal degrees of freedom, the spin connection assigned to the usual affine connection is a necessary constituent. But the final result may be different for particular models of interaction depending on rotational degrees of freedom.

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